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Some new Hermite-Hadamard type conformable fractional integral inequalities for twice differentiable $\mathrm{MT}_{(r;g,m,\varphi)}$-preinvex functions

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Abstract. In the present paper, the notion of $\mathrm{MT}_{(r;g,m,\varphi)}$-preinvex function is introduced and some new integral inequalities for the left-hand side of Gauss-Jacobi type quadrature formula involving $\mathrm{MT}_{(r;g,m,\varphi)}$-preinvex functions are given. Moreover, some generalizations of Hermite-Hadamard type inequalities for $\mathrm{MT}_{(r;g,m,\varphi)}$-preinvex functions that are twice differentiable via conformable fractional integrals are established. At the end, some applications to special means are given.

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1. Introduction and Preliminaries

The following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and I° to denote the interior of I. For any subset $K \subseteq \mathbb{R}^n, K^\circ$ is used to denote the interior of K. $\mathbb{R}^n$ is used to denote a n-dimensional vector space. The set of integrable functions on the interval $[a, b]$ is denoted by $L_1[a, b]$.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{1}$$

Definition 1.

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In (see [32],[35]), Tunç and Yidirim defined the following so-called MT-convex function:

A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to belong to the class of $MT(I)$, if it is nonnegative and for all $x, y \in I$ and $t \in (0,1)$ satisfies the following inequality:

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{2}$$

In recent years, various generalizations, extensions and variants of such inequalities have been obtained.
For other recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions, (see [12],[13],[18]-[27],[33],[34]).

Fractional calculus (see [31]), was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

Definition 2. *Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-u} u^{\alpha-1} du$. *Here* $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.
In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

The following definitions will be used in the sequel.

Definition 3. *The Euler beta function is defined for $a, b > 0$ as*

$$\beta(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Definition 4. *The incomplete beta function is defined for $a, b > 0$ as*

$$\beta_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1} dt, \quad 0 < x \leq 1.$$

For $x = 1$, the incomplete beta function coincides with the complete beta function.

Definition 5. *Let $g : [0,1] \longrightarrow [0,1]$ be a differentiable function. The new generalized incomplete beta function is defined for $a, b > 0$ as*

$$B_{g(x)}(a,b) = \int_{g(0)}^{g(x)} t^{a-1}(1-t)^{b-1} dt.$$

For $g(x) = x$, the new generalized incomplete beta function coincides with the incomplete beta function.

In the following, we give some definitions and properties of conformable fractional integrals which help to obtain main identity and results. Recently, some authors, started to study on conformable fractional integrals (see [1],[2]). In (see [5]), Khalil et al. defined the fractional integral of order $0 < \alpha \leq 1$ only. In (see [6]), Abdeljawad gave the definition of left and right conformable fractional integrals of any order $\alpha > 0$.

Definition 6. *Let $\alpha \in (n, n+1]$ and set $\beta = \alpha - n$, then the left conformable fractional integral starting at a is defined by*

$$\left(I_\alpha^a f\right)(t) = \frac{1}{n!}\int_a^t (t-x)^n (x-a)^{\beta-1} f(x)dx.$$

Analogously, the right conformable fractional integral is defined by

$$\left({}^b I_\alpha f\right)(t) = \frac{1}{n!}\int_t^b (x-t)^n (b-x)^{\beta-1} f(x)dx.$$

Notice that if $\alpha = n+1$, then $\beta = \alpha - n = n+1-n = 1$, where $n = 0, 1, 2, \ldots,$ and hence $\left(I_\alpha^a f\right)(t) = \left(J_{n+1}^a f\right)(t)$.

In (see [7]), Set et al. established a generalization of Hermite-Hadamard type inequality for s-convex functions and gave some remarks to show the relationships with the classical and Riemann-Liouville fractional integrals inequality by using the given properties of conformable fractional integrals.

Theorem 2. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function with $0 \leq a < b$, $s \in (0, 1]$, and $f \in L_1[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for conformable fractional integrals hold*

$$\frac{\Gamma(\alpha-n)}{\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{(b-a)^\alpha 2^s}\left[\left(I_\alpha^a f\right)(b) + \left({}^b I_\alpha f\right)(a)\right]$$

$$\leq \left[\frac{\beta(n+s+1, \alpha-n) + \beta(n+1, \alpha-n+s)}{n!}\right]\frac{f(a)+f(b)}{2^s},$$

with $\alpha \in (n, n+1]$, $n \in \mathbb{N}$, $n = 0, 1, 2, \ldots,$ where Γ is Euler gamma function.

Also Set et al. established some results for some kind of inequalities via conformable fractional integrals (see [8]-[11]).
Due to the wide application of fractional integrals, some authors extended to study fractional Hermite-Hadamard type inequalities for functions of different classes (see [15]-[31]).

Now, let us evoke some definitions.

Definition 7. *(see [4]) A nonnegative function $f : I \subseteq \mathbb{R} \longrightarrow [0, +\infty)$ is said to be P-function or P-convex, if*

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I,\ t \in [0,1].$$

Definition 8. *(see [14]) A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $x + t\eta(y,x) \in K$ for every $x, y \in K$ and $t \in [0,1]$.*

Notice that every convex set is invex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not necessarily true. For more details (see [14],[16]).

Definition 9. *(see [17]) The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect η, if for every $x, y \in K$ and $t \in [0,1]$, we have that*

$$f\left(x + t\eta(y,x)\right) \leq (1-t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not true.

The Gauss-Jacobi type quadrature formula has the following

$$\int_a^b (x-a)^p (b-x)^q f(x)dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R_m^{\star}|f|, \tag{3}$$

for certain $B_{m,k}, \gamma_k$ and rest $R_m^{\star}|f|$ (see [28]).

Recently, Liu (see [29]) obtained several integral inequalities for the left-hand side of (3) under the Definition 7 of P-function.
Also in (see [30]), Özdemir et al. established several integral inequalities concerning the left-hand side of (3) via some kinds of convexity.

Motivated by these results, in Section 2, the notion of $\mathrm{MT}_{(r;g,m,\varphi)}$-preinvex function is introduced and some new integral inequalities for the left-hand side of (3) involving $\mathrm{MT}_{(r;g,m,\varphi)}$-preinvex functions are given. In Section 3, some generalizations of Hermite-Hadamard type inequalities for $\mathrm{MT}_{(r;g,m,\varphi)}$-preinvex functions that are twice differentiable via conformable fractional integrals are given. In Section 4, some applications to special means are given. In Section 5, some conclusions and future research are given. These general inequalities give us some new estimates for Hermite-Hadamard type conformable fractional integral and fractional integral inequalities.

2. New integral inequalities for $\mathrm{MT}_{(r;g,m,\varphi)}$-preinvex functions

Definition 10. *(see [3]) A set $K \subseteq \mathbb{R}^n$ is said to be m-invex with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + t\eta(y,x,m) \in K$ holds for each $x, y \in K$ and any $t \in [0,1]$.*

Remark 1. *In Definition 10, under certain conditions, the mapping $\eta(y,x,m)$ could reduce to $\eta(y,x)$. For example when $m=1$, then the m-invex set degenerates an invex set on K.*

We next give new definition, to be referred as $MT_{(r;g,m,\varphi)}$-preinvex function.

