EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 7, No. 4, 2014, 419-428
ISSN 1307-5543 – www.ejpam.com

Some Approximation Properties of Szasz-Mirakyan-Bernstein Operators

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Abstract. In this study, we have constructed a new sequence of positive linear operators by using Szasz-Mirakyan and Bernstein operators on space of continuous functions on the unit compact interval. We also find order of this approximation by using modulus of continuity and give the Voronovskaya-type theorem.

2010 Mathematics Subject Classifications: 41A25, 41A36, 26A48

Key Words and Phrases: Positive linear operators, Korovkin's Theorem, Szasz-Mirakyan Operators, Bernstein Operators

1. Introduction

Let $\mathbb{N}$ denotes the set of natural numbers and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let f be real-valued function defined on the closed interval $[0,1]$. The n-th Bernstein operator of f, $B_n(f)$ is defined as

$$B_n\left(f;x\right) = \sum_{k=0}^{n} p_{n,k}\left(x\right) f\left(\frac{k}{n}\right), \qquad x \in [0,1], \qquad n \in \mathbb{N} \tag{1}$$

where

$$p_{n,k}\left(x\right) = \binom{n}{k} x^k (1-x)^{n-k}, \qquad 0 \le k \le n.$$

The Bernstein polynomials $B_n(f)$ was introduced to prove the Weierstrass approximation theorem by S. N. Bernstein [2] in 1912. They have been studied intensively and their connection with different branches of analysis, such as convex and numerical analysis, total positivity and the theory of monotone operators have been investigated. Basic facts on Bernstein polynomials and their generalizations can be found in [5, 7, 9, 10, 12, 14] and references therein.

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For the function f which is continuous on $[0,\infty)$, the Szasz-Mirakyan operators which are introduced by G. M. Mirakyan [8] in 1941 and then, are investigated by J. Favard [4] and O. Szasz [15], are defined as

$$S_n(f;x)=\sum_{m=0}^{\infty}q_{n,m}(x)f\left(\frac{m}{n}\right),\qquad x\in[0,\infty),n\in\mathbb{N}$$

where

$$q_{n,m}(x)=e^{-nx}\frac{(nx)^m}{m!},\qquad m\in\mathbb{N}_0. \tag{2}$$

2. Construction of the Generating Operators

Let I is a fixed interval (bounded or not) in $\mathbb{R}$ and ϖ_m be a sequence of density functions on the interval I, that is, the functions ϖ_m have the following properties:

i. ϖ_m non-negative for all $x\in I$ and $m\in\mathbb{N}_0$

ii. $\sum_{m=0}^{\infty}\varpi_m(x)=1$ for all $x\in I$

Let (L_n) be a sequence of positive linear operators defined on the set of the continuous functions on the interval I, say $C(I)$. Now we define the generating operators G_n on $C(I)$. For every $n\in\mathbb{N}$, $x\in I$ and $f\in C(I)$

$$G_n(f;x)=\sum_{m=0}^{\infty}\varpi_m(nx)L_{\varphi_{n,m}}(f;x)\qquad m\in\mathbb{N}_0, \tag{3}$$

where ϖ_m are density functions on I and $\varphi_{n,m}:=\varphi(n,m)=\alpha_n\beta_m$ where (α_n) is a non-decreasing and (β_m) is a strictly increasing natural sequence. It is easy to check that the operators G_n are positive and linear on $C(I)$.

Taking $I=[0,1]$, $\beta_m=m+1$, $\varpi_m=q_{n,m}$ and $L_{\varphi_{n,m}}=B_{\varphi_{n,m}}$ where B_φ and $q_{n,m}$ defined in (1) and (2) respectively, we can rewrite (3) as

$$E_n\left(f;x\right)=\sum_{m=1}^{\infty}\frac{e^{-nx}(nx)^{m-1}}{(m-1)!}\sum_{k=0}^{m\alpha_n}\binom{m\alpha_n}{k}x^k(1-x)^{m\alpha_n-k}f\left(\frac{k}{m\alpha_n}\right). \tag{4}$$

The operators E_n defined in (4) is called the Szasz-Mirakyan-Bernstein (SMB) operators. In this study, we investigate some approximation properties of these operators and find Voronovskya-type theorem and the order of this approximation by using modulus of continuity.

