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A Zariski Topology For Semimodules

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Abstract. Given a very strong multiplication semimidule M over a commutative semiring R, a Zariski topology is defined on the spectrum $\mathrm{Spec}_k(M)$ of prime k-subsemimodules of M. The properties and possible structures of this topology are studied.

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1. Introduction

As a generalization of rings, semirings have been found useful for solving problems in different areas of applied mathematics and information sciences, since the structure of a semiring provides an algebraic framework for modelling and studying the key factors in these applied areas. They play an important role in studying optimization theory, graph theory, theory of discrete event dynamical systems, generalized fuzzy computation, automata theory, coding theory, cryptography theory, and so on (see Golan [16], Glazek [17], Hebisch and Weinert [18], Simon [23]). Ideals of semirings play a central role in the structure theory and are useful for many purposes. However, they do not in general coincide with the usual ring ideals and, for this reason, their use is somewhat limited in trying to obtain analogues of ring theorems for semirings. Indeed, many results in rings apparently have no analogues in semirings using only ideals. In order to overcome this deficiency, the present authors [14] defined a more resttricted class of ideals in semirings, which is called the class of "strong ideals", with the property that they are k-ideals.

Let M be a module over a commutative ring R with identity. The prime spectrum $\mathrm{Spec}(R)$ and the topological space obtained by introducing Zariski topology on the set of prime ideals of R play an important role in the fields of commutative algebra, algebraic geometry and lattice

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theory. Also, recently the notion of prime submodules and Zariski topology on Spec(M), the set of all prime R-submodules of M, are studied by many authors (for example see Lu[19], Lu [20], Lu [21], MacCasland, Moore and Smith [22]). The main dificulty is figuring out what additional hypotheses the semimodule must satisfy to get similar results of modules. The two new key notions are that a "strong subsemimodule" and a "very strong subsemimodule". In this paper, Section 3, we list some basic properties concerning very strong multiplication semimodules. For example, we give some results to characterize the prime k-subsemimodules of very strong multiplication semimodules. In Section 4, we concentrate on Zariski topology of very strong multiplication semimodules M over commutative semirings R with identity and generalize the some well known results of Zariski topoloy on the sets of prime ideals of a commutative ring to Spec(M), the sets of prime k-subsemimodules of M and investigate the basic properties of this topology. In this regard we strongly use the notion of product of subsemimodules of very strong multiplication semimodules. For example, we prove that an open set $X = \mathrm{Spec}_k(M)$ is compact if and only if it is a finite union of basic open sets..

2. Preliminaries

Throughout this paper R is a commutative semiring with identity. In order to make this paper easier to follow, we recall in this section various notions from semimodule theory which will be used in the sequel. For the definitions of monoid, semirings, semimodules and subsemimodules we refer [16, 18, 13, 7]. All semiring in this paper are commutative with non-zero identity.

Definition 1. *(1) Let M be a semimodule over a semiring R. A subtractive subsemimodule (= k-subsemimodule) N is a subsemimodule of M such that if $x, x+y \in N$, then $y \in N$ (so $\{0_M\}$ is a k-subsemimodule of M).*

(2) A prime subsemimodule of M is a proper subsemimodule N of M in which $x \in N$ or $rM \subseteq N$ whenever $rx \in N$. The collection of all prime k-subsemimodules of M is called the k-spectrum of M and denoted by $\mathrm{Spec}_k(M)$. *We define k-ideals and prime k-ideals of a semiring R in a similar fashion.*

(3) A subsemimodule L of M is said to be semiprime *if L is an intersection of prime k-subsemimodules of M.*

(4) Let T be a proper subsemimodule of an R-semimodule M. Then the prime radical $\mathrm{rad}(T)$ *of T (in M) is the intersection of all prime k-subsemimodules of M containing T or, in case there are no such prime k-subsemimodules,* $\mathrm{rad}(T)$ *is M. Note that $T \subseteq \mathrm{rad}(T)$ and that $\mathrm{rad}(T) = M$ or $\mathrm{rad}(T)$ is semiprime k-subsemimodule of M.*

(5) A prime subsemimodule N of M is called extraordinary *if whenever A and B are semiprime k-subsemimodules of M with $A \cap B \subseteq N$, then $A \subseteq N$ or $B \subseteq N$.*

(6) An R-semimodule M is called multiplication semimodule provided that for every subsemimodule N of M there exists an ideal I of R such that $N = IM$.

(7) A proper ideal I of a semiring R is said to be strong ideal (or strongly zero-sum ideal), if for each $a \in I$ there exists $b \in I$ such that $a + b = 0$ (see [14, Example 2.3] and [11]).

Quotient semirings are determined by equivalence relations rather than by ideals as in the ring case. Allen [1] has presented the notion of a partitioning ideal (= Q-ideal) I in the semiring R and constructed the quotient semiring R/I. If I is an ideal of a semiring R, we define a relation $\sim$ on R, given by $r_1 \sim r_2$ if and only if there exist $a_1, a_2 \in I$ satisfying $r_1 + a_1 = r_2 + a_2$. Then $\sim$ is an equivalence relation on R, and we denote the equivalence class of r by $r + I$ and these collection of all equivalence classes by R/I. Golan shows that R/I is a semiring with $(r + I) + (s + I) = r + s + I$ and $(r + I)(s + I) = rs + I$. The semiring R/I has additive identity $0 + I$ and multiplicative identity $1 + I$ (see Golan [16]). In this paper, we will follow Golan's terminology for quotient semirings.