Definition 11. *Let $K \subseteq \mathbb{R}$ be an open m-invex set with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$, $g : [0,1] \longrightarrow (0,1)$ be a differentiable function and $\varphi : I \longrightarrow K$ is a continuous function. The function $f : K \longrightarrow (0,\infty)$ is said to be $MT_{(r;g,m,\varphi)}$-preinvex function with respect to η, if*

$$f(m\varphi(y) + g(t)\eta(\varphi(x),\varphi(y),m)) \leq M_r\left(f(\varphi(x)), f(\varphi(y)), m; g(t)\right) \tag{4}$$

holds for any fixed $m \in (0,1]$ and for all $x,y \in I$, $t \in [0,1]$, where

$$M_r(f(\varphi(x)), f(\varphi(y)), m; g(t))$$

$$= \begin{cases} \left[\frac{m\sqrt{g(t)}}{2\sqrt{1-g(t)}} f^r(\varphi(x)) + \frac{m\sqrt{1-g(t)}}{2\sqrt{g(t)}} f^r(\varphi(y))\right]^{\frac{1}{r}}, & \text{if } r \neq 0; \\ \\ f(\varphi(x))^{\frac{m\sqrt{g(t)}}{2\sqrt{1-g(t)}}} f(\varphi(y))^{\frac{m\sqrt{1-g(t)}}{2\sqrt{g(t)}}}, & \text{if } r = 0, \end{cases}$$

is the weighted power mean of order r for positive numbers $f(\varphi(x))$ and $f(\varphi(y))$.

Remark 2. *In Definition 11, it is worthwhile to note that the class $MT_{(r;g,m,\varphi)}(I)$ is a generalization of the class $MT(I)$ given in Definition 1 for $r = m = 1$ with respect to $\eta(\varphi(x),\varphi(y),m) = \varphi(x) - m\varphi(y)$, $\varphi(x) = x$, $\forall x,y \in I$, $g(t) = t$, $\forall t \in (0,1)$.*

Let give below a nontrivial example for motivation of this new interesting class of $MT_{(r;g,m,\varphi)}$-preinvex functions.

Example 1. $f_1, f_2 : (1,\infty) \longrightarrow (0,\infty)$, $f_1(x) = x^p$, $f_2(x) = (1+x)^p$, $p \in \left(0, \frac{1}{1000}\right)$; $h : [1, 3/2] \longrightarrow (0,\infty)$, $h(x) = (1+x^2)^k$, $k \in \left(0, \frac{1}{100}\right)$, *are simple examples of the new class of $MT_{(1;t,m,x)}$-preinvex functions with respect to $\eta(\varphi(x),\varphi(y),m) = \varphi(x) - m\varphi(y)$, $\varphi(x) = x$, $g(t) = t$, $r = 1$, for any fixed $m \in (0,1]$, but they are not convex.*

In this section, in order to prove our main results regarding some new integral inequalities involving $MT_{(r;g,m,\varphi)}$-preinvex functions, we need the following new interesting Lemma:

Lemma 1. *Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0,1] \longrightarrow [0,1]$ is a differentiable function. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b),\varphi(a),m)] \longrightarrow \mathbb{R}$ is a continuous function on K° with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$, for $m\varphi(a) < m\varphi(a) + \eta(\varphi(b),\varphi(a),m)$. Then for any fixed $m \in (0,1]$ and $p,q > 0$, we have*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b),\varphi(a),m) - x)^q f(x)dx$$

$$= \eta(\varphi(b), \varphi(a), m)^{p+q+1}$$

$$\times \int_0^1 g^p(t)(1-g(t))^q f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m))d[g(t)].$$

Proof. It is easy to observe that

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$

$$= \eta(\varphi(b), \varphi(a), m) \int_0^1 (m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m) - m\varphi(a))^p$$

$$\times (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - m\varphi(a) - g(t)\eta(\varphi(b), \varphi(a), m))^q$$

$$\times f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m))d[g(t)]$$

$$= \eta(\varphi(b), \varphi(a), m)^{p+q+1}$$

$$\times \int_0^1 g^p(t)(1-g(t))^q f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m))d[g(t)].$$

Theorem 3. *Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0,1] \longrightarrow (0,1)$ is a differentiable function. Assume that $f : K = [m\varphi(a), m\varphi(a)+\eta(\varphi(b), \varphi(a), m)] \longrightarrow (0,\infty)$ is a continuous function on K° with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$, for $m\varphi(a) < m\varphi(a) + \eta(\varphi(b), \varphi(a), m)$. Let $k > 1$ and $0 < r \leq 1$. If $f^{\frac{k}{k-1}}$ is a nonnegative $MT_{(r;g,m,\varphi)}$-preinvex function on an open m-invex set K for any fixed $m \in (0,1]$, then for any fixed $p, q > 0$, we have*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$

$$\leq \left(\frac{m}{2}\right)^{\frac{k-1}{rk}} |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} B^{\frac{1}{k}}(g(t); k, p, q)$$

$$\times \left[B^r_{g(1)}\left(1 - \frac{1}{2r}, 1 + \frac{1}{2r}\right) f^{\frac{rk}{k-1}}(\varphi(a)) + A^r(g(t); r) f^{\frac{rk}{k-1}}(\varphi(b)) \right]^{\frac{k-1}{rk}},$$

where

$$B(g(t); k, p, q) = \int_0^1 g^{kp}(t)(1-g(t))^{kq} d[g(t)];$$

$$A(g(t); r) = \int_{1-g(1)}^{1-g(0)} \left(\sqrt{\frac{1-t}{t}}\right)^{\frac{1}{r}} dt.$$

Proof. Let $k > 1$ and $0 < r \leq 1$. Since $f^{\frac{k}{k-1}}$ is a nonnegative $MT_{(r;g,m,\varphi)}$-preinvex function on K, combining with Lemma 1, Hölder inequality and Minkowski inequality for all $t \in [0,1]$ and for any fixed $m \in (0,1]$, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b),\varphi(a),m) - x)^q f(x)dx$$

$$\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1} \left[\int_0^1 g^{kp}(t)(1-g(t))^{kq} d[g(t)] \right]^{\frac{1}{k}}$$

$$\times \left[\int_0^1 f^{\frac{k}{k-1}}(m\varphi(a) + g(t)\eta(\varphi(b),\varphi(a),m)) d[g(t)] \right]^{\frac{k-1}{k}}$$

$$\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1} B^{\frac{1}{k}}(g(t);k,p,q)$$

$$\times \left[\int_0^1 \left(\frac{m\sqrt{g(t)}}{2\sqrt{1-g(t)}} f^r(\varphi(b))^{\frac{k}{k-1}} + \frac{m\sqrt{1-g(t)}}{2\sqrt{g(t)}} f^r(\varphi(a))^{\frac{k}{k-1}} \right)^{\frac{1}{r}} d[g(t)] \right]^{\frac{k-1}{k}}$$

