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On the Lattice of Convex Sublattices of $S(B_n)$ and $S(C_n)$

G. Sheeba Merlin[1,*], A. Vethamanickam[2]

[1] *Department of Mathematics, Karunya University, Coimbatore, India*
[2] *Department of Mathematics, Rani Anna College for Women, Tirunelveli, India*

Abstract. In this paper we prove that $CS[S(B_n)]$ and $CS[S(C_n)]$ are Eulerian lattices under the set inclusion relation but they are neither simplicial nor dual simplicial.

2010 Mathematics Subject Classifications: 06A06, 06A07, 06B10

Key Words and Phrases: Lattices, Convex Sublattices, Dual Simplicial Lattices, Eulerian Lattices

1. Introduction

The study of lattice of convex sublattices of a lattice was started by K. M. Koh[3], in the year 1972. He investigated the internal structure of a lattice L, in relation to $CS(L)$, like so many other authors for various algebraic structures such as groups, Boolean algebras, directed graphs and so on.

In [3], several basic properties of $CS(L)$ have been studied where one of the results proved is "If L is complemented then $CS(L)$ is complemented". Also, the connection of the structure of $CS(L)$ with those of the ideal lattice $I(L)$ and the dual ideal lattice $D(L)$ are examined by K. M. Koh. He also derived the best lower bound and upper bound for the cardinality of $CS(L)$, where L is finite. In a subsequent paper[1], Chen C. K., Koh K. M., proved that

$$CS(L \times K) \cong [(CS(L) - \{\emptyset\}) \times (CS(K) - \{\emptyset\})] \cup \{\emptyset\}.$$

Finally they proved that when L is a finite lattice and $CS(L) \cong CS(M)$ and if L is relatively complemented(complemented) then M is relatively complemented(complemented). This is true for Eulerian lattices, since an Eulerian lattice is relatively complemented. These results gave motivation for us to look into the connection between L and $CS(L)$ for Eulerian lattices which are a class of lattices not defined by identities. A construction of a new Eulerian lattice $S(B_n)$ from a Boolean algebra B_n of rank n is found in the thesis

*Corresponding author.

Email addresses: sheebamerlin@karunya.edu (G. Sheeba Merlin), dr_vethamanickam@yahoo.co.in (A. Vethamanickam)

of V. K. Santhi in 1992[11]. In 2012, Subbarayan had proved in his paper that the lattice of convex sublattices of a boolean algebra B_n, of rank n, $CS(B_n)$ with respect to the set inclusion relation is a dual simplicial Eulerian lattice.

In this paper, we are going to look at the similar structure of $CS(S(B_n))$.

$S(B_4)$ is shown in the following diagram.