3. Some Notations and Auxiliary Facts

In this section we will give some basic definitions, theorems and some elementary properties concerning space of functions and moduli of smoothness of first and second order. For more information see [1] or [11].

1. Let $C[0,1]$ be the space of real-valued continuous function on $[0,1]$ equipped with the uniform norm:
$$\|f\| := \max\{|f(x)| : x \in [0,1]\}$$
and $C^r[0,1], r \in \mathbb{N}_0$, be the set all r-times continuously differentiable functions $f \in C[0,1]$.

2. For the real-valued function f defined on $[0,1]$ and $\delta \geq 0$, the modulus of continuity $\omega(f,\delta)$ and the second modulus of smoothness $\omega_2(f,\delta)$ of f are defined by
$$\omega(f,\delta) := \sup_{|x-y|\leq\delta} \{|f(x)-f(y)|\},$$
$$\omega_2(f,\delta) := \sup_{0\leq h\leq\delta}\ \sup_{0\leq x\leq 1-2h} \{|f(x+2h)-2f(x+h)+f(x)|\},$$
respectively. It is known that, for a function $f \in C[0,1]$, we have $\lim_{\delta\to 0}\omega(f,\delta) = 0$ and, for any $\delta > 0$,
$$\left|f(t)-f(x)\right| \leq \omega(f,\delta)\left(\frac{|t-x|}{\delta}+1\right) \tag{5}$$

3. As usual, a function $f \in Lip_M\mu$, ($M > 0$ and $0 < \mu \leq 1$), if the inequality
$$\left|f(t)-f(x)\right| \leq M\,|t-x|^{\mu} \tag{6}$$
holds for all $t, x \in [0,1]$

4. Let e_i denote the test functions defined by $e_i(t) = t^i$, $t \in \mathbb{R}$, $i = 0,1,2,\ldots$.

Theorem 1 (Korovkin [6]). *Let $L_n : C[a,b] \to C[a,b]$ be a sequence of positive linear operators. If*
$$\lim_{n\to\infty} L_n(e_i;x) = e_i(x), \qquad i = 0,1,2,$$
uniformly on $[a,b]$, then
$$\lim_{n\to\infty} L_n(f;x) = f(x).$$
uniformly on $[a,b]$, for every continuous function f defined on $[a,b]$.

4. Approximation Properties of E_n

In this section we give some classical approximation properties of the operators E_n. By simple calculations, we get the following lemmas.

Lemma 1. *For $x \in [0,1]$ and $n \in \mathbb{N}$, we have*
$$\begin{aligned}
E_n(e_0;x) =& 1;\\
E_n(e_1;x) =& x;\\
E_n(e_2;x) =& x^2 + \frac{(1-x)(1-e^{-nx})}{n\alpha_n};
\end{aligned}$$

$$E_n(e_3;x) = x^3 + \frac{3x(1-x)\left(1-e^{-nx}\right)}{n\alpha_n} + \frac{(1-x)(1-2x)}{n\alpha_n^2}\sum_{m=1}^{\infty}\frac{e^{-nx}(nx)^m}{m.m!};$$

$$E_n(e_4;x) = x^4 + \frac{6x^2(1-x)\left(1-e^{-nx}\right)}{n\alpha_n} + \frac{x(1-x)(7-11x)}{n\alpha_n^2}\sum_{m=1}^{\infty}\frac{e^{-nx}(nx)^m}{m.m!} + \frac{(1-x)\left(6x^2-6x+1\right)}{n\alpha_n^3}\sum_{m=1}^{\infty}\frac{e^{-nx}(nx)^m}{m^2m!}.$$

Lemma 2. *For $x \in [0,1]$ and $n \in \mathbb{N}$, the following holds:*

$$E_n(e_1-x;x) = 0,$$

$$E_n((e_1-x)^2;x) = \frac{(1-x)(1-e^{-nx})}{n\alpha_n},$$

$$E_n((e_1-x)^3;x) = \frac{(1-x)(1-2x)}{n\alpha_n^2}\sum_{m=1}^{\infty}\frac{e^{-nx}(nx)^m}{m.m!},$$

$$E_n((e_1-x)^4;x) = \frac{3x(1-x)^2}{n\alpha_n^2}\sum_{m=1}^{\infty}\frac{e^{-nx}(nx)^m}{m.m!} + \frac{(1-x)(6x^2-6x+1)}{n\alpha_n^3}\sum_{m=1}^{\infty}\frac{e^{-nx}(nx)^m}{m^2m!}.$$

