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Some results on K-contact and Trans-Sasakian Manifolds

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Abstract. We obtain results on the vanishing of divergence of Pseudo projective curvature tensor $\widetilde{P}$ with respect to semi-symmetric metric connection on k-contact and trans-Sasakian manifolds.

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1. Introduction

In 1924, Friedman and Schouten [11] introduced the notion of semi-symmetric linear connection on a differentiable manifold. Then in 1932, Hayden [14] introduced the idea of metric connection with a torsion on a Riemannain manifold. A systematic study of semi-symmetric metric connection on a Riemannain manifold has been given by Yano [18] in 1970 and later studied by K.S.Amur and S.S.Pujar [1], C.S.Bagewadi [2], U.C.De et al [10], Sharafuddin and Hussain [16] and others. The authors U.C.De [10] and C.S.Bagewadi et al [[3, 12]] have obtained results on the conservativeness of Projective, Pseudo projective, Conformal, Concircular, Quasi conformal curvature tensors on k-contact, Kenmotsu and trans-sasakian manifolds.

In this paper we extend the conservativeness of Pseudo projective curvature tensor to k-contact and trans-Sasakian manifolds admitting semi-symmetric metric connection. After preliminaries in section 2, we study in section 3 the Pseudo projective curvature tensor with respect to semi-symmetric metric connection on k-contact manifold. In the section 4 we study some properties regarding Pseudo projective curvature tensor with respect to this connection on trans-Sasakian manifold under the condition $\phi(grad\alpha) = (n-2)grad\beta$ and obtained some interesting results.

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2. Preliminaries

Let M^n be an almost contact metric manifold [9] with an almost contact metric structure (ϕ, ξ, η, g), that is, ϕ is a $(1,1)$ tensor field, ξ is a vector field; η is a 1-form and g is a compatible Riemannian metric such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta.\phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X), \tag{2.3}$$

for all $X, Y \in TM$.

If M^n is a k-contact Riemannian manifold, then besides (2.1), (2.2) and (2.9) the following relations hold [15]:

$$\nabla_X \xi = -\phi X, \tag{2.4}$$

$$(\nabla_X \eta)(Y) = -g(\phi X, Y), \tag{2.5}$$

$$S(X, \xi) = (n-1)\eta(X), \tag{2.6}$$

$$\eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{2.7}$$

for any vector fields X, Y, where R and S denote respectively the curvature tensor of type $(1,3)$ and the Ricci tensor of type $(0,2)$.

An almost contact metric structure (ϕ, ξ, η, g) in M is called trans-Sasakian structure [14] if $(M \times R, J, G)$ belongs to the class w_4 [[8], [13]] where J is the almost complex structure on $M \times R$ defined by $J(X, \lambda d/dt) = (\phi X - \lambda\xi, \eta(X)d/dt)$ for all vector fields X on M and smooth functions λ on $M \times R$ and G is the product metric on $M \times R$. This may be expressed by the condition [8]

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{2.8}$$

for some smooth functions functions α and β on M, and we say that the trans-Sasakian structure is of type (α, β).

Let M be a n-dimensional trans-Sasakian manifold. From (2.8)it is easy to see that

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{2.9}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.10}$$

In a n-dimensional trans-Sasakian manifold, we have

$$R(\xi, X)\xi = (\alpha^2 - \beta^2 - \xi\beta)(\eta(X)\xi - X), \tag{2.11}$$

$$2\alpha\beta + \xi\alpha = 0, \tag{2.12}$$

$$S(X, \xi) = ((n-1)(\alpha^2 - \beta^2) - \xi\beta)\eta(X) - (n-2)X\beta - (\phi X)\alpha. \tag{2.13}$$

If in a n-dimensional trans Sasakian manifold of type(α, β), we have [4]

$$\phi(grad\alpha) = (n-2)grad\beta, \tag{2.14}$$

then (2.11) and (2.13) reduces to

$$R(\xi, X)\xi = (\alpha^2 - \beta^2)(\eta(X)\xi - X), \tag{2.15}$$

$$S(X, \xi) = (n-1)(\alpha^2 - \beta^2)\eta(X). \tag{2.16}$$

In this paper we study trans Sasakian manifold under the condition (2.14).