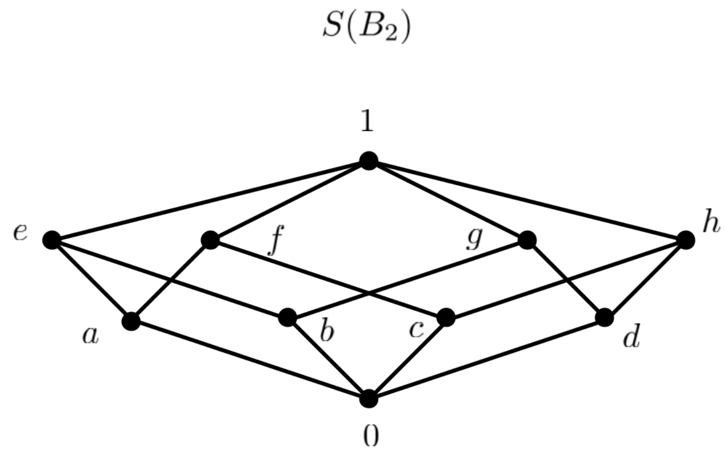

Figure 1: $S(B_2)$

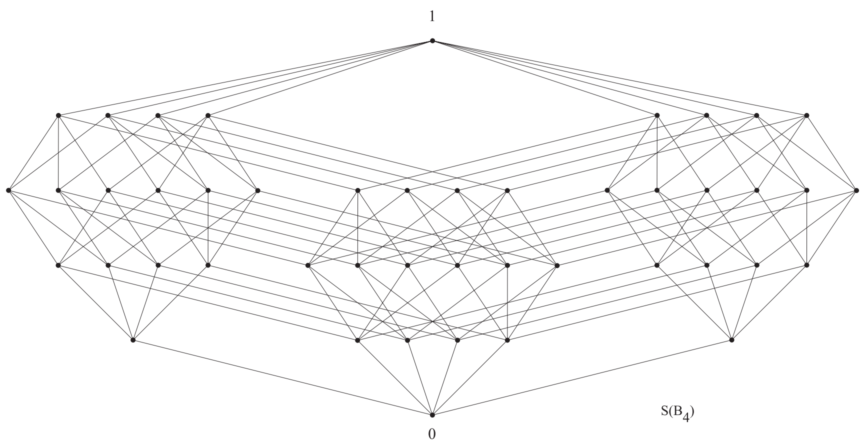

Figure 2: $S(B_4)$

2. Preliminaries

Throughout this section $CS(L)$ is equipped with the partial order of set inclusion relation.

Definition 2.1. *A finite graded poset P is said to be Eulerian if its Möbius function assumes the value $\mu(x, y) = (-1)^{l(x,y)}$ for all $x \leq y$ in P, where $l(x, y) = \rho(y) - \rho(x)$ and ρ is the rank function on P.*

An equivalent definition for an Eulerian poset is as follows:

Lemma 2.2. *[5] A finite graded poset P is Eulerian if and only if all intervals $[x, y]$ of length $l \geq 1$ in P contain an equal number of elements of odd and even rank.*

Example 2.3. *Every Boolean algebra of rank n is Eulerian and the lattice C_4 of Figure 2 is an example for a non-modular Eulerian lattice.*
Also, every C_n is Eulerian for $n \geq 4$.

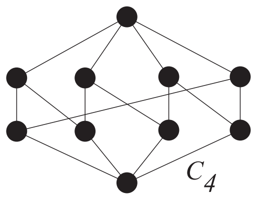

Figure 3: Non-modular Eulerian lattice

Lemma 2.4. *[12] If L_1 and L_2 are two Eulerian lattices then $L_1 \times L_2$ is also Eulerian.*

We note that any interval of an Eulerian lattice is Eulerian and an Eulerian lattice cannot contain a three element chain as an interval.

Definition 2.5. *A poset P is called Simplicial if for all $t \neq 1 \in P$, $[0, t]$ is a Boolean algebra and P is called Dual Simplicial if for all $t \neq 0 \in P$, $[t, 1]$ is a Boolean algebra.*

Lemma 2.6. *[1] Let L and K be any two lattices. Then*

$$CS(L \times K) \approxeq [(CS(L) - \{\emptyset\}) \times (CS(K) - \{\emptyset\})] \cup \{\emptyset\}.$$

Lemma 2.7. *[14] Let B_n be a Boolean lattice of rank n. Then $CS(B_n)$ is a dual simplicial Eulerian lattice.*

3. Convex Sublattices of $S(B_n)$

Theorem 3.1. *The lattice of convex sublattices of $S(B_n)$, $CS(S(B_n))$ with respect to the set inclusion relation is an Eulerian lattice.*

Proof. It is clear that the rank of $CS(S(B_n))$ is $n + 2$.
We are going to prove that $CS(S(B_n))$ is Eulerian.
That is, to prove that this interval $[\emptyset, B_n]$ has the same number of elements of odd and even rank.