Lemma 3. *For all $j \in \mathbb{N}_0$, we have*

$$\sum_{m=1}^{\infty}\frac{e^{-x}x^m}{m^jm!} \le \frac{\left(j+1\right)!}{x^j}, \qquad x \in (0,\infty).$$

Lemma 4. *For all $n \in \mathbb{N}$, we have*

$$E_n\left(\left(e_1-x\right)^4;x\right) \le c_n(x)\left(\frac{1}{n\alpha_n}\right)^2, \qquad x \in (0,1]$$

where $\lim_{n\to\infty} c_n(x) = 6$.

Proof. For $x \in (0,1]$. By Lemma 2 and Lemma 3, it results that

$$E_n\left(\left(e_1-x\right)^4;x\right) \le \frac{3x(1-x)^2}{n\alpha_n^2}\frac{2!}{nx} + \frac{(1-x)\left(6x^2-6x+1\right)}{n\alpha_n^3}\frac{3!}{n^2x^2}$$
$$\le \left(\frac{1}{n\alpha_n}\right)^2\left(6 + \frac{6x^2-6x+1}{x^2n\alpha_n}\right) = \left(\frac{1}{n\alpha_n}\right)^2 c_n(x)$$

□

Theorem 2. *If $f \in C[0,1]$, then the sequence of positive linear operators $\left\{E_n\right\}$ converges uniformly to f on $[0,1]$.*

Proof. From Lemma 1, we get

$$E_n\left(e_i\right) \overset{[0,1]}{\rightrightarrows} e_i \qquad i=0,1,2 \qquad n\to\infty.$$

Then, using Korovkin's theorem, we can conclude that

$$E_n\left(f\right) \overset{[0,1]}{\rightrightarrows} f, \qquad n\to\infty.$$

Where, the symbol $\overset{[0,1]}{\rightrightarrows}$ shows the uniform convergence on $[0,1]$. □

5. Voronovskaya-Type Theorem

The Voronovskaya theorem for the Bernstein operators is given in [7] or [6]. Also, for the sequence of positive linear operators can be found in [3, 13].

Theorem 3. *If $f\in C^2[0,1]$, then*

$$\lim_{n\to\infty} n.\alpha_n\left[E_n\left(f;x\right)-f\left(x\right)\right]=\frac{1}{2}\left(1-x\right)f''\left(x\right)$$

for every fixed $x\in[0,1]$.

Proof. We use the Taylor formula for a fixed point $x_0\in[0,1]$. For all $t\in[0,1]$, we have

$$f\left(t\right)=f\left(x_0\right)+f'\left(x_0\right)\left(t-x_0\right)+\frac{1}{2}f''\left(x_0\right)\left(t-x_0\right)^2+g\left(t;x_0\right)\left(t-x_0\right)^2$$

where $g(t;x_0)$ is the Peano form of the remainder, $g(.;x_0)\in C^2[0,1]$ and

$$\lim_{t\to x_0} g(t;x_0)=0.$$

Because $E_n(e_0;x)=1$, then

$$\begin{aligned}E_n\left(f;x_0\right)-f\left(x_0\right)=&f'\left(x_0\right)E_n\left(\left(e_1-x_0\right);x_0\right)\\&+\frac{1}{2}f''\left(x_0\right)E_n\left(\left(e_1-x_0\right)^2;x_0\right)+E_n\left(g\left(\cdot,x_0\right)\cdot\left(e_1-x_0\right)^2;x_0\right)\end{aligned}$$

By Cauchy-Schwartz's inequality, we have

$$n\alpha_nE_n\left(g\left(\cdot,x_0\right)\left(e_1-x_0\right)^2;x_0\right)\le\left[n^2\alpha_n^2E_n\left(\left(e_1-x_0\right)^4;x_0\right)\right]^{1/2}\cdot\left[E_n\left(g^2\left(\cdot,x_0\right);x_0\right)\right]^{1/2}$$