Let (M^n, g) be an n-dimensional Riemannian manifold of class C^∞ with metric tensor g and let ∇ be the Levi-Civita connection on M^n. A linear connection $\widetilde{\nabla}$ on (M^n, g) is said to be semi symmetric [16] if the torsion tensor T of the connection $\widetilde{\nabla}$ satisfies

$$T(X, Y) = \pi(Y)X - \pi(X)Y, \tag{2.17}$$

where π is a 1-form on M^n with ρ as associated vector field, i.e., $\pi(X) = g(X, \rho)$ for any differentiable vector field X on M^n.

A semi-symmetric connection $\widetilde{\nabla}$ is called semi-symmetric metric connection [5] if it further satisfies $\widetilde{\nabla} g = 0$.

In an almost contact manifold semi-symmetric metric connection is defined by identifying the 1-form π of (2.17) with the contact-form η, i.e., by setting [16]

$$T(X, Y) = \eta(Y)X - \eta(X)Y \tag{2.18}$$

with ξ as associated vector field. i.e., $g(X, \xi) = \eta(X)$.

The relation between the semi-symmetric metric connection $\widetilde{\nabla}$ and the Levi-Civita connection ∇ of (M^n, g) has been obtained by K.Yano [18], which is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi, \tag{2.19}$$

where $\eta(Y) = g(Y, \xi)$.

Further, a relation between the curvature tensor R and $\widetilde{R}$ of type $(1,3)$ of the connections ∇ and $\widetilde{\nabla}$ respectively is given by [18].

$$\widetilde{R}(X, Y)Z = R(X, Y)Z - K(Y, Z)X + K(X, Z)Y - g(Y, Z)FX + g(X, Z)FY. \tag{2.20}$$

where K is a tensor field of type $(0,2)$ defined by

$$K(Y, Z) = g(FY, Z) \quad = \quad (\nabla_Y \eta)(Z) - \eta(Y)\eta(Z) + \frac{1}{2}\eta(\xi)g(Y, Z), \tag{2.21}$$

for any vector fields X and Y.

From (2.20), it follows that

$$\widetilde{S}(Y, Z) = S(Y, Z) - (n-2)K(Y, Z) - a.g(Y, Z) \tag{2.22}$$

where $\widetilde{S}$ denotes the Ricci tensor with respect to $\widetilde{\nabla}$ and $a = \mathrm{Tr}.K$. Differentiating (2.22) covariantly with respect to X, we obtain [6]

$$\begin{aligned}(\widetilde{\nabla}_X \widetilde{S})(Y, Z) \quad = \quad & (\nabla_X S)(Y, Z) - (n-2)(\nabla_X K)(Y, Z) - \eta(Y)S(X, Z) - \eta(Z)S(X, Y) \\ & +(n-2)\eta(Y)K(X, Z) + (n-2)\eta(Z)K(Y, X) + g(X, Y)S(\xi, Z) \\ & +g(X, Z)S(Y, \xi) - (n-2)g(X, Z)K(Y, \xi) - (n-2)g(X, Y)K(Z, \xi)\end{aligned} \tag{2.23}$$

Now let e_i be an orthogonal basis of the tangent space at each point of the manifold M^n for $i = 1, 2, ..., n$. Putting $Y = Z = e_i$ in (2.23) and then taking summation over the index i, we get

$$\widetilde{\nabla}_X \widetilde{r} = \nabla_X r - (n-2)(\nabla_X a) \tag{2.24}$$