Quotient semimodules over a semiring R have already been introduced and studied by present authors in [13]. Chaudhari and Bonde [7] extended the definition of Q_M-subsemimodule of a semimodule to a more general case: A subsemimodule N of a semimodule M over a semiring R is called a partitioning subsemimodule (= Q_M-subsemimodule) if there exists a subset Q_M of M such that $M = \cup\{q + N : q \in Q_M\}$ and if $q_1, q_2 \in Q_M$ then $(q_1 + N) \cap (q_2 + N) \neq \emptyset$ if and only if $q_1 = q_2$. Let N be a Q_M-subsemimodule of M and let
$M/N = \{q + N : q \in Q_M\}$. Then M/N forms an R-semimodule under the operations $\oplus$ and $\odot$ defined as follows: $(q_1 + N) \oplus (q_2 + N) = q_3 + N$, where $q_3 \in Q_M$ is the unique element such that $q_1 + q_2 + N \subseteq q_3 + N$ and $r \odot (q_1 + N) = q_4 + I$, where $r \in R$ and $q_4 \in Q_M$ is the unique element such that $rq_1 + N \subseteq q_4 + N$. This R-semimodule M/N is called the quotient semimodule of M by N [7]. By [7, Lemma 2.3], there exists a unique element $q_0 \in Q_M$ such that $q_0 + N = N$. Thus $q_0 + N$ is the zero element of M/N. Also, [7, Theorem 2.4] show that the structure $(M/N, \oplus, \odot)$ is essentially independent of Q_M (see [7, Example 2.6]).

Lemma 1. *Let M be a semimodule over a semiring R. If $\{M_i\}_{i\in\Lambda}$ is a collection of subsemimodules of M, then $\sum_{i\in\Lambda} M_i$ and $\bigcap_{i\in\Lambda} M_i$ are subsemimodules of M.*

3. Properties of Strong Multiplication Semimodules

In this section, we list some basic properties concerning very strong multiplication semimodules. Our starting point is the following lemma.

Lemma 2. *Let N be a Q_M-subsemimodule of a semimodule M over a semiring R. If T is a k-subsemimodule of M containing N, then $(T :_R M) = (T/N :_R M/N)$.*

Proof. Let $r \in (T : M)$. If $q + N \in M/N$, then there exists a unique element q' of Q_M such that $r(q + N) = q' + N$, where $rq + N \subseteq q' + N$; so $q' \in T \cap Q_M$ since $rq \in T$ and T is a k-subsemimodule. Thus $(T : M) \subseteq (T/N : M/N)$. For the other inclusion, assume that $a \in (T/N : M/N)$ and $m \in M$. Then $m = q_1 + n$ for some $q_1 \in Q_M$ and $n \in N$; so there is a unique element q_2 of Q_M with $a(q_1 + N) = q_2 + N \in T/N$, where $aq_1 + N \subseteq q_2 + N$. Thus T being a k-subsemimodule gives $aq_1 \in T$. As $am = aq_1 + an \in T$, we have $a \in (T : M)$.

Theorem 1. *Let R be a semiring with identity, M an R-semimodule and N an Q_M-subsemimodule of M. Then there is a one-to-one correspondence between prime k-subsemimodules of R-semimodule M/N and prime k-subsemimodules of M containing N.*

Proof. Let T be a prime k-subsemimodule of M containing N. It then follows from [7, Theorem 3.6] that T/N is a proper k-susemimodule of M/N. Let $a(q_1+N)=q_2+N\in T/N$, where $q_2\in Q_M\cap T$ and $aq_1+N\subseteq q_2+N$, so $aq_1\in T$ since T is a k-subsemimodule. Then T prime gives either $q_1\in T$ (so $q_1+N\in T/N$) or $a\in(T:M)=(T/N:M/N)$ by Lemma 2. Thus, T/N is a prime k-subsemimodule of M/N. Conversely, assume that T/N is a prime k-subsemimodule of M/N. To show that T is a prime k-subsemimodule of M, suppose that $rm\in T$, where $r\in R$ and $m\in M$. We may assume that $r\neq 0$. There are elements $q\in Q_M$ and $n\in N$ such that $m=q+n$, so $rm=rq+rn\in T$; hence $rq\in T$ since T is a k-subsemimodule. Therefore, there exists a unique element $q'\in Q_M$ such that $r(q+N)=q'+N$, where $rq+N\subseteq q'+N$; hence $q'\in T$. Thus $r(q+N)\in T/N$. Then T/N prime gives either $q+N\in T/N$ (so $m\in T$) or $r\in(T/N:M/N)=(T:M)$, and the proof is complete.

Corollary 1. *Let R be a semiring with identity, M an R-semimodule and N an Q_M-subsemimodule of M. Then there is a one-to-one correspondence between semiprime k-subsemimodules of R-semimodule M/N and semiprime k-subsemimodules of M containing N.*

Proof. Apply Theorem 1.

Definition 2. *Let M be a semimodule over a semiring R. A subsemimodule N of M is said to be a strong subsemimodule if for each $x\in N$ there exists $y\in N$ such that $x+y=0$.*

Example 1. *(1) Clearly, every submodule of a module over a ring R is a strong subsemimodule.*

(2) Let R denote the semiring of non-negative integers with the usual operations of addition and multiplication, and let $M=Z_6$ denote the monoid of integers modulo 6. Then M is a semimodule over R by [16, p. 151], and an inspection will show that $N=\{\bar{0},\bar{2},\bar{4}\}$ and M are strong subsemimodules of M.