The function $\varphi(t;x_0)=g^2(t;x_0)$, $t\ge0$, satisfies the conditions of Teorem 2; therefore

$$\lim_{n\to\infty} E_n(g^2(t;x_0);x_0)=0$$

Moreover, by Lemma 4, we have

$$n\alpha_n E_n\left(g\left(\cdot,x_0\right)\left(e_1-x_0\right)^2;x_0\right)\leq\left(n^2\alpha_n^2\left(c_n\left(x_0\right)\left(\frac{1}{n\alpha_n}\right)^2\right)\right)^{1/2}\cdot\left(E_n\left(\left(g^2\left(\cdot,x_0\right)\right);x_0\right)\right)^{1/2}$$

It results that $\lim_{n\to\infty} n\alpha_n E_n\left(g\left(\cdot,x_0\right)\left(e_1-x_0\right)^2;x_0\right)=0$. By the above results and by Lemma 2, we obtain

$$\lim_{n\to\infty} n.\alpha_n\left[E_n\left(f;x_0\right)-f\left(x_0\right)\right]=\frac{1}{2}\left(1-x_0\right)f''\left(x_0\right).$$

□

6. Rates of Convergence

In this section we shall give error estimates, the for $f\in C[0,1]$ and $f\in C^1[0,1]$.

Theorem 4. *If $f\in C[0,1]$, then*

$$\left\|E_n(f)-f\right\|\leq 2\omega\left(f;\frac{1}{\sqrt{n\alpha_n}}\right). \tag{7}$$

Proof. Let $f\in C[0,1]$. By linearity and positivity of the operators E_n we get, for all $n\in\mathbb{N}$ and $x\in[0,1]$, that

$$\left|E_n\left(f;x\right)-f\left(x\right)\right|\leq E_n\left(\left|f-f\left(x\right)\right|;x\right). \tag{8}$$

Now using (5) in inequality (8) we have, for any $\delta>0$, that

$$\left|E_n\left(f;x\right)-f\left(x\right)\right|\leq\left(1+\frac{1}{\delta}E_n\left(\left|e_1-x\right|;x\right)\right)\omega\left(f;\delta\right) \tag{9}$$

Applying the Cauchy-Schwartz inequality for positive linear operators it follows from (9) that

$$\left|E_n\left(f;x\right)-f\left(x\right)\right|\leq\left(1+\frac{1}{\delta}\sqrt{E_n\left(\left(e_1-x\right)^2;x\right)}\right)\omega\left(f;\delta\right)$$

Using Lemma 1 in the last inequality, we can write

$$\left|E_n\left(f;x\right)-f\left(x\right)\right|\leq\left(1+\frac{1}{\delta}\left(\frac{(1-x)\left(1-e^{-nx}\right)}{n\alpha_n}\right)^{1/2}\right)\omega\left(f;\delta\right)\leq\left(1+\frac{1}{\delta}\left(\frac{1}{n\alpha_n}\right)^{1/2}\right)\omega\left(f;\delta\right).$$

Choosing $\delta_n=\left(\frac{1}{n\alpha_n}\right)^{1/2}$, we have the inequality

$$\left|E_n\left(f;x\right)-f\left(x\right)\right|\leq 2\omega\left(f;\frac{1}{\sqrt{n\alpha_n}}\right).$$

□

Theorem 5. *For all $f \in Lip_M \mu$ and $x \in [0,1]$, we have*

$$\left\|E_n(f) - f\right\| \leq M\left(\frac{1}{n\alpha_n}\right)^{\mu/2}.$$

Proof. Applying E_n to the inequality (6), we have

$$\left|E_n(f;x) - f(x)\right| \leq E_n\left(\left|f - f(x)\right|;x\right) \leq ME_n\left(\left|e_1 - x\right|^{\mu};x\right)$$