$$\leq \left(\frac{m}{2}\right)^{\frac{k-1}{rk}} |\eta(\varphi(b),\varphi(a),m)|^{p+q+1} B^{\frac{1}{k}}(g(t);k,p,q)$$

$$\times \left\{ \left(\int_0^1 \left(\frac{\sqrt{g(t)}}{\sqrt{1-g(t)}} \right)^{\frac{1}{r}} f^{\frac{k}{k-1}}(\varphi(b)) d[g(t)] \right)^r \right.$$

$$\left. + \left(\int_0^1 \left(\frac{\sqrt{1-g(t)}}{\sqrt{g(t)}} \right)^{\frac{1}{r}} f^{\frac{k}{k-1}}(\varphi(a)) d[g(t)] \right)^r \right\}^{\frac{k-1}{rk}}$$

$$= \left(\frac{m}{2}\right)^{\frac{k-1}{rk}} |\eta(\varphi(b),\varphi(a),m)|^{p+q+1} B^{\frac{1}{k}}(g(t);k,p,q)$$

$$\times \left[B_{g(1)}^r \left(1 - \frac{1}{2r}, 1 + \frac{1}{2r} \right) f^{\frac{rk}{k-1}}(\varphi(a)) + A^r(g(t);r) f^{\frac{rk}{k-1}}(\varphi(b)) \right]^{\frac{k-1}{rk}}.$$

Corollary 1. *Under the same conditions as in Theorem 3 for $r = 1$ and $g(t) = t$, we get*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b),\varphi(a),m) - x)^q f(x)dx$$

$$\leq \left(\frac{m\pi}{4}\right)^{\frac{k-1}{k}} |\eta(\varphi(b),\varphi(a),m)|^{p+q+1} \beta^{\frac{1}{k}}(kp+1,kq+1) \left[f^{\frac{k}{k-1}}(\varphi(a)) + f^{\frac{k}{k-1}}(\varphi(b)) \right]^{\frac{k-1}{k}}.$$

Theorem 4. *Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0,1] \longrightarrow (0,1)$ is a differentiable function. Assume that $f : K = [m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)] \longrightarrow (0,\infty)$ is a continuous function on K° with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$, for $m\varphi(a) < m\varphi(a) + \eta(\varphi(b),\varphi(a),m)$. Let $l \geq 1$ and $0 < r \leq 1$. If f^l is a nonnegative $MT_{(r;g,m,\varphi)}$-preinvex function on an open m-invex set K for any fixed $m \in (0,1]$, then for any fixed $p,q > 0$, we have*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b),\varphi(a),m) - x)^q f(x)dx$$

$$\leq \left(\frac{m}{2}\right)^{\frac{1}{rl}} |\eta(\varphi(b),\varphi(a),m)|^{p+q+1} B^{\frac{l-1}{l}}(g(t);1,p,q)$$

$$\times \left[B^r\left(g(t);\frac{1}{2r}, 2pr-1, 2qr+1\right) f^{rl}(\varphi(a)) \right.$$

$$\left. + B^r\left(g(t);\frac{1}{2r}, 2pr+1, 2qr-1\right) f^{rl}(\varphi(b)) \right]^{\frac{1}{rl}}.$$

Proof. Let $l \geq 1$ and $0 < r \leq 1$. Since f^l is a nonnegative $MT_{(r;g,m,\varphi)}$-preinvex function on K, combining with Lemma 1, the well-known power mean inequality and Minkowski inequality for all $t \in [0,1]$ and for any fixed $m \in (0,1]$, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b),\varphi(a),m) - x)^q f(x)dx$$

$$= \eta(\varphi(b),\varphi(a),m)^{p+q+1} \int_0^1 \left[g^p(t)(1-g(t))^q\right]^{\frac{l-1}{l}} \left[g^p(t)(1-g(t))^q\right]^{\frac{1}{l}}$$

$$\times f(m\varphi(a) + g(t)\eta(\varphi(b),\varphi(a),m))d[g(t)]$$

$$\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1} \left[\int_0^1 g^p(t)(1-g(t))^q d[g(t)]\right]^{\frac{l-1}{l}}$$

$$\times \left[\int_0^1 g^p(t)(1-g(t))^q f^l(m\varphi(a) + g(t)\eta(\varphi(b),\varphi(a),m))d[g(t)]\right]^{\frac{1}{l}}$$

$$\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1} B^{\frac{l-1}{l}}(g(t);1,p,q)$$

$$\times \left[\int_0^1 g^p(t)(1-g(t))^q \left(\frac{m\sqrt{g(t)}}{2\sqrt{1-g(t)}} f^r(\varphi(b))^l + \frac{m\sqrt{1-g(t)}}{2\sqrt{g(t)}} f^r(\varphi(a))^l\right)^{\frac{1}{r}} d[g(t)]\right]^{\frac{1}{l}}$$

$$\leq \left(\frac{m}{2}\right)^{\frac{1}{rl}} |\eta(\varphi(b),\varphi(a),m)|^{p+q+1} B^{\frac{l-1}{l}}(g(t);1,p,q)$$

$$\times\Bigg\{\left(\int_0^1 g^{p+\frac{1}{2r}}(t)(1-g(t))^{q-\frac{1}{2r}}f^l(\varphi(b))d[g(t)]\right)^r$$

$$+\left(\int_0^1 g^{p-\frac{1}{2r}}(t)(1-g(t))^{q+\frac{1}{2r}}f^l(\varphi(a))d[g(t)]\right)^r\Bigg\}^{\frac{1}{rl}}$$

$$=\left(\frac{m}{2}\right)^{\frac{1}{rl}}|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}B^{\frac{l-1}{l}}(g(t);1,p,q)$$

$$\times\Bigg[B^r\left(g(t);\frac{1}{2r},2pr-1,2qr+1\right)f^{rl}(\varphi(a))$$

$$+B^r\left(g(t);\frac{1}{2r},2pr+1,2qr-1\right)f^{rl}(\varphi(b))\Bigg]^{\frac{1}{rl}}.$$

Corollary 2. *Under the same conditions as in Theorem 4 for $r=1$ and $g(t)=t$, we get*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq\left(\frac{m}{2}\right)^{\frac{1}{l}}|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\beta^{\frac{l-1}{l}}(p+1,q+1)$$

$$\times\left[\beta\left(p+\frac{1}{2},q+\frac{3}{2}\right)f^l(\varphi(a))+\beta\left(p+\frac{3}{2},q+\frac{1}{2}\right)f^l(\varphi(b))\right]^{\frac{1}{l}}.$$

3. Some new Hermite-Hadamard type conformable fractional integral inequalities for twice differentiable $\mathrm{MT}_{(r;g,m,\varphi)}$-preinvex functions

In this section, in order to prove our main results regarding some generalizations of Hermite-Hadamard type inequalities for twice differentiable $\mathrm{MT}_{(r;g,m,\varphi)}$-preinvex functions via conformable fractional integrals, we need the following new interesting integral identity:

Lemma 2. *Let $\varphi: I\longrightarrow K$ be a continuous function and $g:[0,1]\longrightarrow[0,1]$ is a differentiable function. Suppose $K\subseteq\mathbb{R}$ be an open m-invex subset with respect to $\eta: K\times K\times(0,1]\longrightarrow\mathbb{R}$ for any fixed $m\in(0,1]$ and let $m\varphi(a)<m\varphi(a)+\eta(\varphi(b),\varphi(a),m)$. Assume that $f:K=[m\varphi(a),m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]\longrightarrow\mathbb{R}$ be a twice differentiable function on K° and $f''\in L_1[m\varphi(a),m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]$. Then for $\alpha>0$, we have*

$$\frac{\eta^{\alpha+2}(\varphi(x),\varphi(a),m)}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\Bigg\{\frac{\beta(n+2,\alpha-n)}{\eta(\varphi(x),\varphi(a),m)}$$

$$\times\Big[f'(m\varphi(a)+g(1)\eta(\varphi(x),\varphi(a),m))-f'(m\varphi(a)+g(0)\eta(\varphi(x),\varphi(a),m)\Big]$$

$$-\frac{B_{g(1)}(n+2,\alpha-n)f'(m\varphi(a)+g(1)\eta(\varphi(x),\varphi(a),m))}{\eta(\varphi(x),\varphi(a),m)}$$

$$+\frac{1}{\eta^2(\varphi(x),\varphi(a),m)}$$

$$\times\Big[g^{n+1}(1)(1-g(1))^{\alpha-n-1}f(m\varphi(a)+g(1)\eta(\varphi(x),\varphi(a),m))$$

$$-g^{n+1}(0)(1-g(0))^{\alpha-n-1}f(m\varphi(a)+g(0)\eta(\varphi(x),\varphi(a),m))\Big]$$

$$+\frac{1}{\eta^{\alpha+2}(\varphi(x),\varphi(a),m)}$$

$$\times\Bigg[(n+1)\int_{m\varphi(a)+g(0)\eta(\varphi(x),\varphi(a),m)}^{m\varphi(a)+g(1)\eta(\varphi(x),\varphi(a),m)}(t-m\varphi(a))^n$$

$$\times(m\varphi(a)+\eta(\varphi(x),\varphi(a),m)-t)^{\alpha-n-1}f(t)dt$$

$$-(\alpha-n-1)\int_{m\varphi(a)+g(0)\eta(\varphi(x),\varphi(a),m)}^{m\varphi(a)+g(1)\eta(\varphi(x),\varphi(a),m)}(t-m\varphi(a))^{n+1}$$

$$\times(m\varphi(a)+\eta(\varphi(x),\varphi(a),m)-t)^{\alpha-n-2}f(t)dt\Bigg]\Bigg\}$$

$$+\frac{\eta^{\alpha+2}(\varphi(x),\varphi(b),m)}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\Bigg\{\frac{\beta(n+2,\alpha-n)}{\eta(\varphi(x),\varphi(b),m)}$$

$$\times\Big[f'(m\varphi(b)+g(1)\eta(\varphi(x),\varphi(b),m))-f'(m\varphi(b)+g(0)\eta(\varphi(x),\varphi(b),m)\Big]$$

$$-\frac{B_{g(1)}(n+2,\alpha-n)f'(m\varphi(b)+g(1)\eta(\varphi(x),\varphi(b),m))}{\eta(\varphi(x),\varphi(b),m)}$$

$$+\frac{1}{\eta^2(\varphi(x),\varphi(b),m)}$$

$$\times\Big[g^{n+1}(1)(1-g(1))^{\alpha-n-1}f(m\varphi(b)+g(1)\eta(\varphi(x),\varphi(b),m))$$

$$-g^{n+1}(0)(1-g(0))^{\alpha-n-1}f(m\varphi(b)+g(0)\eta(\varphi(x),\varphi(b),m))\Big]$$

$$+\frac{1}{\eta^{\alpha+2}(\varphi(x),\varphi(b),m)}$$

$$\times\Bigg[(n+1)\int_{m\varphi(b)+g(0)\eta(\varphi(x),\varphi(b),m)}^{m\varphi(b)+g(1)\eta(\varphi(x),\varphi(b),m)}(t-m\varphi(b))^{n}$$

$$\times(m\varphi(b)+\eta(\varphi(x),\varphi(b),m)-t)^{\alpha-n-1}f(t)dt$$

$$-(\alpha-n-1)\int_{m\varphi(b)+g(0)\eta(\varphi(x),\varphi(b),m)}^{m\varphi(b)+g(1)\eta(\varphi(x),\varphi(b),m)}(t-m\varphi(b))^{n+1}$$

$$\times(m\varphi(b)+\eta(\varphi(x),\varphi(b),m)-t)^{\alpha-n-2}f(t)dt\Bigg]\Bigg\}$$

$$=\frac{\eta^{\alpha+2}(\varphi(x),\varphi(a),m)}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\int_0^1(\beta(n+2,\alpha-n)-B_{g(t)}(n+2,\alpha-n))f''(m\varphi(a)+g(t)\eta(\varphi(x),\varphi(a),m))d[g(t)]$$

$$+\frac{\eta^{\alpha+2}(\varphi(x),\varphi(b),m)}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\int_0^1(\beta(n+2,\alpha-n)-B_{g(t)}(n+2,\alpha-n))f''(m\varphi(b)+g(t)\eta(\varphi(x),\varphi(b),m))d[g(t)]. \quad (5)$$

Proof. A simple proof of the equality can be done by performing two integration by parts in the integrals from the right side, changing the variable and using Definition 5. The details are left to the interested reader.

Throughout this paper we denote

$$I_{f,g,\eta,\varphi}(x;\alpha,n,m,a,b)=\frac{\eta^{\alpha+2}(\varphi(x),\varphi(a),m)}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\int_0^1(\beta(n+2,\alpha-n)-B_{g(t)}(n+2,\alpha-n))f''(m\varphi(a)+g(t)\eta(\varphi(x),\varphi(a),m))d[g(t)]$$

$$+\frac{\eta^{\alpha+2}(\varphi(x),\varphi(b),m)}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\int_0^1(\beta(n+2,\alpha-n)-B_{g(t)}(n+2,\alpha-n))f''(m\varphi(b)+g(t)\eta(\varphi(x),\varphi(b),m))d[g(t)]. \quad (6)$$

Using relation (6), the following results can be obtained for the corresponding version for power of the second derivative.