Further, since ξ is a killing vector in k-contact manifold. S, α, r, and a are invariant under it, i.e.,

$$L_\xi S = 0, \quad L_\xi r = 0 \tag{2.25}$$

$$L_\xi K = 0, \quad L_\xi a = 0 \tag{2.26}$$

We recall some definitions which are used in later section, A Riemannian manifold is said to be η-Einstein manifold if the Ricci tensor S is of the form

$$S(X, Y) = \lambda g(X, Y) + \mu\eta(X)\eta(Y)$$

where λ, μ are the associated functions on the manifold. A Riemannian manifold is said to be cyclic-Ricci tensor, if the Ricci tensor S satisfies the condition

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0$$

3. k-contact Manifold Admitting a Semi-symmetric Metric Connection With $Div.\widetilde{P} = 0$

The pseudo projective curvature tensor on a Riemannian manifold is given by ([7], [17])

$$\begin{aligned} \widetilde{P}(X, Y)Z &= aR(X, Y)Z + b(S(Y, Z)X - S(X, Z)Y] \\ &\quad \frac{r}{n}\left[\frac{a}{n-1} + b\right][g(Y, Z)X - g(X, Z)Y]. \end{aligned} \tag{3.1}$$

In this section we prove the following: If a k-contact manifold M^n $(n > 2)$ admits a semi-symmetric metric connection and if the Pseudo projective curvature tensor with respect to this connection is conservative, then the manifold is η-Einstein; the scalar curvature of such a manifold is given by (3.14).

Proof. : Let us suppose that in a k-contact Manifold M^n with respect to semi-symmetric metric connection $Div.C = 0$ where Div denotes the divergence.
Differentiate (3.1) covariantly and then contracting we get $Div.\widetilde{P}$. By virtue of conserva-

tiveness of $\widetilde{P}$ i.e.,$div.\widetilde{P}=0$, we obtain

$$\begin{aligned}
&(a+b)[(\nabla_X S)(Y,Z)-(\nabla_Y S)(X,Z)]-[a+b(n-2)][(\nabla_X K)(Y,Z)-(\nabla_Y K)(X,Z)]\\
=\ &(a-b)[S(Y,Z)\eta(X)-S(X,Z)\eta(Y)]-a(n-1)\eta(R(X,Y)Z)+a.S(X,Y)\eta(Z)\\
&+a(n-A-1)[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]+b[g(Y,Z)S(X,\xi)-g(X,Z)S(Y,\xi)] \quad (3.2)\\
&+(a+b(n-2))[K(X,Y)\eta(Z)-K(X,Z)\eta(Y)+K(Y,X)\eta(Z)-K(Y,Z)\eta(X)]\\
&-b(n-2)[g(Y,Z)K(X,\xi)+g(X,Z)K(Y,\xi)]+\frac{1}{n}\left[\frac{a+(n-1)}{(n-1)}\right][g(Y,Z)\nabla_X r\\
&-g(X,Z)\nabla_Y r]+\left[a+\frac{1}{n}\frac{a+(n-1)}{(n-1)}\right][g(Y,Z)\nabla_X A-g(X,Z)\nabla_Y A].
\end{aligned}$$

By virtue of (2.1) and (2.4) we obtain from (2.21) that

$$K(X,Y) = g(X,\phi Y)-\eta(X)\eta(Y)+\frac{1}{2}g(X,Y). \quad (3.3)$$

$$K(X,\xi) = -\frac{1}{2}\eta(X) \quad (3.4)$$

$$LX = -\phi X-\eta(X)\xi+\frac{1}{2}X. \quad (3.5)$$

Now putting $X=\xi$ in (3.2), then using (2.1), (2.6),(2.7),(3.3) and (3.4), we get