If we consider the Hölder inequality with $p = \frac{2}{\mu}$, $q = \frac{2}{2-\mu}$ and by Lemma 2 for the last inequality, we get

$$\begin{aligned}\left|E_n(f;x) - f(x)\right| \leq& M\left(E_n\left(\left(e_1 - x\right)^2;x\right)\right)^{\mu/2} = M\left(\frac{(1-x)\left(1-e^{-nx}\right)}{n\alpha_n}\right)^{\mu/2}\\ \leq& M\left(\frac{1}{n\alpha_n}\right)^{\mu/2}.\end{aligned}$$

□

Theorem 6. *If $f \in C^1[0,1]$, then*

$$\left\|E_n(f) - f\right\| \leq \frac{2}{\sqrt{n\alpha_n}}\omega\left(f'; \frac{1}{\sqrt{n\alpha_n}}\right).$$

Proof. By the mean value theorem, there exists $\xi \in (t;x)$:

$$f(t) - f(x) = (t-x) f'(\xi).$$

As the operators E_n are linear and positive and on the fact that Lemma 2 it follows immediately the equality

$$E_n(f;x) - f(x) = E_n\left(\left(e_1 - x\right)f'\left(\xi_{t,x}\right);x\right) = E_n\left(\left(e_1 - x\right)\left(f'\left(\xi_{t,x}\right) - f'(x)\right);x\right)$$

where $\xi_{t,x} \in (\min\{t,x\}, \max\{t,x\})$. Using the property of modulus of continuity, we get

$$\begin{aligned}\left|f'\left(\xi_x(t)\right) - f'(x)\right| \leq& \omega\left(f'; \left|\xi_x(t) - x\right|\right) \leq \omega\left(f';\delta\right)\left(1 + \frac{1}{\delta}\left|\xi_x(t) - x\right|\right)\\ \leq& \omega\left(f';\delta\right)\left(1 + \frac{1}{\delta}|t-x|\right).\end{aligned}$$

Consequently,

$$\begin{aligned}\left|E_n(f;x) - f(x)\right| \leq& \omega\left(f';\delta\right)E_n\left(\left|e_1 - x\right|\left(1 + \frac{1}{\delta}\left|e_1 - x\right|\right);x\right)\\ =& \omega\left(f';\delta\right)E_n\left(\left|e_1 - x\right| + \frac{1}{\delta}\left(e_1 - x\right)^2;x\right)\end{aligned}$$

$$=\omega\left(f';\delta\right)\left(E_n\left(\left|e_1-x\right|;x\right)+\frac{1}{\delta}E_n\left(\left(e_1-x\right)^2;x\right)\right)$$

Using the Cauchy-Schwartz inequality and Lemma 2 for the last inequality, we have

$$\begin{aligned}\left|E_n\left(f;x\right)-f\left(x\right)\right| \leq & \omega\left(f';\delta\right)\left(\sqrt{E_n\left(\left(e_1-x\right)^2;x\right)}+\frac{1}{\delta}E_n\left(\left(e_1-x\right)^2;x\right)\right)\\ =&\omega\left(f';\delta\right)\left(\sqrt{\frac{(1-x)(1-e^{-nx})}{n\alpha_n}}+\frac{1}{\delta}\frac{(1-x)\left(1-e^{-nx}\right)}{n\alpha_n}\right)\\ \leq&\omega\left(f';\delta\right)\left(\frac{1}{\sqrt{n\alpha_n}}+\frac{1}{\delta}\frac{1}{n\alpha_n}\right)\end{aligned}$$

Choosing $\delta_n=\left(\frac{1}{n\alpha_n}\right)^{1/2}$, we have the inequality

$$\left|E_n\left(f;x\right)-f(x)\right| \leq \frac{2}{\sqrt{n\alpha_n}}\omega\left(f';\frac{1}{\sqrt{n\alpha_n}}\right).$$

□

Theorem 7. *If $f\in C^2[0,1]$, then for all $n\in\mathbb{N}$ the following inequality holds:*

$$\left|E_n(f;x)-f(x)\right| \leq \frac{\left\|f''\right\|}{2n\alpha_n}. \tag{10}$$