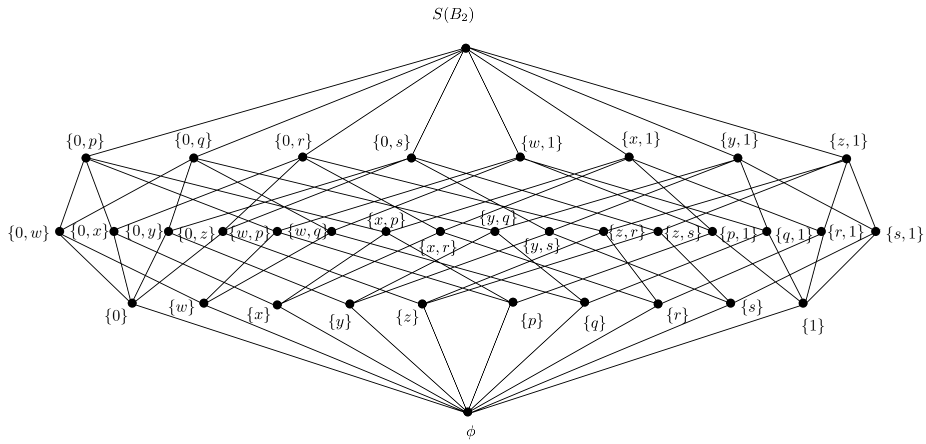

Figure 4: $CS[S(B_2)]$

Let A_i be the number of elements of rank i in $CS(S(B_n))$.

$$
\begin{aligned}
A_1 &= \text{The number of singleton subsets of } CS[S(B_n)] \\
&= 2+n+2+2n+\binom{n}{2}+2\binom{n}{2}+\binom{n}{3}+2\binom{n}{3}+\binom{n}{4} \\
&\qquad +\ldots+2\binom{n}{n-2}+\binom{n}{n-1}+2\binom{n}{n-1} \\
&= 2+\binom{n}{1}+2\binom{n}{0}+2\binom{n}{1}+\binom{n}{2}+2\binom{n}{2}+\binom{n}{3} \\
&\qquad +2\binom{n}{3}+\binom{n}{4}+\ldots+2\binom{n}{n-2}+\binom{n}{n-1}+2\binom{n}{n-1} \qquad (1)
\end{aligned}
$$

$$
\begin{aligned}
A_2 &= \text{The number of rank 2 elements in } CS(S(B_n)) \\
&= \text{The number of edges in } S(B_n) \\
&= \text{number of edges containing } 0 + \text{number of edges containing the atoms} \\
&\qquad + \text{number of edges from the rank 2 elements} \\
&\qquad + \ldots + \text{number of edges containing the coatoms of } S(B_n).
\end{aligned}
$$

$$\text{Number of edges containing } 0 \qquad = n+2 \qquad (2)$$

Number of edges containing an extreme atom $=n$

There are 2 such extreme atoms. Therefore total number of such edges $= 2\begin{pmatrix} n \\ 1 \end{pmatrix}$.
From an atom of a middle copy, the number of edges $= n - 1 + 2 = n + 1$.
There are n such atoms.
Therefore total number of such type of edges $= n(n + 1)$.
Totally from the atoms, the number of edges is equal to

$$2\begin{pmatrix} n \\ 1 \end{pmatrix} + \begin{pmatrix} n \\ 1 \end{pmatrix}(n+1). \tag{3}$$

Number of edges from a rank 2 element in an extreme copy $= n - 1$.
There are $2n$ such elements.
Therefore the number of edges from these elements $= 2\begin{pmatrix} n \\ 1 \end{pmatrix}(n-1)$.

The number of edges from the rank 2 elements in the middle copy $= \begin{pmatrix} n \\ 2 \end{pmatrix} \times (n - 2 + 2) = \begin{pmatrix} n \\ 2 \end{pmatrix} \times n$.
The total number of edges from rank 2 elements is

$$2\begin{pmatrix} n \\ 1 \end{pmatrix}(n-1) + \begin{pmatrix} n \\ 2 \end{pmatrix} \times n. \tag{4}$$

The number of edges from the rank 3 elements in the middle copy is $n-3+2 = n-1$.
There are $\begin{pmatrix} n \\ 3 \end{pmatrix}$ such elements.
Therefore the number of edges from the rank 3 elements in the middle copy $= \begin{pmatrix} n \\ 3 \end{pmatrix}(n-1)$.
The number of edges from a rank 3 element in an extreme copy is $n - 2$.
There are $2\begin{pmatrix} n \\ 2 \end{pmatrix}$ such elements.