$$\begin{aligned}
&(a+b)[(\nabla_\xi S)(Y,Z)-(\nabla_Y S)(\xi,Z)]-[a+n(n-2)][(\nabla_\xi K)(Y,Z)-(\nabla_Y K)(\xi,Z)]\\
=\ &[a+n(n-2)])g(\phi Y,Z)+(a-b)S(Y,Z)+\left[a\left(A+\frac{1}{2}\right)+b(2n-3)\right]g(Y,Z)\\
&-\left[a\left(A+\frac{1}{2}\right)+b(n-2)\right]\eta(Y)\eta(Z)+\frac{1}{n}\left[\frac{a+(n-1)}{(n-1)}\right][g(Y,Z)\nabla_\xi r-\eta(Z)\nabla_Y r] \quad (3.6)\\
&+\left[a+\frac{1}{n}\frac{a+(n-1)}{(n-1)}\right][g(Y,Z)\nabla_\xi A-\eta(Z)\nabla_Y A].
\end{aligned}$$

From (2.25) and (2.26), we obtain

$$(\nabla_\xi S)(Y,Z) = -S(\nabla_Y\xi,Z)-S(Y,\nabla_Z\xi), \quad (\nabla_\xi r)=0, \quad (3.7)$$

$$(\nabla_\xi K)(Y,Z) = -K(\nabla_Y\xi,Z)-K(Y,\nabla_Z\xi), \quad (\nabla_\xi a)=0, \quad (3.8)$$

respectively.

By using(3.7)and(3.8) in (3.6), we have

$$\begin{aligned}
&(a+b)[-S(\nabla_Y\xi,Z)-S(Y,\nabla_Z\xi)-(\nabla_Y S)(\xi,Z)]\\
=\ &[a+n(n-2)][-K(\nabla_Y\xi,Z)-K(Y,\nabla_Z\xi)-(\nabla_Y K)(\xi,Z)]\\
&+[a+n(n-2)])g(\phi Y,Z)+(a-b)S(Y,Z)+\left[a\left(A+\frac{1}{2}\right)+b(2n-3)\right]g(Y,Z)\\
&-\left[a\left(A+\frac{1}{2}\right)+b(n-2)\right]\eta(Y)\eta(Z)+\frac{1}{n}\left[\frac{a+(n-1)}{(n-1)}\right][g(Y,Z)\nabla_\xi r-\eta(Z)\nabla_Y r] \quad (3.9)\\
&+\left[a+\frac{1}{n}\frac{a+(n-1)}{(n-1)}\right][g(Y,Z)\nabla_\xi A-\eta(Z)\nabla_Y A].
\end{aligned}$$

Using (2.4) ,(2.6) and (3.4) in (3.9), we get

$$\begin{aligned}
& -(a+b)S(\phi Y,Z)-(a-b)S(Y,Z) \\
= & \left[a\left(A+\frac{3}{2}\right)+b(3n-5)\right]g(Y,Z)-\left[a\left(A+\frac{3}{2}\right)-2b(n-2)\right]\eta(Y)\eta(Z) \\
& +[a(n+1)+b(3n-5)]g(\phi Y,Z)-\frac{1}{n}\left[\frac{a+(n-1)}{(n-1)}\right]\eta(Z)\nabla_Y r \\
& -\left[a+\frac{1}{n}\frac{a+(n-1)}{(n-1)}\right]\eta(Z)\nabla_Y A. \qquad (3.10)
\end{aligned}$$

Next, by replacing Z by ϕZ in above and then using (2.1), we obtain

$$\begin{aligned}
& -(a+b)S(\phi Y,\phi Z)-(a-b)S(Y,\phi Z) \\
& \left[a\left(A+\frac{3}{2}\right)+b(3n-5)\right]g(Y,\phi Z)+\left[a(n+1)+b(3n-5)\right]g(\phi Y,\phi Z) \qquad (3.11)
\end{aligned}$$

Interchanging Y and Z in (3.11), we have

$$\begin{aligned}
& -(a+b)S(\phi Y,\phi Z)-(a-b)S(\phi Y,Z) \\
& \left[a\left(A+\frac{3}{2}\right)+b(3n-5)\right]g(\phi Y,Z)+\left[a(n+1)+b(3n-5)\right]g(\phi Y,\phi Z) \qquad (3.12)
\end{aligned}$$