Definition 3. *A semimodule M over a semiring R is called a strong multiplication semimodule whenever N is a k-subsemimodule of M, then there exists a strong ideal I of R such that $N=IM$.*

Definition 4. *A semimodule M over a semiring R is called a very strong semimodule if I is an ideal of R and $m\in M$, then the ideal $\{r\in R: rm\in IM\}$ is a strong k-ideal of R.*

Definition 5. *A very strong semimodule M over a semiring R is called a very strong multiplication semimodule whenever N is a k-subsemimodule of M, then there exists a strong ideal I of R such that $N=IM$.*

Proposition 1. *Let M be a semimodule over a semirig R. Then the following statements hold:*

(i) If N is a strong subsemimodule of M, then N is a k-subsemimodule.

(ii) If I is a strong ideal of R, then IM is a strong k-subsemimodule.

(iii) If N is a strong subsemimodule of M, then $N + L$ is a strong k-subsemimodule of M for every strong subsemimodule L of M.

(iv) If I is a strong ideal of R and N is a strong Q_M-subsemimodule of M, then $I(M/N) = (IM + N)/N$.

Proof.

(i) Let $a, a + b \in N$ for some $a, b \in M$. Then $a + a' = 0$ for some $a' \in N$; hence $b = a + a' + b \in N$.

(ii) Let $z = \sum_{i=1}^{n} r_i m_i \in IM$. Then there exists $s_i \in I$ such that $r_i + s_i = 0$ for every $i = 1, \ldots, n$; so $z + \sum_{i=1}^{n} s_i m_i = 0$. Now the assertion follows from (i).

(iii) Let $a + b \in N + L$, where $a \in N$ and $b \in L$. Then $a + a' = 0$ for some $a' \in N$ and $b + b' = 0$ for some $b' \in L$; hence $(a + b) + (a' + b') = 0$. Thus $N + L$ is a strong ideal. Now the assertion follows from (i).

(iv) First we show that $I(M/N) \subseteq (IM + N)/N$. It is enough to show that for each $r \in I$ and for each $q + N \in M/N$ we have $r.(q + N) \in (IM + N)/N$. Let $r.(q + N) = q' + N$, where $q' \in Q_M$ is the unique element such that $rq + N \subseteq q' + N$, so $rq + n = q' + n'$ for some $n, n' \in N$. It follows that $q' \in Q_M \cap (IM + N)$ since $IM + N$ is a k-subsemimodule by (ii) and (iii). Thus $r.(q + N) \in (IM + N)/N$. For the reverse inclusion, assume that $q_1 + N \in (IM + N)/N$, where $q_1 \in Q_M \cap (IM + N)$. Then there are elements $q'_i \in Q_M$, $t_i \in N$, $r_i \in R$ and $n \in N$ such that $q_1 = \sum_{i=1}^{s} r_i(q'_i + n_i) + n$; hence $q_1 = \sum_{i=1}^{s} r_i q'_i + x$, where $x \in N$. Let $q_0 + N$ is the zero in M/N. Clearly $x + N \subseteq N$. Assume that $y \in N$. Since $N = q_0 + N$ by [7, Lemma 2.3], there exist $a, b, c \in N$ with $y = q_0 + a$, $x = q_0 + b$ and $b + c = 0$; so $y = q_0 + b + a + c = x + a + c \in x + N$; hence $x + N = N$. An inspection will show that $q_1 + N = \sum_{i=1}^{s} r_i.(q'_i + N) \in I(M/N)$. Thus $(IM + N)/N \subseteq I(M/N)$, and so we have equality.

Theorem 2. *Let N be a strong Q_M-subsemimodule of a strong multiplication semimodule M over a semiring R. Then M/N is a strong multiplication R-semimodule.*

Proof. Let L be a k-subsemimodule of M/N. Then by [7, Theorem 3.6], $L = T/N$ for some k-subsemimodule T of M with $N \subseteq T$, so there exists a strong ideal I of R such that $T = IM$. Therefore $I(M/N) = (IM + N)/N = T/N = L$ by Proposition 1 (iv), as needed.

Theorem 3. *Let N be a strong Q_M-subsemimodule of a very strong multiplication semimodule M over a semiring R. Then M/N is a very strong multiplication R-semimodule.*

Proof. By Theorem 2 and Definition 5, it suffices to show that M/N is a very strong semimodule. Let I be an ideal of R and $q + N \in M/N$, where $q \in Q_M$ and set $J = \{r \in R : r.(q + N) \in I(M/N)\}$; we show that J is a strong k-ideal of R. Let $r, r' \in J$. There are unique elements $q_1, q_2 \in Q_M$ such that $r.(q + N) = q_1 + N \in I(M/N)$ and

$r'.(q+N) = q_2 + N \in I(M/N)$, where $rq + N \subseteq q_1 + N$, $r'q + N \subseteq q_2 + N$. Then there exists a unique element $q_3 \in Q_M$ such that $(q_1+N) \oplus (q_2+N) = q_3+N$, where $q_1+q_2+N \subseteq q_3+N$, so $(r+r')q+N \subseteq q_1+q_2+N \subseteq q_3+N \in I(M/N)$; hence $r+r' \in J$. Similarly, if $s \in R$, then $sr \in J$. Thus J is an ideal of R. By assumption, there must exists a strong ideal I' of R such that $N = I'M$, and the set $J_1 = \{r \in R : rq \in (I+I')M\}$ is a strong k-ideal of R since M is a very strong semimodule. Let $t \in J$. Then $tq + N \subseteq q' + N \in I(M/N)$ for some $q' \in (IM + I'M) \cap Q_M$, so $tq \in (I+I')M$ since it is a k- subsemimodule by Proposition 1 (iii); hence $t \in J_1$. Thus $J \subseteq J_1$. It follows that J is a strong k-ideal of R, and this completes the proof.