Therefore number of edges from rank 3 elements in the extreme copies $= 2\begin{pmatrix} n \\ 2 \end{pmatrix}(n-2)$.
Therefore total number of edges from rank 3 elements of $CS(S(B_n))$ is

$$2\begin{pmatrix} n \\ 2 \end{pmatrix}(n-2) + \begin{pmatrix} n \\ 3 \end{pmatrix}(n-1) \tag{5}$$

Proceeding like this we get the number of edges from the co-atoms $= 2n = 2\begin{pmatrix} n \\ n-1 \end{pmatrix}(n - \overline{n-1})$

From (2), (3), (4) and (5), the total number of edges in $S(B_n)$ is

$$\begin{aligned}
A_2 &= n+2+2\binom{n}{1}+\binom{n}{1}(n+1)+2\binom{n}{1}(n-1) \\
&\qquad +\binom{n}{2}n+\binom{n}{3}(n-1)+2\binom{n}{2}(n-2) \\
&\qquad +\ldots+2\binom{n}{n-1}(n-\overline{n-1}) \\
&= 2+\binom{n}{1}+2\binom{n}{1}+\binom{n}{1}(n+2-1) \\
&\qquad +2\binom{n}{1}(n-1)+\binom{n}{2}(n+2-2)+2\binom{n}{2}(n-2) \\
&\qquad +\binom{n}{3}(n+2-3)+\ldots+2\binom{n}{n-1}(n-\overline{n-1}) \qquad (6)
\end{aligned}$$

$A_3 =$ The number of 4-element sublattices.

$$\text{The number of 4-element sublattices from } 0 = 2\binom{n}{1}+\binom{n}{2}. \qquad (7)$$

Fix an atom $a \in S(B_n)$.
If a is the bottom element of the left copy of $S(B_n)$ then $[a,1] \simeq B_n$.
Therefore the number of B_2's containing a is $\binom{n}{2}$.

Similarly the number of B_2's containing the bottom element of the right copy is $\binom{n}{2}$.
If a is in the middle copy of $S(B_n)$ then $[a,1] \simeq S(B_{n-1})$.
In this $S(B_{n-1})$, we have two extreme copies and a middle copy.
Therefore the number of B_2's containing a is $2(n-1)+\binom{n-1}{2}$.

There are $\binom{n}{1}$ such atoms. Therefore the total number of B_2's containing all the atoms in the middle copy is

$$\binom{n}{1}\left[2(n-1)+\binom{n-1}{2}\right].$$

Therefore the number of B_2's containing all the atoms of $S(B_n)$ is

$$2\binom{n}{2}+\binom{n}{1}\left[2(n-1)+\binom{n-1}{2}\right]. \qquad (8)$$

Fix a rank 2 element x in $S(B_n)$. If x is in the left copy of $S(B_n)$, we have,

$$[x,1] \simeq B_{n-1}.$$

A B_2 containing x emanates from a rank 2 element in that B_{n-1}.
There are $\binom{n-1}{2}$ rank 2 elements in B_{n-1}.
Therefore the number of B_2's containing x in the left copy is $\binom{n-1}{2}$.
There are n such rank 2 elements x in the left copy.
The number of B_2's in the left copy containing all the rank 2 elements is $\binom{n-1}{2}n$.
Similarly the same number in the right copy.
If x is in the middle copy of $S(B_n)$, then

$$[x, 1] \simeq S(B_{n-2}).$$

The number of B_2's containing x in the left copy of that $S(B_{n-2})$ is $n-2$. Similarly the number in the right copy is $n-2$.