Proof. Using the Taylor formula, we write

$$f\left(t\right)=f\left(x\right)+f'\left(x\right)\left(t-x\right)+R_{f,x}\left(t\right) \tag{11}$$

where

$$R_{f,x}\left(t\right)=\int_x^t(t-v)f''(v)\,dv.$$

By the mean value theorem that there exist $\xi_{t,x}\in\left(\min\left\{x,t\right\},\max\left\{x,t\right\}\right)$, which satisfies

$$R_{f,x}\left(t\right)=\frac{f''\left(\xi_{t,x}\right)}{2}(t-x)^2.$$

we can rewrite (11) as

$$f\left(t\right)=f\left(x\right)+f'\left(x\right)\left(t-x\right)+\frac{f''\left(\xi_{t,x}\right)}{2}(t-x)^2 \tag{12}$$

Applying E_n to the formula (12), by Lemma 2, we have

$$\left|E_n\left(f;x\right)-f\left(x\right)\right| \leq E_n\left(\left|\frac{f''\left(\xi_{t,x}\right)}{2}\right|(e_1-x)^2;x\right) \leq \frac{\left\|f''\right\|}{2}E_n\left((e_1-x)^2;x\right)$$

$$=\frac{\left\|f''\right\|}{2}\frac{(1-x)\left(1-e^{-nx}\right)}{n\alpha_n}\leq\frac{\left\|f''\right\|}{2n\alpha_n}.$$

□

Theorem 8. *If $f\in C[0,1]$, then for all $n\in\mathbb{N}$ the following inequality holds:*

$$\|E_n(f)-f\|\leq 3\omega_2\left(f;\frac{1}{\sqrt{n\alpha_n}}\right).$$

Proof. Let $x\in[0,1]$. For $0<h\leq\frac{1}{2}\min\{x,1-x\}$ we define

$$g_h(x)=\frac{1}{h^2}\int_{-h/2}^{h/2}\int_{-h/2}^{h/2}\left\{2f\left(x+t_1+t_2\right)-f\left(x+2t_1+2t_2\right)\right\}dt_1dt_2.$$

Consequently

$$\begin{aligned}\left|g''(x)\right|&=\left|\left\{f(x+2h)-2f(x+h)+f(x)\right\}+\left\{f(x-2h)-2f(x-h)+f(x)\right\}\right|\\&\leq\frac{2}{\delta^2}\omega_2\left(f;\delta\right)\end{aligned}$$

also $\left|\int_{-a}^{a}h(t)\,dt\right|\leq 2a\sup\limits_{u\in[-a,a]}|h(u)|$. Therefore

$$\begin{aligned}\left|f(x)-g_h(x)\right|&=\left|\frac{1}{h^2}\int_{-h/2}^{h/2}\int_{-h/2}^{h/2}\left\{f\left(x+2t_1+2t_2\right)-2f\left(x+t_1+t_2\right)+f(x)\right\}dt_1dt_2\right|\\&=\frac{1}{h^2}\int_{-h/2}^{h/2}\int_{-h/2}^{h/2}\left|f\left(x+2t_1+2t_2\right)-2f\left(x+t_1+t_2\right)+f(x)\right|dt_1dt_2\\&\leq\omega_2\left(f;\delta\right).\end{aligned}$$

Using these inequalities and (10) we have

$$\begin{aligned}\left\|E_n(f)-f\right\|&\leq\left\|E_n\left(f-g_h\right)\right\|+\left\|E_n\left(g_h\right)-g_h\right\|+\left\|f-g_h\right\|\\&\leq\left\|E_n\right\|\left\|f-g_h\right\|+\left\|E_n\left(g_h\right)-g_h\right\|+\left\|f-g_h\right\|\\&=2\left\|f-g_h\right\|+\left\|E_n\left(g_h\right)-g_h\right\|\leq 2\omega_2\left(f;\delta\right)+\frac{\|g_h''\|}{2n\alpha_n}\\&\leq 2\omega_2\left(f;\delta\right)+\frac{1}{2n\alpha_n}\frac{2}{\delta^2}\omega_2\left(f;\delta\right)=\omega_2\left(f;\delta\right)\left\{2+\frac{1}{n\alpha_n\delta^2}\right\}\end{aligned}$$

Choosing $\delta_n=\left(\frac{1}{n\alpha_n}\right)^{1/2}$ we get the desired estimate. □

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