Proposition 2. *Let M be a non-strong semimodule over a semiring R with $M \neq 0$. Then M has at least one strong maximal Q_M-subsemimodule.*

Proof. Since $\{0_M\}$ is a proper strong Q_M-subsemimodule of M with respect to the set $Q_M = M - \{0_M\}$, the set Δ of all proper strong Q_M-subsemimodules of M is not empty. Of course, the relation of inclusion, $\subseteq$, is a partial order on Δ. If $\{M_i\}_{i \in I}$ is a chain of strong Q_M-subsemimidules of M, then $N = \bigcup_{i \in I} M_i$ is a strong Q_M-subsemimodule of M. Furthermore, N is proper since M is not strong. So by Zorn's Lemma Δ has a maximal element, i.e., M has a strong maximal Q_M-subsemimodule.

Theorem 4. *Let M be a non-strong semimodule over a semiring R with $M \neq 0$. Then* $\mathrm{Spec}_k(M) \neq \emptyset$.

Proof. By Proposition 2, there exists a strong maximal Q_M-subsemimodule N of M; so it is a k-prime subsemimodule of M by [13, Theorem 14] (since every Q_M-subsemimodule is a k-subsemimodule by [7, Theorem 3.2]), as required.

Recall that we follows Golan's terminology for quotient semirings in the following lemma.

Lemma 3. *Let I be an ideal of a semiring R with $1 \neq 0$. Then the following hold:*

(i) If L is a k-ideal of R/I, then $L = J/I$ for some k-ideal J of R.

(ii) If $I \subseteq P$, then P is a maximal k-ideal of R if and only P/I is a maximal k-ideal of R/I.

(iii) R has at least one strong maximal k-ideal.

(iv) If J is a proper strong k-ideal of R, then $J \subseteq P$ for some strong maximal k-ideal P of R.

Proof.

(i) Assume that $J = \{r \in R : r + I \in L\}$ and let $a \in I$. Since $a + I = 0 + I \in L$, we have $I \subseteq J$. Let $a, b \in J$ and $r \in R$. Then $(a+I)+(b+I) = a+b+I \in L$; so $a+b \in J$. Similarly, $ra \in J$. Thus J is a k-ideal of R. Finally, it is easy to see that $L = J/I$.

(ii) Suppose that P is a maximal k-ideal of R and let L be a k-ideal of R/I such that $P/I \subsetneqq L$. There exists a k-ideal J of R such that $P/I \subsetneqq L = J/I$ by (i), so $P \subsetneqq J$; hence $J = R$. Thus $L = R/I$. The other implication is similar.

(iii) Since $\{0\}$ is a proper strong k-ideal of R, the set Δ of all proper strong k-ideals of R is not empty. So by Zorn's Lemma Δ has a maximal element (with respect to $\subseteq$), i.e., R has a proper strong maximal k-ideal.

(iv) Since R/I is non-trivial, and so, by (iii), has a strong maximal k-ideal L, which, by (i), will have to have the form P/I for some k-ideal P of R with $I \subseteq P$. It now follows from (ii) that P is a maximal k-ideal of R. It remains to show that P is a strong ideal of R. Let $a \in P$. Then $a + I \in P/I$. By assumption, $(a + I) + (b + I) = a + b + I = 0 + I = I$ for some $b + I \in P/I$, so $a + b \in I$. Then there is an element $c \in I$ such that $a + b + c = 0$, as required.

Theorem 5. *Let M be a non-zero very strong multiplication semimodule over a semiring R. Then every proper strong Q_M-subsemimodule of M is contained in a strong maximal k-subsemimodule of M.*

Proof. Assume that N is a proper strong Q_M-subsemimodule of M and let $q_0 + N$ is the zero in M/N. Then there exists $x \in M - N$ such that $x = q + n$ for some $q \in Q_M$ with $q \notin N$ (since every Q_M-subsemimodule is a k-subsemimodule by [7, Theorem 3.2]) and $n \in N$; hence $q_0 + N \neq q + N \in M/N$. Then M/N is a non-zero very strong multiplication semimodule. Thus it is sufficient to prove that any non-zero very strong multiplication semimodule contains a maximal k-subsemimodule. Let $0 \neq m \in M$. If $I = \{0\}$, then the ideal $J = \{r \in R : rm = 0\}$ is a proper strong k-ideal of R since M is a very strong semimodule and hence $J \subseteq P$ for some strong maximal k-ideal P of R by Lemma 3 (iv). If $M = PM$, then $Rm = TM$ for some ideal T of R, so $Rm = TPM = PRm = Pm$ and hence $m = pm$ for some $p \in P$. There exists $p' \in P$ such that $p + p' = 0$. But this implies $(1 + p')m = 0$; so $1 + p' \in I \subseteq P$, a contradiction. Thus $M \neq PM$. Since P is a strong ideal of R, we have PM is a strong subsemimodule of M by Proposition 1 (i). Let $PM \subsetneqq N = LM \subseteq M$ for some ideal L of R. It follws that there is an element $a \in L$ with $a \notin P$; so $P + Ra = R$. Therefore there exist $t \in P$ and $r \in R$ such that $t + ra = 1$; whence $m = tm + ram \in PM + LM = N$. Thus $M = N$ and hence PM is a strong maximal k-subsemimodule of M.