Come to the middle copy $\simeq B_{n-2}$. Therefore the number of B_2's containing x in the middle copy of that $S(B_{n-2})$ is $\binom{n-2}{2}$.

Therefore the total number of B_2's containing x in this $S(B_{n-2})$ is $2(n-2)+\binom{n-2}{2}$.

There are $\binom{n}{2}$ such x's. Therefore the total number of B_2's containing all the rank 2 elements in the middle copy is

$$\binom{n}{2}\left[2(n-2)+\binom{n-2}{2}\right]+2\binom{n}{1}\binom{n-1}{2}. \tag{9}$$

Fix a rank 3 element x of $S(B_n)$. If x is in the left copy of $S(B_n)$, then

$$[x, 1] \simeq B_{n-2}.$$

A B_2 containing x emanates from a rank 4 element in that B_{n-2}.
Therefore the number of B_2's containing x in the left copy of $S(B_n)$ is $\binom{n}{2}\binom{n-2}{2}$.
Similarly to the right copy.
Come to the middle copy. If x is in the middle copy of $S(B_n)$, then

$$\therefore [x, 1] \simeq S(B_{n-3}).$$

In this $S(B_{n-3})$ we have to calculate the number of B_2's containing x.
The number of B_2's containing x in the left copy of $S(B_{n-3})$ is $n-3$.
Similarly the number in the right copy of this $S(B_{n-3})$ is $n-3$.
The number of B_2's containing x in the middle copy of this $S(B_{n-3})$ is $\binom{n-3}{2}$.

Therefore, the number of B_2's in this $S(B_{n-2})$ containing x is $2(n-3)+\binom{n-3}{2}$.
There are $\binom{n}{3}$ such rank 3 elements in x in the middle copy of $S(B_n)$.
Therefore the total number of B_2's containing x in $S(B_n)$ is

$$\binom{n}{3}\left[2(n-3)+\binom{n-3}{2}\right].$$

Therefore the total number of B_2's containing all the rank 3 elements is

$$2\binom{n}{2}\binom{n-2}{2}+\binom{n}{3}\left[2(n-3)+\binom{n-3}{2}\right] \tag{10}$$

Continuing like this, we get, the number of B_2's containing all the rank $(n-2)$ elements in $S(B_n)$ is

$$2\binom{n}{n-3}\times 3+\binom{n}{n-2}\times 4. \tag{11}$$

The number of B_2's containing rank $(n-1)$ elements is

$$2\binom{n}{n-2}+\binom{n}{n-1}. \tag{12}$$

From (7), (8), (9), (10), (11) and (12) we get,

$$\begin{aligned}
A_3 = 2\binom{n}{1}+\binom{n}{2}+2\binom{n}{2}+\binom{n}{1}\left[2(n-1)+\binom{n-1}{2}\right]& \\
+\binom{n}{2}\left[2(n-2)+\binom{n-2}{2}\right]+2\binom{n}{1}\binom{n-1}{2}& \\
+2\binom{n}{2}\binom{n-2}{2}+\binom{n}{3}\left[2(n-3)+\binom{n-3}{2}\right]+\dots& \\
+2\binom{n}{n-3}\times 3+\binom{n}{n-2}\times 4+2\binom{n}{n-2}+\binom{n}{n-1}&.
\end{aligned}$$

That is,

$$\begin{aligned}
A_3 = 2\binom{n}{1}+\binom{n}{2}+2\binom{n}{2}+2\binom{n}{1}\binom{n-1}{1}+\binom{n}{1}\binom{n-1}{2}& \\
+2\binom{n}{1}\binom{n-1}{2}+2\binom{n}{2}\binom{n-2}{1}+\binom{n}{2}\binom{n-2}{2}& \\
+2\binom{n}{2}\binom{n-2}{2}+2\binom{n}{3}\binom{n-3}{1}+\binom{n}{3}\binom{n-3}{2}& \\
+\dots+2\binom{n}{n-2}+\binom{n}{n-1}&. \qquad (13)
\end{aligned}$$