By adding (3.11) with (3.12), and then by using the skew-symmetric property of ϕ, one can get

$$S(Y,Z)=P_1.g(Y,Z)+Q_1.\eta(Y)\eta(Z) \qquad (3.13)$$

$$\begin{aligned}
\text{where} P_1 &= \left[-\frac{a}{a+b}(n-1)-\frac{b}{a+b}(3n-5)\right] \quad \text{and} \\
Q_1 &= \left[\frac{2a+b}{a+b}(n-1)+\frac{b}{a+b}(3n-5)\right].
\end{aligned}$$

There fore the manifold is η-Einstein.
Let e_i be an orthogonal basis of the tangent space at each point of the manifold M^n for $i=1,2,....,n$. Putting $Y=Z=e_i$ in (3.13) and then taking summation over the index i, we get

$$r=-(n-1)(n-2)\frac{(a+3b)}{(a+b)}. \qquad (3.14)$$

This proves the theorem.

Suppose in k-contact manifold admitting a semi-symmetric metric connection, the Pseudo projective curvature tensor with respect to this connection is conservative. Then the manifold has a cyclic-Ricci tensor with respect to Levi-Civita connection; and moreover the scalar curvature of the manifold is constant if and only if the vector field ξ is harmonic provided $(a+b)\neq 0$.

Proof. Differentiating (3.13) covariantly with respect to X, we have

$$(\nabla_X S)(Y,Z) = \left[\frac{2a+b}{a+b}(n-1) + \frac{b}{a+b}(3n-5)\right][g(\phi Y,X)\eta(Z) + g(\phi X,Z)\eta(Y)] \quad (3.15)$$

Similarly

$$(\nabla_Y S)(Z,X) = \left[\frac{2a+b}{a+b}(n-1) + \frac{b}{a+b}(3n-5)\right][g(\phi Y,Z)\eta(X) + g(\phi X,Y)\eta(Z)] \quad (3.16)$$

$$(\nabla_Z S)(X,Y) = \left[\frac{2a+b}{a+b}(n-1) + \frac{b}{a+b}(3n-5)\right][g(\phi X,Z)\eta(Y) + g(\phi Z,Y)\eta(X)] \quad (3.17)$$

Adding the equations (3.15), (3.16) and (3.17), then using skew-symmetry of ϕ, we obtain

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0 \quad (3.18)$$

Thus the manifold has a cyclic-Ricci tensor.
Taking an orthonormal frame field and contracting (3.15) over X and Z, we obtain

$$dr(Y) = \left[\frac{2a+b}{a+b}(n-1) + \frac{b}{a+b}(3n-5)\right]\psi\eta(Y) \quad (3.19)$$

where $\psi = Tr.\phi$. From (3.19), it follows that

$$dr(Y) = 0 \;\; if \;\; and \;\; only \;\; \psi = 0 \;\; provided \;\; (a+b) \neq 0. \quad (3.20)$$

4. Trans-Sasakian Manifold Admitting a Semi-symmetric Metric Connection With $Div.\widetilde{P} = 0$

Here we recall some results which will be used in further. [5]: In a trans-Sasakian manifold under the condition (2.14), we have

$$\begin{aligned}[(\nabla_\xi S)(Y,Z) - (\nabla_Y S)(\xi,Z)] &= \beta S(Y,Z) - (n-1)(\alpha^2-\beta^2)\beta g(Y,Z) \\ &\quad -(n-1)(\alpha^2-\beta^2)\alpha g(Y,\phi Z) + \alpha S(Y,\phi Z).\end{aligned} \quad (4.1)$$