Definition 6. *Let M be a non-zero semimodule over a semiring R. An element u of M is said to be unit provided that u is not contained in any strong maximal k-subsemimodule of M.*

Theorem 6. *Let M be a non-zero very strong multiplication semimodule over a semiring R. Then $u \in M$ is unit if and only if $M = Ru$.*

Proof. The sufficiency is clear. Conversely, suppose that u is an unit element of M. Then Ru is not contained in any strong maximal k-subsemimodule of M; hence $M = Ru$ by Theorem 5.

Assume that P is a strong maximal k-ideal of a semiring R and let M be a semimodule over R. We say that M is a P-cyclic provided there exist $p \in P$ and $m \in M$ such that $(1+p)M \subseteq Rm$. We say that a subset $T_P(M)$ of M is P-torsion precisely when
$T_P(M) = \{m \in M : (1 + p)m = 0 \text{ for some } p \in P\}$. The Definition is the same as that introduced by Z. El-Bast and P. F. Smith in [6]. It is easy to see that $T_P(M)$ is a subsemimodule of M.

Proposition 3. *(i) If M is a strong multiplication semimodule over a semiring R, then for every strong maximal k-ideal P of R either $M = T_P(M)$ or M is P-cyclic.*

(ii) If M is a faithful very strong multiplication semimodule over a semiring R, then $\bigcap_{i\in\Lambda}(I_iM) = (\bigcap_{i\in\Lambda} I_i)M$ for any non-empty collection of strong ideals I_i $(i \in \Lambda)$ of R.

(iii) Let P be a strong prime k-ideal of a semiring R and M a faithful very strong multiplication semimodule over R. Let $a \in R$, $x \in M$ satisfy $ax \in PM$. Then $a \in P$ or $x \in PM$. In particular, if $M \neq PM$, then PM is a strong prime subsemimodule of M.

Proof.

(i) Let P be a strong maximal k-ideal of R. Suppose $M = PM$. Let $m \in M$. Then $Rm = IM$ for some strong ideal I of R by [15, Proposition 2.4]. Hence $Rm = IM = IPM = Pm$ and $m = pm$ for some $p \in P$. By assumption, there exists $p' \in P$ such that $pm + p'm = (1 + p')m = 0$ and $m \in T_P(M)$. It follows that $T_P(M) = M$. Now suppose that $PM \neq M$. There exists $y \in M$ such that $y \notin PM$. There is a strong ideal J of R such that $Ry = JM$. Clearly, $J \nsubseteq P$. Since, $J + P$ is a strong ideal of R, we must have $J + P = R$, so $1 = e + q$ for some $e \in J$ and $q \in P$. There exists $q' \in P$ such that $q + q' = 0$; hence $1 + q' \in J$. It follows that $(1 + q')M \subseteq Ry$ and M is P-cyclic.

(ii) Let I_i $(i \in \Lambda)$ be any non-empty collection of strong ideals of R. Set $I = \bigcap_{i\in\Lambda} I_i$. Clearly, $IM \subseteq \bigcap_{i\in\Lambda}(I_iM)$. For the reverse inclusion, assume that $x \in \bigcap_{i\in\Lambda}(I_iM)$. Then $K = \{r \in R : rx \in IM\}$ is a strong k-ideal of R. Suppose $K \neq R$. Then by Lemma 3 (iv), there exists a strong maximal k-ideal P of R such that $K \subseteq P$. Clearly, $x \notin T_P(M)$. For if $x \in T_P(M)$, then $(1 + p)x = 0 \in IM$ for some $p \in P$; hence $(1 + p) \in K \subseteq P$, a contradiction. Therefore, M is P-cyclic by (i). There exist $p \in P$ and $m \in M$ such that $(1 + p)M \subseteq Rm$. Then $(1 + p)x \in \bigcap_{i\in\Lambda}(I_im)$. For each $i \in \Lambda$, there is an element $a_i \in I_i$ such that $(1 + p)x = a_im$. Choose $j \in \Lambda$. Then for each $i \in \Lambda$, $a_jm = a_im$. By assumption, $a_i + a'_i = 0$ for some $a'_i \in I_i$; hence $a_jm + a'_im = 0$. Now $(1 + p)(a_j + a'_i)M \subseteq (a_j + a'_i)Rm = 0$ implies $(1 + p)(a_j + a'_i) = 0$ since M is faithful. Therefore, $a_j + pa_j + a'_i + pa'_i = 0$, so $(1 + p)a_j = (1 + p)a_i \in I_i$; hence $(1 + p)a_j \in I$. Thus $(1 + p)^2x = (1 + p)(a_im) \in IM$. It follows that $(1 + p)^2 \in K \subseteq P$, a contradiction. So $K = R$; hence $x \in IM$, and (ii) is proved.

(iii) Let $a \notin P$. Then the ideal $K = \{r \in R : rx \in PM\}$ is a strong k-ideal of R. Suppose $K \neq R$. Then there exists a strong maximal k-ideal P' of R such that $K \subseteq P'$. Clearly, $x \notin T_{P'}(M)$. By (i), M is P'-cyclic, that is, there exist $m \in M$ and $q \in P'$ such that $(1 + q)M \subseteq Rm$. In particular, $(1 + q)x = sm$ for some $s \in R$. Therefore we have $(1 + q)ax \in (1 + q)PM \subseteq PRm = Pm$; hence $asm = pm$ for some $p \in P$. By assumption, $p + p' = 0$ for some $p \in P'$; hence $(as + p')m = 0$. Since $(1 + q)\text{ann}(m)M \subseteq R\text{ann}(m)m = 0$, we must have $(1 + q)(as + p') = 0$, $(1 + q)as = (1 + q)p \in P$. But $P \subseteq K \subseteq P'$ so that $s \in P$ and $(1 + q)x = sm \in PM$. Thus $(1 + q) \in K \subseteq P'$, which is a contradiction. It follows that $K = R$ and $x \in PM$, as required.