Similar argument will give,
$A_4 =$ the number of rank 3 sublattices.

$$\begin{aligned}A_4 = &\left[2\binom{n}{2}+\binom{n}{3}\right]+2\binom{n}{3}+\binom{n}{1}\left[2\binom{n-1}{2}+\binom{n-1}{3}\right]\\ &+\binom{n}{2}\left[2\binom{n-2}{2}+\binom{n-2}{3}\right]+2\binom{n}{2}\binom{n-2}{3}\\ &+2\binom{n}{1}\binom{n-1}{3}+\binom{n}{3}\left[2\binom{n-3}{2}+\binom{n-3}{3}\right]\\ &+\ldots+2\binom{n}{n-3}+\binom{n}{n-2}.\end{aligned}$$

That is,

$$\begin{aligned}A_4 = &\,2\binom{n}{2}+\binom{n}{3}+2\binom{n}{2}+2\binom{n}{1}\binom{n-1}{2}+\binom{n}{1}\binom{n-1}{3}\\ &+2\binom{n}{1}\binom{n-1}{3}+2\binom{n}{2}\binom{n-2}{2}+\binom{n}{2}\binom{n-2}{3}\\ &+2\binom{n}{2}\binom{n-2}{3}+2\binom{n}{3}\binom{n-3}{2}+\binom{n}{3}\binom{n-3}{3}\\ &+\ldots+2\binom{n}{n-3}+\binom{n}{n-2}. \qquad (14)\end{aligned}$$

$A_5 =$ the number of rank 4 sublattices.

$$\begin{aligned}A_5 = &\,2\binom{n}{3}+\binom{n}{4}+2\binom{n}{4}+2\binom{n}{1}\binom{n-1}{3}+\binom{n}{1}\binom{n-1}{4}\\ &+2\binom{n}{1}\binom{n-1}{4}+2\binom{n}{2}\binom{n-2}{3}+\binom{n}{2}\binom{n-2}{4}\\ &+2\binom{n}{2}\binom{n-2}{4}+2\binom{n}{3}\binom{n-3}{3}+\binom{n}{3}\binom{n-3}{4}\\ &+\ldots+2\binom{n}{n-4}+\binom{n}{n-3} \qquad (15)\end{aligned}$$

and so on.

Finally, we get

$$A_n = 2\binom{n}{n-2}+\binom{n}{n-1}+2\binom{n}{n-1}+2\binom{n}{1}\binom{n-1}{n-2}. \qquad (16)$$

$$A_{n+1} = 2\binom{n}{n-1}+(n+2). \qquad (17)$$

Case (i): Suppose n is even.

$$\begin{aligned}
&A_1 - A_2 + A_3 - A_4 + \ldots + A_{n+1}\\
&= \binom{n}{0}[2+2-2] + \binom{n}{1}\Bigg[1+2-1-2-n-2+1-2\binom{n-1}{1}+2\\
&\quad +2\binom{n-1}{1}+\binom{n-1}{2}+2\binom{n-1}{2}-2\binom{n-1}{2}-\binom{n-1}{3}\\
&\quad -2\binom{n-1}{3}+2\binom{n-1}{3}+\binom{n-1}{4}+2\binom{n-1}{4}+\ldots\\
&\quad +\binom{n-1}{n-1}\Bigg]+\binom{n}{2}\Bigg[1+2-n-2+2\binom{n-2}{1}+1+2+2\binom{n-2}{1}\\
&\quad +\binom{n-2}{2}+2\binom{n-2}{2}-2-2\binom{n-2}{2}-\binom{n-2}{3}-2\binom{n-2}{3}\\
&\quad +2\binom{n-2}{3}+\binom{n-2}{4}+2\binom{n-2}{4}+\ldots+\binom{n-2}{n-2}\Bigg]+\binom{n}{3}\Bigg[1\\
&\quad +2-n-2+3+2\binom{n-3}{1}+\binom{n-3}{2}-1-2-2\binom{n-3}{2}+2\\
&\quad -\binom{n-3}{3}+1+2\binom{n-3}{3}+\binom{n-3}{n-3}\Bigg]+\ldots-\binom{n}{n-1}\Bigg[1+2-2\\
&\quad -1+1-1-2+2-\binom{n-\overline{n-1}}{n-\overline{n-1}}\Bigg]+\binom{n}{n}[2]\\
&=2[2^{n-1}]-\left[\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n-1}\right]\\
&=2^n-[2^n-2]\\
&=2.
\end{aligned}$$