Theorem 5. *Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0,1] \longrightarrow (0,1)$ is a differentiable function. Suppose $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $m \in (0,1]$ and let $m\varphi(a) < m\varphi(a)+\eta(\varphi(b),\varphi(a),m)$. Assume that $f : K = [m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)] \longrightarrow (0,\infty)$ be a twice differentiable function* *on K°. If f''^q is a nonnegative $MT_{(r;g,m,\varphi)}$-preinvex function on K, $q>1$, $p^{-1}+q^{-1}=1$, then for $\alpha>0$ and $0<r\le 1$, we have*

$$|I_{f,g,\eta,\varphi}(x;\alpha,n,m,a,b)| \le \left(\frac{m}{2}\right)^{\frac{1}{rq}} \frac{\delta^{\frac{1}{p}}(g(t);p,\alpha,n)}{|\eta(\varphi(b),\varphi(a),m)|}$$

$$\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\left[B^r_{g(1)}\left(1-\frac{1}{2r},1+\frac{1}{2r}\right)f''(\varphi(a))^{rq}\right.$$

$$\left.+B^r_{g(1)}\left(1+\frac{1}{2r},1-\frac{1}{2r}\right)f''(\varphi(x))^{rq}\right]^{\frac{1}{rq}}$$

$$+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\left[B^r_{g(1)}\left(1-\frac{1}{2r},1+\frac{1}{2r}\right)f''(\varphi(b))^{rq}\right.$$

$$\left.+B^r_{g(1)}\left(1+\frac{1}{2r},1-\frac{1}{2r}\right)f''(\varphi(x))^{rq}\right]^{\frac{1}{rq}}\Bigg\}, \tag{7}$$

where $\delta(g(t);p,\alpha,n)=\displaystyle\int_0^1\left[\beta(n+2,\alpha-n)-B_{g(t)}(n+2,\alpha-n)\right]^p d[g(t)]$.

Proof. Suppose that $q>1$ and $0<r\le 1$. Using relation (6), Hölder inequality, the fact that f''^q is a nonnegative $\mathrm{MT}_{(r;g,m,\varphi)}$-preinvex function on an open m-invex set K°, combining with Minkowski inequality for all $t\in[0,1]$ and for any fixed $m\in(0,1]$ and taking the modulus, we have

$$|I_{f,g,\eta,\varphi}(x;\alpha,n,m,a,b)| \le \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}$$

$$\times\int_0^1\big(\beta(n+2,\alpha-n)-B_{g(t)}(n+2,\alpha-n)\big)f''(m\varphi(a)+g(t)\eta(\varphi(x),\varphi(a),m))d[g(t)]$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}$$

$$\times\int_0^1\big(\beta(n+2,\alpha-n)-B_{g(t)}(n+2,\alpha-n)\big)f''(m\varphi(b)+g(t)\eta(\varphi(x),\varphi(b),m))d[g(t)]$$

$$\le\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}\left(\int_0^1\left[\beta(n+2,\alpha-n)-B_{g(t)}(n+2,\alpha-n)\right]^p d[g(t)]\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 f''(m\varphi(a)+g(t)\eta(\varphi(x),\varphi(a),m))^q d[g(t)]\right)^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}\left(\int_0^1\left[\beta(n+2,\alpha-n)-B_{g(t)}(n+2,\alpha-n)\right]^p d[g(t)]\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 f''(m\varphi(b)+g(t)\eta(\varphi(x),\varphi(b),m))^q d[g(t)]\right)^{\frac{1}{q}}$$

$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}\left(\int_0^1 \left[\beta(n+2,\alpha-n)-B_{g(t)}(n+2,\alpha-n)\right]^p d[g(t)]\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 \left[\frac{m\sqrt{g(t)}}{2\sqrt{1-g(t)}}f''(\varphi(x))^{rq}+\frac{m\sqrt{1-g(t)}}{2\sqrt{g(t)}}f''(\varphi(a))^{rq}\right]^{\frac{1}{r}} d[g(t)]\right)^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}\left(\int_0^1 \left[\beta(n+2,\alpha-n)-B_{g(t)}(n+2,\alpha-n)\right]^p d[g(t)]\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 \left[\frac{m\sqrt{g(t)}}{2\sqrt{1-g(t)}}f''(\varphi(x))^{rq}+\frac{m\sqrt{1-g(t)}}{2\sqrt{g(t)}}f''(\varphi(b))^{rq}\right]^{\frac{1}{r}} d[g(t)]\right)^{\frac{1}{q}}$$

$$\leq \left(\frac{m}{2}\right)^{\frac{1}{rq}}\delta^{\frac{1}{p}}(g(t);p,\alpha,n)\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}$$

$$\times\left[\left(\int_0^1\left(\sqrt{\frac{1-g(t)}{g(t)}}\right)^{\frac{1}{r}} f''(\varphi(a))^q d[g(t)]\right)^r\right.$$

$$\left.+\left(\int_0^1\left(\sqrt{\frac{g(t)}{1-g(t)}}\right)^{\frac{1}{r}} f''(\varphi(x))^q d[g(t)]\right)^r\right]^{\frac{1}{rq}}$$

$$+\left(\frac{m}{2}\right)^{\frac{1}{rq}}\delta^{\frac{1}{p}}(g(t);p,\alpha,n)\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}$$

$$\times\left[\left(\int_0^1\left(\sqrt{\frac{1-g(t)}{g(t)}}\right)^{\frac{1}{r}} f''(\varphi(b))^q d[g(t)]\right)^r\right.$$

$$\left.+\left(\int_0^1\left(\sqrt{\frac{g(t)}{1-g(t)}}\right)^{\frac{1}{r}} f''(\varphi(x))^q d[g(t)]\right)^r\right]^{\frac{1}{rq}}$$

$$=\left(\frac{m}{2}\right)^{\frac{1}{rq}}\frac{\delta^{\frac{1}{p}}(g(t);p,\alpha,n)}{|\eta(\varphi(b),\varphi(a),m)|}$$

$$\times\left\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\left[B^r_{g(1)}\left(1-\frac{1}{2r},1+\frac{1}{2r}\right)f''(\varphi(a))^{rq}\right.\right.$$

$$\left.+B^r_{g(1)}\left(1+\frac{1}{2r},1-\frac{1}{2r}\right)f''(\varphi(x))^{rq}\right]^{\frac{1}{rq}}$$

$$+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\left[B^r_{g(1)}\left(1-\frac{1}{2r},1+\frac{1}{2r}\right)f''(\varphi(b))^{rq}\right.$$

$$\left.+B^r_{g(1)}\left(1+\frac{1}{2r},1-\frac{1}{2r}\right)f''(\varphi(x))^{rq}\right]^{\frac{1}{rq}}\Bigg\}.$$

Corollary 3. *Under the same conditions as in Theorem 5, if we choose $\alpha \in (n, n+1]$ where $n = 0, 1, 2, \ldots$ and $g(t) = t$, we get the following inequality for conformable fractional integrals:*

$$\left|\frac{-\eta^{\alpha+1}(\varphi(x),\varphi(a),m)f'(m\varphi(a))-\eta^{\alpha+1}(\varphi(x),\varphi(b),m)f'(m\varphi(b))}{\eta(\varphi(b),\varphi(a),m)}\right.$$

$$-\frac{(n+2-\alpha)(n+1)!}{\eta(\varphi(b),\varphi(a),m)}$$

$$\left.\times\left[\left({}^{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))}I_\alpha f\right)(m\varphi(a))+\left({}^{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))}I_\alpha f\right)(m\varphi(b))\right]\right|$$

$$\leq\left(\frac{m}{2}\right)^{\frac{1}{rq}}\left(\frac{\pi}{2r\sin\left(\frac{\pi}{2r}\right)}\right)^{\frac{1}{q}}\frac{\delta^{\frac{1}{p}}(p,\alpha,n)}{|\eta(\varphi(b),\varphi(a),m)|}$$

$$\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\left[f''(\varphi(a))^{rq}+f''(\varphi(x))^{rq}\right]^{\frac{1}{rq}}$$

$$+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\left[f''(\varphi(b))^{rq}+f''(\varphi(x))^{rq}\right]^{\frac{1}{rq}}\Bigg\},$$

where $\delta(p,\alpha,n)=\int_0^1\left[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\right]^p dt.$