Remark 1. *(i) (Change of semirings) Assume that I is an ideal of a semiring R with $I \subseteq (0 : M)$ and let M be an R-semimodule. We show now how M can be given a natural structure as a semimodule over R/I. Let $r, s \in R$ such that $r + I = s + I$, and let $m \in M$. Then $r + a = s + b$ for some $a, b \in I$, and $rm = sm$. Hence we can unambiguously define a mapping $R/I \times M$ into M (sending $(r + I, m)$ to rm) and it is routine to check that this turns the commutative additive semigroup with a zero element M into an R/I-semimodule. It should be noted that a subset of M is an R-subsemimodule if and only if it is an R/I-subsemimodule.*

(ii) Assume that N is a proper k-subsemimodule of a semimodule M over a semiring R and let I be an ideal of R with $I \subseteq (0 : M)$. Then N is a prime R-subsemimodule of M if and only if N is a prime subsemimodule of M as an R/I-semimodule.

Theorem 7. *The following statements are equivalent for a proper k-subsemimodule N of a very strong multiplication semimodule M over a semiring R.*

(i) N is a strong prime subsemimodule of M.

(ii) $(N : M)$ is a strong prime k-ideal of R.

item $N = PM$ for some strong prime ideal P of R with $(0 : M) \subseteq P$.

Proof. $(i) \Rightarrow (ii)$. By [13, Lemma 4], $(N : M)$ is a prime ideal of R. If $0 \neq m \in M$, then $J = \{r \in R : rm \in (N : M)M\}$ is a proper strong k-ideal of R (since M is very strong multiplication) with $(N : M) \subseteq J$; hence $(N : M)$ is a strong prime k-ideal of R. $(ii) \Rightarrow (iii)$ is clear.

$(iii) \Rightarrow (i)$. Since $N = PM \neq M$ and as an $R/(0 : M)$-semimodule, N is a strong prime subsemimodule by Proposition 3 (iii), so is a strong prime as an R-subsemimodule of M by Remark 1.

Theorem 8. *Let R be a semiring, N a proper subsemimodule of a very strong multiplication R-semimodule M and $A = (N : M)$. Then* $\text{rad}(N) = \text{rad}(A)M$.

Proof. Without loss of generality M is a faithful R-semimodule. Let $\mathscr{B}$ denote the collection of all strong prime ideals P of R such that $A \subseteq P$ and $\mathscr{C}$ denote the collection of all prime ideals P of R such that $A \subseteq P$. Clearly, $\mathscr{C} \subseteq \mathscr{B}$. If $B = \text{rad}(A)$, then $B = \bigcap_{P \in \mathscr{C}} P$ [see, 1], and hence by Proposition 3 (ii), $BM = \bigcap_{P \in \mathscr{C}} (PM) \subseteq \bigcap_{P \in \mathscr{B}} PM$. Let $P \in \mathscr{B}$. If $M = PM$, then $\text{rad}(N) \subseteq PM$. If $M \neq PM$, then $N = AM \subseteq PM$ implies $\text{rad}(N) \subseteq PM$ by Theorem 7. It follows that $\text{rad}(N) \subseteq BM$. Conversely, suppose that L is a strong prime subsemimodule of M containing N. By Theorem 7, there exists a strong prime ideal P' of R such that $L = P'M$. Since $AM = N \subseteq L = P'M \neq M$ it follows that $A \subseteq P'$ by [13, Theorem 7], and hence $B \subseteq P'$. Thus $BM \subseteq L$. It follows that $BM \subseteq \text{rad}(N)$, and so we have equality.

4. Prime spectrum

Assume that R is a semiring and let M be an R-semimodule and N be a subsemimodule of M such that $N = IM$ for some ideal I of R. Then we say that I is a presentation ideal of N. Clearly, every subsemimodule of M has a presentation ideal if and only if M is a multiplication semimodule. Let N and K be subsemimodules of a multiplication R-semimodule M with $N = I_1M$ and $K = I_2M$ for some ideals I_1 and I_2 of R. The product N and K denoted by NK is defined by $NK = I_1I_2M$. Let $N = I_1M = I_2M = N'$ and $K = J_1M = J_2M = K'$ for some ideals I_1, I_2, J_1 and J_2 of R. It is easy to show that $NK = N'K'$, that is, NK is independent of presentation ideals of N and K (The proof is similar to Ameri [4, Theorem 3.4]). It is easy to see that NK is a subsemimodule of M and $NK \subseteq N \cap K$. For $m, m' \in M$ by mm', we mean the product of Rm and Rm', that is, $mm' = IJM$, where I and J are presentations for m and m', respectively.

Lemma 4. *Let N be a proper subsemimodule of a multiplication semimodule M over a semiring R. Then the follwing statements hold:*

(i) N is prime if and only if whenever $UV \subseteq N$ for some subsemimodules U and V of M, then $U \subseteq N$ or $V \subseteq N$.