Case (ii): Suppose n is odd.

$$\begin{aligned}
&A_1 - A_2 + A_3 - A_4 + \ldots - A_{n+1}\\
&= \binom{n}{0}[2+2-2] + \binom{n}{1}\Bigg[1+2-1-2-n-2+1-2\binom{n-1}{1}+2\\
&\quad +2\binom{n-1}{1}+\binom{n-1}{2}+2\binom{n-1}{2}-2\binom{n-1}{2}-\binom{n-1}{3}\\
&\quad -2\binom{n-1}{3}+2\binom{n-1}{3}+\binom{n-1}{4}+2\binom{n-1}{4}+\ldots\\
&\quad +\binom{n-1}{n-1}\Bigg]+\binom{n}{2}\Bigg[1+2-n-2+2\binom{n-2}{1}+1+2+2\binom{n-2}{1}\\
&\quad +\binom{n-2}{2}+2\binom{n-2}{2}-2-2\binom{n-2}{2}-\binom{n-2}{3}-2\binom{n-2}{3}
\end{aligned}$$

$$+2\binom{n-2}{3}+\binom{n-2}{4}+2\binom{n-2}{4}+\ldots+\binom{n-2}{n-2}\Bigg]+\binom{n}{3}\Bigg[1$$
$$+2-n-2+3+2\binom{n-3}{1}+\binom{n-3}{2}-1-2-2\binom{n-3}{2}+2$$
$$-\binom{n-3}{3}+1+2\binom{n-3}{3}+\binom{n-3}{n-3}\Bigg]+\ldots-\binom{n}{n-1}\Bigg[1+2-2$$
$$-1+1-1-2+2-\binom{n-\overline{n-1}}{n-\overline{n-1}}\Bigg]-\binom{n}{n}[2]$$
$$=2\left[\binom{n}{0}+\binom{n}{2}+\ldots+\binom{n}{n-1}\right]-\Bigg[\binom{n}{1}+\binom{n}{2}+\ldots$$
$$+\binom{n}{n-1}\Bigg]$$
$$=2[2^{n-1}-1]-[2^n-2]$$
$$=0.$$

Hence the interval $[\emptyset, S(B_n)]$ has the same number of elements of odd and even rank.

Though in the above theorem we have proved that $CS(S(B_n))$ is Eulerian, it is not dual simplicial. For example, $CS(S(B_2))$ itself is not dual simplicial. For a general non-Boolean Eulerian lattice it seems difficult to decide the structure, but for C_n we give the proof in the next section.

4. Convex Sublattices of $S(C_n)$

Theorem 4.1. *The lattice of convex sublattices of $S(C_n)$ with respect to the set inclusion relation is an Eulerian lattice.*

Proof. We are going to prove that $CS(S(C_n))$ is Eulerian. That is to prove the interval $[5, S(C_n)]$ has the same number of elements of odd and even rank.