Corollary 4. *Under the same conditions as in Corollary 3, if we choose $\alpha = n+1$ where $n = 0, 1, 2, \ldots$, $r = 1$ and $f'' \leq K$, we get the following inequality for fractional integrals:*

$$\left|\frac{-\eta^{\alpha+1}(\varphi(x),\varphi(a),m)f'(m\varphi(a))-\eta^{\alpha+1}(\varphi(x),\varphi(b),m)f'(m\varphi(b))}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}\right.$$

$$+\frac{\eta^{\alpha}(\varphi(x),\varphi(a),m)f(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))+\eta^{\alpha}(\varphi(x),\varphi(b),m)f(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))}{\eta(\varphi(b),\varphi(a),m)}$$

$$-\frac{\Gamma(\alpha+1)}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\left[J^{\alpha}_{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))-}f(m\varphi(a))+J^{\alpha}_{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))-}f(m\varphi(b))\right]\Bigg|$$

$$\leq K\left(\frac{m\pi}{2}\right)^{\frac{1}{q}}\left(\frac{\Gamma(p+1)\Gamma\left(\frac{1}{\alpha+1}\right)}{\Gamma\left(p+1+\frac{1}{\alpha+1}\right)}\right)^{\frac{1}{p}}$$

$$\times\left[\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}\right].$$

Theorem 6. *Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0,1] \longrightarrow (0,1)$ is a differentiable function. Suppose $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K\times K\times(0,1] \longrightarrow \mathbb{R}$ for any fixed $m\in(0,1]$ and let $m\varphi(a) < m\varphi(a)+\eta(\varphi(b),\varphi(a),m)$. Assume that $f : K = [m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)] \longrightarrow (0,\infty)$ be a twice differentiable function on K°. If f''^{q} is a nonnegative $MT_{(r;g,m,\varphi)}$-preinvex function on K, $q\geq 1$, then for $\alpha>0$ and $0<r\leq 1$, we have*

$$|I_{f,g,\eta,\varphi}(x;\alpha,n,m,a,b)| \leq \left(\frac{m}{2}\right)^{\frac{1}{rq}}\frac{H^{1-\frac{1}{q}}}{|\eta(\varphi(b),\varphi(a),m)|}$$

$$\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}$$

$$\times\Bigg[\left(\beta(n+2,\alpha-n)B_{g(1)}\left(1-\frac{1}{2r},1+\frac{1}{2r}\right)-D(g(t);\alpha,n,r)\right)^{r}f''(\varphi(a))^{rq}$$

$$+\left(\beta(n+2,\alpha-n)B_{g(1)}\left(1+\frac{1}{2r},1-\frac{1}{2r}\right)-C(g(t);\alpha,n,r)\right)^{r}f''(\varphi(x))^{rq}\Bigg]^{\frac{1}{rq}}$$

$$+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}$$

$$\times\Bigg[\left(\beta(n+2,\alpha-n)B_{g(1)}\left(1-\frac{1}{2r},1+\frac{1}{2r}\right)-D(g(t);\alpha,n,r)\right)^{r}f''(\varphi(b))^{rq}$$

$$+\left(\beta(n+2,\alpha-n)B_{g(1)}\left(1+\frac{1}{2r},1-\frac{1}{2r}\right)-C(g(t);\alpha,n,r)\right)^{r}f''(\varphi(x))^{rq}\Bigg]^{\frac{1}{rq}}\Bigg\}, \quad (8)$$

where

$$H=\int_0^1\left[\beta(n+2,\alpha-n)-B_{g(t)}(n+2,\alpha-n)\right]d[g(t)]$$

$$=\beta(n+2,\alpha-n)(g(1)-g(0))-g(1)B_{g(1)}(n+2,\alpha-n)+B_{g(1)}(n+3,\alpha-n);$$

$$C(g(t);\alpha,n,r)=\int_0^1 B_{g(t)}(n+2,\alpha-n)\left(\sqrt{\frac{g(t)}{1-g(t)}}\right)^{\frac{1}{r}}d[g(t)];$$

$$D(g(t);\alpha,n,r)=\int_0^1 B_{g(t)}(n+2,\alpha-n)\left(\sqrt{\frac{1-g(t)}{g(t)}}\right)^{\frac{1}{r}}d[g(t)].$$

Proof. Suppose that $q\geq 1$ and $0<r\leq 1$. Using relation (6), the well-known power mean inequality, the fact that f''^q is a nonnegative $MT_{(r;g,m,\varphi)}$-preinvex function on an open m-invex set K°, combining with Minkowski inequality for all $t\in[0,1]$ and for any fixed $m\in(0,1]$ and taking the modulus, we have

$$|I_{f,g,\eta,\varphi}(x;\alpha,n,m,a,b)|\leq\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}$$

$$\times\int_0^1\big(\beta(n+2,\alpha-n)-B_{g(t)}(n+2,\alpha-n)\big)f''(m\varphi(a)+g(t)\eta(\varphi(x),\varphi(a),m))d[g(t)]$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}$$

$$\times\int_0^1\big(\beta(n+2,\alpha-n)-B_{g(t)}(n+2,\alpha-n)\big)f''(m\varphi(b)+g(t)\eta(\varphi(x),\varphi(b),m))d[g(t)]$$

$$\leq\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}\left(\int_0^1\big[\beta(n+2,\alpha-n)-B_{g(t)}(n+2,\alpha-n)\big]d[g(t)]\right)^{1-\frac{1}{q}}$$

$$\times\Bigg[\int_0^1\big[\beta(n+2,\alpha-n)-B_{g(t)}(n+2,\alpha-n)\big]$$

$$\times f''(m\varphi(a)+g(t)\eta(\varphi(x),\varphi(a),m))^q d[g(t)]\Bigg]^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}\left(\int_0^1\big[\beta(n+2,\alpha-n)-B_{g(t)}(n+2,\alpha-n)\big]d[g(t)]\right)^{1-\frac{1}{q}}$$

$$\times\Bigg[\int_0^1\big[\beta(n+2,\alpha-n)-B_{g(t)}(n+2,\alpha-n)\big]$$

$$\times f''(m\varphi(b)+g(t)\eta(\varphi(x),\varphi(b),m))^q d[g(t)]\Bigg]^{\frac{1}{q}}$$

$$\leq\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}\left(\int_0^1\big[\beta(n+2,\alpha-n)-B_{g(t)}(n+2,\alpha-n)\big]d[g(t)]\right)^{1-\frac{1}{q}}$$