(ii) N is prime if and only if whenever $m.m' \subseteq N$ for some $m, m' \in M$, then $Rm \subseteq N$ or $Rm' \subseteq N$.

Proof. The proofs are straightforward (the proofs are similar to Ameri [4, Theorem 3.16 and Corollary 3.17]).

Let M be a non-strong semimodule over a semiring R with $M \neq 0$. Then by Theorem 4, the k-spectrum $X = \mathrm{Spec}_k(M)$ is non-empty. For any subsemimodule N of a semimodule M by $V(N)$ we mean the set of all prime k-subsemimodules of M containing N. Clearly, $V(M) = \emptyset$ and $V(\{0\}) = \mathrm{Spec}_k(M) = X$. Throughout this section we may assume that $\mathrm{Spec}_k(M)$ is non-empty.

Lemma 5. *Let M be a semimodule over a semiring R. Then the following statements hold:*

(i) If N is a subsemimodule of M, then $V(N) = V(\mathrm{rad}(N))$.

(ii) If $\{N_i\}_{i\in I}$ is a family of subsemimodules of M, then $V(\sum_{i\in I} N_i) = \bigcap_{i\in I} V(N_i)$.

Proof.

(i) Since $N \subseteq \mathrm{rad}(N)$, we have $V(\mathrm{rad}(N)) \subseteq V(N)$. For the reverse inclusion, assume that $P \in V(N)$. Then $N \subseteq P$; hence $\mathrm{rad}(N) \subseteq P$, and so we have equality.

(ii) Let $P \in \bigcap_{i\in I} V(N_i)$. Then $N_i \subseteq P$ for every $i \in I$, so $\sum_{i\in I} N_i \subseteq P$, which implies that $\bigcap_{i\in I} V(N_i) \subseteq V(\sum_{i\in I} N_i)$. The reverse inclusion is similar.

If $\zeta(M)$ denotes the collection of all subsets $V(N)$ of $\mathrm{Spec}_k(M)$, then $\zeta(M)$ contains the empty set and $\mathrm{Spec}(M)$ and is closed under arbitrary intersection by Lemma 5 (ii). If also $\zeta(M)$ is closed under finite union, that is, for every subsemimodules N and L of M such that $V(N) \cup V(L) = V(T)$ for some subsemimodule T of M, for in this case $\zeta(M)$ satisfies the axioms of closed subsetes of a topological spaces, which is called Zariski topology. In MacCasland, Moore and Smith [22] a module with Zariski topology is called a top module.

Lemma 6. *The following statements are equivalent for a semimodule M over a semiring R.*

(i) M is a top semimodule.

(ii) Every prime k-subsemimodule of M is extraordinary.

(iii) $V(T) \cup V(L) = V(T \cap L)$ for any semiprime subsemimodules T and L of M.

Proof. $(i) \Rightarrow (ii)$. Let N be any prime k-subsemimodule of M and let T and L be semiprime subsemimodules of M such that $T \cap L \subseteq N$. By (i), there exists a subsemimodule U of M such that $V(T) \cup V(L) = V(U)$. Now set $T = \bigcap_{i \in I} N_i$, where N_i is a prime k-subsemimodule of M $(i \in I)$. For each $i \in I$, $N_i \in V(T) \subseteq V(U)$, so that $U \subseteq N_i$. Thus $U \subseteq T$. Similarly, $U \subseteq L$. Thus $U \subseteq T \cap L$. Now we have $V(T) \cup V(L) \subseteq V(T \cap L) \subseteq V(U) = V(T) \cup V(L)$, that is, $V(T) \cup V(L) = V(T \cap L)$. Now $N \in V(T \cap L)$ gives $T \subseteq N$ or $L \subseteq N$ by Lemma 5. $(ii) \Rightarrow (iii)$ is clear.

$(iii) \Rightarrow (i)$. Let A and B be any k-subsemimodules of M. If $V(A) = \emptyset$, we are done. So we may assume that $V(A)$ and $V(B)$ are both non-empty. Then $V(A) \cup V(B) = V(\mathrm{rad}(A)) \cup V(\mathrm{rad}(B)) = V(\mathrm{rad}(A) \cap \mathrm{rad}(B))$ by Lemma 5 and (iii), as required.

Theorem 9. *If N is a Q_M-subsemimodule of a top semimodule M over a semiring R, then M/N is a top semimodule.*

Proof. Note that any semiprime k-subsemimodule of M/N has the form U/N where U is a semiprime k-subsemimodule of M containing N by Corollary 1. Let T/N be any prime k-subsemimodule of M/N and let U/N and L/N be semiprime k-subsemimodules of M/N such that $(L/N) \cap (U/N) \subseteq T/N$. Then $(L \cap U)/N \subseteq (L/N) \cap (U/N) \subseteq T/N$, so $U \cap L \subseteq T$; hence either $U \subseteq T$ or $L \subseteq T$ since T is extraordinary by Lemma 6. Thus either $U/N \subseteq T/N$ or $L/N \subseteq T/N$. Now the assertion follows from Lemma 6 (ii).

Theorem 10. *Let N, L be k-subsemimodules of a semimodule M over a semiring R. Then the following statements hold:*

(i) If S is a subset of M, then $V(S) = V(< S >)$.

(ii) $V(N) \cup V(IM) = V(IN) = V(N \cap IM)$ for every strong ideal I of R .

(iii) $V(IM) \cup V(JM) = V(IJM) = V(IM \cap JM)$ for every strong ideals I and J of R.

(iv) If $V(N) \subseteq V(L)$, then $L \subseteq \mathrm{rad}(N)$.