[5]: For trans-Sasakian manifold under the condition (2.14), the following results are

true

$$\begin{aligned}
(i)\quad K(Y,Z) &= \alpha g(Y,\phi Z)+\left(\beta+\frac{1}{2}\right)g(Y,Z)-(\beta+1)\eta(Y)\eta(Z)\\
(ii)\quad K(Y,\xi) &= K(\xi,Y)=-\frac{1}{2}\eta(Y)\\
(iii)\quad K(\nabla_Y\xi,Z) &= -\alpha^2[g(Y,Z)-\eta(Y)\eta(Z)]-2\alpha\beta g(\phi Y,Z)\\
&\quad -\frac{\alpha}{2}g(\phi Y,Z)+\beta\left(\beta+\frac{1}{2}\right)[g(Y,Z)-\eta(Y)\eta(Z)] \qquad (4.2)\\
(iv)\quad K(Y,\nabla_Z\xi) &= \alpha^2[g(Y,Z)-\eta(Y)\eta(Z)]+\frac{\alpha}{2}g(\phi Y,Z)\\
&\quad +\beta\left(\beta+\frac{1}{2}\right)[g(Y,Z)-\eta(Y)\eta(Z)].
\end{aligned}$$

[5]: In a trans-Sasakian manifold under the condition (2.14), we have

$$\begin{aligned}
[(\nabla_\xi K)(Y,Z)-(\nabla_Y K)(\xi,Z)] &= \alpha g(Y,\phi Z)-2\alpha\beta g(\phi Y,Z)\\
&\quad -[(\alpha^2-\beta^2)-(2\beta+1)][g(Y,Z)-\eta(Y)\eta(Z)] \qquad (4.3)
\end{aligned}$$

In this section we prove the following: Let in a trans-Sasakian manifold M^n $(n>2)$ under the condition (2.14) admits a semi-symmetric metric connection the Pseudo projective curvature tensor with respect to this connection is conservative. Then the manifold M^n is η-Einstein with respect to Levi-Civita connection; the scalar curvature of such a manifold is given by (4).

Proof.

Let us suppose that in a trans-Sasakian Manifold M^n under the condition (2.14)with respect to semi-symmetric metric connection $Div.\widetilde{P}=0$.
Putting $X=\xi$ in (3.2) then using (2.1),(2.3), (2.16) and (4.2(ii)) we get

$$\begin{aligned}
& (a+b)[(\nabla_\xi S)(Y,Z)-(\nabla_Y S)(\xi,Z)]-(a+b(n-2))[(\nabla_\xi K)(Y,Z)-(\nabla_Y K)(\xi,Z)]\\
= \; & (a-b)S(Y,Z)+(a+b(n-2))\alpha g(Y,\phi Z)-a(n-1)\eta(R(\xi,Y)Z)\\
& +\left[(a+b(n-2))\left(\beta+\frac{1}{2}\right)+a(n-A-1)+b(n-1)(\alpha^2-\beta^2)+\frac{b}{2}(n-2)\right]g(Y,Z)\\
& -\left[a(n-A-1)+(a+b(n-2))\left(\beta+\frac{1}{2}\right)+\frac{b}{2}(n-2)\right]\eta(Y)\eta(Z) \qquad (4.4)\\
& +\left[\frac{1}{n}\frac{a+(n-1)}{(n-1)}(n-2)-a\right][\eta(Z)\nabla_Y A-g(Y,Z)\nabla_\xi A]\\
& -\frac{1}{n}\left[\frac{a+(n-1)}{(n-1)}\right][\eta(Z)\nabla_Y r-g(Y,Z)\nabla_\xi r].
\end{aligned}$$