$$\times\Bigg[\int_0^1\big[\beta(n+2,\alpha-n)-B_{g(t)}(n+2,\alpha-n)\big]$$

$$\times\left[\frac{m\sqrt{g(t)}}{2\sqrt{1-g(t)}}f''(\varphi(x))^{rq}+\frac{m\sqrt{1-g(t)}}{2\sqrt{g(t)}}f''(\varphi(a))^{rq}\right]^{\frac{1}{r}}d[g(t)]\Bigg]^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}\left(\int_0^1\left[\beta(n+2,\alpha-n)-B_{g(t)}(n+2,\alpha-n)\right]d[g(t)]\right)^{1-\frac{1}{q}}$$

$$\times\Bigg[\int_0^1\left[\beta(n+2,\alpha-n)-B_{g(t)}(n+2,\alpha-n)\right]$$

$$\times\left[\frac{m\sqrt{g(t)}}{2\sqrt{1-g(t)}}f''(\varphi(x))^{rq}+\frac{m\sqrt{1-g(t)}}{2\sqrt{g(t)}}f''(\varphi(b))^{rq}\right]^{\frac{1}{r}}d[g(t)]\Bigg]^{\frac{1}{q}}$$

$$\leq\left(\frac{m}{2}\right)^{\frac{1}{rq}}H^{1-\frac{1}{q}}\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}$$

$$\times\Bigg[\left(\int_0^1\left(\sqrt{\frac{1-g(t)}{g(t)}}\right)^{\frac{1}{r}}\left[\beta(n+2,\alpha-n)-B_{g(t)}(n+2,\alpha-n)\right]f''(\varphi(a))^q d[g(t)]\right)^r$$

$$+\left(\int_0^1\left(\sqrt{\frac{g(t)}{1-g(t)}}\right)^{\frac{1}{r}}\left[\beta(n+2,\alpha-n)-B_{g(t)}(n+2,\alpha-n)\right]f''(\varphi(x))^q d[g(t)]\right)^r\Bigg]^{\frac{1}{rq}}$$

$$+\left(\frac{m}{2}\right)^{\frac{1}{rq}}H^{1-\frac{1}{q}}\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}$$

$$\times\Bigg[\left(\int_0^1\left(\sqrt{\frac{1-g(t)}{g(t)}}\right)^{\frac{1}{r}}\left[\beta(n+2,\alpha-n)-B_{g(t)}(n+2,\alpha-n)\right]f''(\varphi(b))^q d[g(t)]\right)^r$$

$$+\left(\int_0^1\left(\sqrt{\frac{g(t)}{1-g(t)}}\right)^{\frac{1}{r}}\left[\beta(n+2,\alpha-n)-B_{g(t)}(n+2,\alpha-n)\right]f''(\varphi(x))^q d[g(t)]\right)^r\Bigg]^{\frac{1}{rq}}$$

$$=\left(\frac{m}{2}\right)^{\frac{1}{rq}}\frac{H^{1-\frac{1}{q}}}{|\eta(\varphi(b),\varphi(a),m)|}$$

$$\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}$$

$$\times\Bigg[\left(\beta(n+2,\alpha-n)B_{g(1)}\left(1-\frac{1}{2r},1+\frac{1}{2r}\right)-D(g(t);\alpha,n,r)\right)^r f''(\varphi(a))^{rq}$$

$$+\left(\beta(n+2,\alpha-n)B_{g(1)}\left(1+\frac{1}{2r},1-\frac{1}{2r}\right)-C(g(t);\alpha,n,r)\right)^r f''(\varphi(x))^{rq}\Bigg]^{\frac{1}{rq}}$$

$$+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}$$

$$\times\Bigg[\left(\beta(n+2,\alpha-n)B_{g(1)}\left(1-\frac{1}{2r},1+\frac{1}{2r}\right)-D(g(t);\alpha,n,r)\right)^{r}f''(\varphi(b))^{rq}$$

$$+\left(\beta(n+2,\alpha-n)B_{g(1)}\left(1+\frac{1}{2r},1-\frac{1}{2r}\right)-C(g(t);\alpha,n,r)\right)^{r}f''(\varphi(x))^{rq}\Bigg]^{\frac{1}{rq}}\Bigg\}.$$

Corollary 5. *Under the same conditions as in Theorem 6, if we choose $\alpha\in(n,n+1]$ where $n=0,1,2,\ldots$ and $g(t)=t$, we get the following inequality for conformable fractional integrals:*

$$\Bigg|\frac{-\eta^{\alpha+1}(\varphi(x),\varphi(a),m)f'(m\varphi(a))-\eta^{\alpha+1}(\varphi(x),\varphi(b),m)f'(m\varphi(b))}{\eta(\varphi(b),\varphi(a),m)}$$

$$-\frac{(n+2-\alpha)(n+1)!}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\left[\left({}^{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))}I_{\alpha}f\right)(m\varphi(a))+\left({}^{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))}I_{\alpha}f\right)(m\varphi(b))\right]\Bigg|$$

$$\leq\left(\frac{m}{2}\right)^{\frac{1}{rq}}\frac{\beta^{1-\frac{1}{q}}(n+3,\alpha-n)}{|\eta(\varphi(b),\varphi(a),m)|}$$

$$\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}$$

$$\times\Bigg[\left(\frac{\pi}{2r\sin\left(\frac{\pi}{2r}\right)}\beta(n+2,\alpha-n)-D(t;\alpha,n,r)\right)^{r}f''(\varphi(a))^{rq}$$

$$+\left(\frac{\pi}{2r\sin\left(\frac{\pi}{2r}\right)}\beta(n+2,\alpha-n)-C(t;\alpha,n,r)\right)^{r}f''(\varphi(x))^{rq}\Bigg]^{\frac{1}{rq}}$$

$$+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}$$

$$\times\Bigg[\left(\frac{\pi}{2r\sin\left(\frac{\pi}{2r}\right)}\beta(n+2,\alpha-n)-D(t;\alpha,n,r)\right)^{r}f''(\varphi(b))^{rq}$$

$$+\left(\frac{\pi}{2r\sin\left(\frac{\pi}{2r}\right)}\beta(n+2,\alpha-n)-C(t;\alpha,n,r)\right)^{r}f''(\varphi(x))^{rq}\Bigg]^{\frac{1}{rq}}\Bigg\}.$$

Corollary 6. *Under the same conditions as in Corollary 5, if we choose $\alpha = n+1$ where $n = 0,1,2,\ldots,$ $r = 1$ and $f'' \le K$, we get the following inequality for fractional integrals:*

$$\Bigg|\frac{-\eta^{\alpha+1}(\varphi(x),\varphi(a),m)f'(m\varphi(a)) - \eta^{\alpha+1}(\varphi(x),\varphi(b),m)f'(m\varphi(b))}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}$$

$$+\frac{\eta^{\alpha}(\varphi(x),\varphi(a),m)f(m\varphi(a)+\eta(\varphi(x),\varphi(a),m)) + \eta^{\alpha}(\varphi(x),\varphi(b),m)f(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))}{\eta(\varphi(b),\varphi(a),m)}$$

$$-\frac{\Gamma(\alpha+1)}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\left[J^{\alpha}_{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))-}f(m\varphi(a)) + J^{\alpha}_{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))-}f(m\varphi(b))\right]\Bigg|$$

$$\le \left(\frac{m}{2}\right)^{\frac{1}{q}} \frac{K}{(\alpha+2)^{1-\frac{1}{q}}}\left[\frac{\pi}{\alpha+1} - (C(t;\alpha,n,1)+D(t;\alpha,n,1))\right]^{\frac{1}{q}}$$

$$\times\left[\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2} + |\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}\right].$$