(v) $V(N) = V(L)$ *if and only if* $\mathrm{rad}(N) = \mathrm{rad}(L)$

(vi) *If M is a strong multiplication semimodule, then* $V(N) \cup V(L) = V(NL) = V(N \cap L)$.

(vii) *Every strong multiplication semimodule is a top module.*

(viii) *If M is a strong multiplication semimodule, then every prime k-subsemimodule of M is extraordinary.*

Proof.

(i) Obvious.

(ii) It is clear that $V(N) \cup V(IM) \subseteq V(N \cap IM) \subseteq V(IN)$. Let $P \in V(IN)$. Then $IN \subseteq P$ and hence $N \subseteq P$ or $IM \subseteq P$ by [13, Theorem 7]. Thus $P \in V(N)$ or $P \in V(IM)$, i.e. $P \in V(N) \cup V(IM)$. Hence $V(IN) \subseteq V(N) \cup V(IM)$.

(iii) Follows from (ii).

(iv) Obvious.

(v) Let $V(N) = V(L)$. By Lemma 5, we have $V(N) \subseteq V(\mathrm{rad}(L))$; hence $\mathrm{rad}(L) \subseteq \mathrm{rad}(N)$ by (iv). Similarly, $\mathrm{rad}(N) \subseteq \mathrm{rad}(L)$, and so we have equality. The other implication is similar.

(vi) Apply (iii).

(vii) Follows from (vi).

(viii) Follows from (vii) and Lemma 6.

Remark 2. *Assume that M is a semimodule over a semiring R and let $X = \mathrm{Spec}_k(M)$. For each subset S of M, by X_S we mean $X - V(S) = \{P \in X : S \nsubseteq P\}$. If $S = \{m\}$, we denote by $X_m = \{P \in X : Rm \nsubseteq P\} = \{P \in X : m \nsubseteq P\}$. Clearly, the sets X_m are open, and they are called* basic open sets.

Lemma 7. *Let M be a strong multiplication semimodule over a semiring R. Then the following statements hold:*

(i) $X_{IM} \cap X_{JM} = X_{IJM}$ *for every strong ideals I and J of R.*

(ii) *The set* $\mathscr{A} = \{X_m : m \in M\}$ *forms a base for the Zariski topology on X.*

Proof.

(i) Immediately follows from Theorem 10 (iii) (taking complements).

(ii) Suppose that U is an open set in X. Then $U = X - V(N)$ for some k-subsemimodule N of M. Let $N =< \{m_i : i \in I\} >$, where $\{m_i : i \in I\}$ is a generator set of N. Then $V(N) = V(\sum_{i\in I} Rm_i) = \bigcap_{i\in I} V(Rm_i)$ by Lemma 5 (ii). It follows that $U = X - V(N) = X - \bigcap_{i\in I} V(Rm_i) = \bigcup_{i\in I} X_{m_i}$. Thus $\mathscr{A}$ is a base for the Zariski topology on X.

Proposition 4. *Let M be a very strong multiplication semimodule over a semiring R. Then every basic open set of X is compact.*

Proof. By Lemma 7 (ii), it suffices to show that every cover of basic open sets has a finite subcover. Suppose that $X_m \subseteq \bigcup_{t\in I} X_{m_t}$, and let N be the subsemimodule of M generated by $\{m_t : t \in I\}$. It follows that $\bigcap_{t\in I} V(Rm_t) = V(N) \subseteq V(Rm)$, so $V(\text{rad}(N)) \subseteq V(\text{rad}(< m >))$ by Lemma 5 (i); hence $\text{rad}(< m >) \subseteq \text{rad}(N)$ by Theorem 10 (iv). Moreover, by Theorem 8, $\text{rad}(N) = \text{rad}(A)M$, where $A = (N : M)$. By assumption, there exists a finite subset J of I and $r_i \in \text{rad}(A)$ $(i \in J)$ such that $m = \sum_{t\in J} r_t m_t$. For $r_i \in \text{rad}(A)$, there is a positive integer s_i such that $r_i^{s_i} \in A$. If $s = \sum_{i\in J} s_i$, then $r_i^s \in A$ for every $i \in J$. For each $i \in J$, there exists a strong ideal I_i of R such that $Rm_i = I_i M$; so by Proposition 3 (ii), $m \in \sum_{i\in J} r_i I_i M = (\sum_{i\in J}(r_i I_i))M$. Thus $m^s \subseteq (\sum_{i\in J}(r_i I_i)^s)M \subseteq AM$. Therefore Theorem 10 gives $V(N) = \bigcap_{i\in I} V(Rm_i) \subseteq \bigcap_{i\in J} V(Rm_i) \subseteq V(m) = V(Rm) = V(m^s)$. Taking complements, we have $X_m \subseteq \bigcup_{i\in J} X_{m_i}$, and so the proof is complete.

Theorem 11. *Let M be a very strong multiplication semimodule over a semiring R. Then an open set of X is compact if and only if it is a finite union of basic open sets.*

Proof. Apply Lemma 7 and Proposition 4.

Corollary 2. *Let M be a finitely generated very strong multiplication semimodule over a semiring R. Then X is compact.*

Proof. Let $M = \sum_{i=1}^n Rm_i$. Then $V(M) = \emptyset$; hence $X_M = X$, that is, $X = \bigcup_{i=1}^n X_{m_i}$. Thus X is compact.

Question: Assume that M is a very strong multiplication semimodule over a semiring R and let X be compact. Is M finitely generated?

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