Let A_i be the number of elements of rank i in $CS(S(C_n))$.

$$\begin{aligned} A_1 &= \text{The number of singleton subsets of } CS(S(C_n)) \\ &= 1+n+2+3n+2n+1 \\ &= 6n+4. \end{aligned} \tag{18}$$

$$\begin{aligned} A_2 &= \text{The number of rank 2 elements in } CS(S(C_n)) \\ &= 2+n+2n+4n+4n+2n+2n \\ &= 15n+2. \end{aligned} \tag{19}$$

$$A_3 = \text{The number of 4-element sublattices}$$

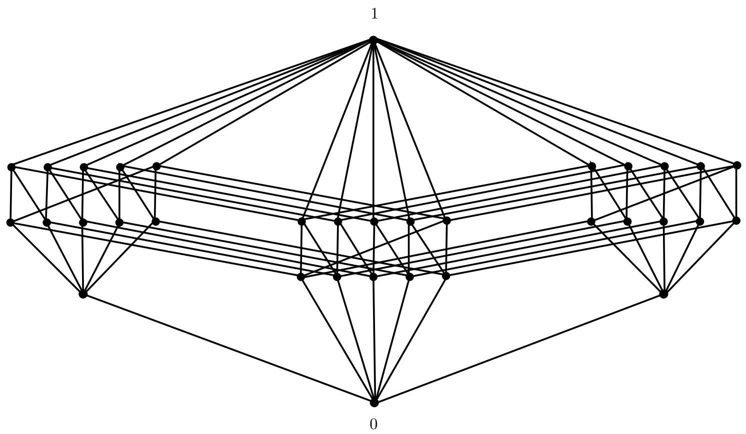

Figure 5: $S(C_5)$

$$
\begin{aligned}
&= 2n + n + 2n + 2n + 2n + 2n + n \\
&= 12n. \qquad (20)
\end{aligned}
$$

$$
\begin{aligned}
A_4 &= \text{The number of rank 3 sublattices} \\
&= 2n + n + 2 \\
&= 3n + 2. \qquad (21)
\end{aligned}
$$

Therefore,

$$A_1 - A_2 + A_3 - A_4 = 6n + 4 - 15n - 2 + 12n - 3n - 2 = 0.$$

Hence the interval $[5, S(C_n)]$ has a same number of elements of odd and even rank.

References

[1] Chen C. K., Koh K. M., *On the lattice of convex sublattices of a finite lattice*, Nanta Math., **5** (1972), 92–95.

[2] Gratzer G., *General Lattice Theory*, **Birkhauser Verlag, Basel**, 1978.

[3] Koh K. M., *On the lattice of convex sublattices of a finite lattice*, Nanta Math., **5** (1972), 18–37.

[4] Lavanya S., Parameshwara Bhatta S., *A new approach to the lattice of convex sublattices of a lattice*, Algebra Univ., **35** (1996), 63–71.

[5] Paffenholz A., *Constructions for Posets, Lattices and Polytopes*, Doctoral Dissertation, School of Mathematics and Natural Sciences, Technical University of Berlin, (2005).

[6] Ramana Murty P. V., *On the lattice of convex sublattices of a lattice*, Southeast Asian Bulletin of Mathematics, **26** (2002), 51–55.

[7] Rota C. G., *On the foundations of Combinatorial theory I, Theory of Mobius functions*, Z. Wahrschainlichkeitstheorie, **2** (1964), 340–368.

[8] Stanley R.P., *Some aspects of groups acting on finite posets*, J. Combinatoria theory, **A. 32** (1982), 131–161.

[9] Stanley R.P., *A survey of Eulerian posets, Polytops: abstract, convex and computational*, Kluwer Acad. Publi., Dordrecht, (1994), 301–333.

[10] Stanley R.P., *Enumerative Combinatorics*, Woodsworth & Brooks, Cole, **Vol 1**, 1986.

[11] Santhi V. K., *Topics in Commutative Algebra*, Ph. D thesis, Madurai Kamaraj University, 1992.

[12] Vethamanickam A., *Topics in Universal Algebra*, Ph. D thesis, Madurai Kamaraj University, 1994.

[13] Vethamanickam A., Subbarayan R., *Some simple extensions of Eulerian lattices*, Acta Math. Univ., Comenianae, **79(1)** (2010), 47–54.

[14] Subbarayan R., Vethamanickam A., *On the lattice of convex sublattices*, Elixir Dis. Math., Comenianae, **50** (2012), 10471–10474.