Using (2.15) ,(4.1), (4.2(i)) and (4.3) in above, we get

$$\begin{aligned}
& (a+b)(\beta-1)S(Y,Z)+(a+b)\alpha S(Y,\phi Z) \\
= \; & -[2\alpha(\beta+1)(a+b(n-2))+(a+b)(n-1)(\alpha^2-\beta^2)\alpha]g(\phi Y,Z) \\
& +\grave{P}.g(Y,Z)+\grave{Q}.\eta(Y)\eta(Z) \\
& +\left[\frac{1}{n}\frac{a+(n-1)}{(n-1)}(n-2)-a\right][\eta(Z)\nabla_Y A-g(Y,Z)\nabla_\xi A] \qquad (4.5) \\
& -\frac{1}{n}\left[\frac{a+(n-1)}{(n-1)}\right][\eta(Z)\nabla_Y r-g(Y,Z)\nabla_\xi r].
\end{aligned}$$

$$\begin{aligned}
\text{where } \grave{P} &= [a(\beta-1)+b(\beta+1)](n-1)(\alpha^2-\beta^2)+a(n-A-1) \\
& +\frac{b}{2}(n-2)+[a+b(n-2)]\left[2\left(\beta+\frac{1}{4}\right)-(\alpha^2-\beta^2)\right] \quad \text{and} \\
\grave{Q} &= [(a-b)(\alpha^2-\beta^2)-b](n-2)-a\left(n-A-\frac{1}{2}\right).
\end{aligned}$$

Next, by replacing Z by ϕZ in (4.5) and then using (2.1), we obtain

$$\begin{aligned}
& -(a+b)\alpha S(Y,Z)-(a+b)(\beta-1)S(\phi Y,Z) \\
= \; & -[2\alpha(\beta+1)[a+b(n-2)+(a+b)(n-1)(\alpha^2-\beta^2)\alpha]g(Y,Z) \\
& -\grave{P}.g(Y,\phi Z)+[2\alpha(\beta+1)(a+b(n-2))]\eta(Y)\eta(Z) \qquad (4.6) \\
& +\left[\frac{1}{n}\frac{a+(n-1)}{(n-1)}(n-2)-a\right]g(Y,\phi Z)\nabla_\xi A-\frac{1}{n}\left[\frac{a+(n-1)}{(n-1)}\right]g(Y,\phi Z)\nabla_\xi r.
\end{aligned}$$

Interchanging Y and Z in above, we have

$$\begin{aligned}
& -(a+b)\alpha S(Y,Z)-(a+b)(\beta-1)S(Y,\phi Z) \\
= \; & -[2\alpha(\beta+1)[a+b(n-2)+(a+b)(n-1)(\alpha^2-\beta^2)\alpha]g(Y,Z) \\
& -\grave{P}.g(\phi Y,Z)+[2\alpha(\beta+1)(a+b(n-2))]\eta(Y)\eta(Z) \qquad (4.7) \\
& +\left[\frac{1}{n}\frac{a+(n-1)}{(n-1)}(n-2)-a\right]g(\phi Y,Z)\nabla_\xi A-\frac{1}{n}\left[\frac{a+(n-1)}{(n-1)}\right]g(\phi Y,Z)\nabla_\xi r.
\end{aligned}$$

By adding (4.6) and (4.7), then by using skew-symmetric property of ϕ, one can obtain

$$S(Y,Z)=P_2.g(Y,Z)+Q_2.\eta(Y)\eta(Z). \qquad (4.8)$$

$$\begin{aligned}
\text{where } P_2 &= \left[2\frac{(\beta+1)}{(a+b)}[a+b(n-2)]+(n-1)(\alpha^2-\beta^2)\right] \text{ and} \\
Q_2 &= -2\frac{(\beta+1)}{(a+b)}[a+b(n-2)].
\end{aligned}$$

Therefore the manifold is η-Einstein.
Let e_i be an orthogonal basis of the tangent space at each point of the manifold M^n for $i = 1, 2, ..., n$. Putting $Y = Z = e_i$ in (4.8) and then taking summation over the index i, we get

$$r = (n-1)\left[2\frac{(\beta+1)}{(a+b)}[a + b(n-2)] + n(\alpha^2 - \beta^2)\right].$$

This proves the theorem.

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