Remark 3. *If we choose $\alpha = (n, n+1]$ where $n = 0,1,2,\ldots,$ for different choices of positive value $r = \frac{1}{2}, \frac{1}{3}$ etc., for any fixed $m \in (0,1]$, for a particular choices of a differentiable function $g(t)$, for example: $e^{-(t+1)}$, $\sin\left(\frac{\pi(t+1)}{3}\right)$, $\cos\left(\frac{\pi(t+1)}{3}\right)$, etc., and a particular choices of a continuous function $\varphi(x) = e^x$ for all $x \in \mathbb{R}$, x^n for all $x > 0$ and for all $n \in \mathbb{N}$, etc., by Theorem 5 and Theorem 6 we can get some special kinds of Hermite-Hadamard type conformable fractional integral inequalities. In particular for $\alpha = n+1$, $n = 0,1,2,\ldots,$ we get some special kinds of Hermite-Hadamard type fractional integral inequalities.*

4. Applications to special means

In the following we give certain generalizations of some notions for a positive valued function of a positive variable.

Definition 12. *(see [36]) A function $M : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$, is called a Mean function if it has the following properties:*

(i) Homogeneity: $M(ax, ay) = aM(x,y)$, for all $a > 0$,

(ii) Symmetry: $M(x,y) = M(y,x)$,

(iii) Reflexivity: $M(x,x) = x$,

(iv) Monotonicity: If $x \le x'$ and $y \le y'$, then $M(x,y) \le M(x',y')$,

(v) Internality: $\min\{x, y\} \le M(x, y) \le \max\{x, y\}$.

We consider some means for arbitrary positive real numbers α, β ($\alpha \neq \beta$).

(i) The arithmetic mean:

$$A := A(\alpha, \beta) = \frac{\alpha + \beta}{2}$$

(ii) The geometric mean:

$$G := G(\alpha, \beta) = \sqrt{\alpha\beta}$$

(iii) The harmonic mean:

$$H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}}$$

(iv) The power mean:

$$P_r := P_r(\alpha, \beta) = \left(\frac{\alpha^r + \beta^r}{2}\right)^{\frac{1}{r}}, \quad r \ge 1.$$

(v) The identric mean:

$$I := I(\alpha, \beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right), & \alpha \neq \beta; \\ \alpha, & \alpha = \beta. \end{cases}$$

(vi) The logarithmic mean:

$$L := L(\alpha, \beta) = \frac{\beta - \alpha}{\ln(\beta) - \ln(\alpha)}.$$

(vii) The generalized log-mean:

$$L_p := L_p(\alpha, \beta) = \left[\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta - \alpha)}\right]^{\frac{1}{p}}; \quad p \in \mathbb{R} \setminus \{-1, 0\}.$$

(viii) The weighted p-power mean:

$$M_p\begin{pmatrix} \alpha_1, & \alpha_2, & \cdots & , \alpha_n \\ u_1, & u_2, & \cdots & , u_n \end{pmatrix} = \left(\sum_{i=1}^{n} \alpha_i u_i^p\right)^{\frac{1}{p}}$$

where $0 \le \alpha_i \le 1$, $u_i > 0\,(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} \alpha_i = 1$.

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \le G \le L \le I \le A$. Now, let a and b be positive real numbers such that $a < b$. Consider the function $M := M(\varphi(x), \varphi(y)) : [\varphi(x), \varphi(x) + \eta(\varphi(y), \varphi(x))] \times [\varphi(x), \varphi(x) + \eta(\varphi(y), \varphi(x))] \longrightarrow \mathbb{R}_+$, which is one of the above mentioned means, $\varphi : I \longrightarrow K$ be a continuous function and $g : [0, 1] \longrightarrow (0, 1)$ is a differentiable function. Therefore one can obtain various inequalities using the results of Section 3 for these means as follows: Replace $\eta(\varphi(x),\varphi(y),m)$ with $\eta(\varphi(x),\varphi(y))$ and setting $\eta(\varphi(x),\varphi(y)) = M(\varphi(x),\varphi(y))$, $\forall x,y\in I$, for value $m=1$ in (7) and (8), one can obtain the following interesting inequalities involving means:

$$|I_{f,g,M(\cdot,\cdot),\varphi}(x;\alpha,n,1,a,b)| = \Bigg|\frac{M^{\alpha+2}(\varphi(a),\varphi(x))}{M(\varphi(a),\varphi(b))}$$

$$\times\Bigg\{\frac{\beta(n+2,\alpha-n)}{M(\varphi(a),\varphi(x))}$$

$$\times\Big[f'(\varphi(a)+g(1)M(\varphi(a),\varphi(x))) - f'(\varphi(a)+g(0)M(\varphi(a),\varphi(x))\Big]$$

$$-\frac{B_{g(1)}(n+2,\alpha-n)f'(\varphi(a)+g(1)M(\varphi(a),\varphi(x)))}{M(\varphi(a),\varphi(x))}$$

$$+\frac{1}{M^2(\varphi(a),\varphi(x))}$$

$$\times\Big[g^{n+1}(1)(1-g(1))^{\alpha-n-1}f(\varphi(a)+g(1)M(\varphi(a),\varphi(x)))$$

$$-g^{n+1}(0)(1-g(0))^{\alpha-n-1}f(\varphi(a)+g(0)M(\varphi(a),\varphi(x)))\Big]$$

$$+\frac{1}{M^{\alpha+2}(\varphi(a),\varphi(x))}$$

$$\times\Bigg[(n+1)\int_{\varphi(a)+g(0)M(\varphi(a),\varphi(x))}^{\varphi(a)+g(1)M(\varphi(a),\varphi(x))}(t-\varphi(a))^{n}$$

$$\times(\varphi(a)+M(\varphi(a),\varphi(x))-t)^{\alpha-n-1}f(t)dt$$

$$-(\alpha-n-1)\int_{\varphi(a)+g(0)M(\varphi(a),\varphi(x))}^{\varphi(a)+g(1)M(\varphi(a),\varphi(x))}(t-\varphi(a))^{n+1}$$

$$\times(\varphi(a)+M(\varphi(a),\varphi(x))-t)^{\alpha-n-2}f(t)dt\Bigg]\Bigg\}$$

$$+\frac{M^{\alpha+2}(\varphi(b),\varphi(x))}{M(\varphi(a),\varphi(b))}$$

$$\times\Bigg\{\frac{\beta(n+2,\alpha-n)}{M(\varphi(b),\varphi(x))}$$

$$\times\Big[f'(\varphi(b)+g(1)M(\varphi(b),\varphi(x))) - f'(\varphi(b)+g(0)M(\varphi(b),\varphi(x))\Big]$$

$$-\frac{B_{g(1)}(n+2,\alpha-n)f'(\varphi(b)+g(1)M(\varphi(b),\varphi(x)))}{M(\varphi(b),\varphi(x))}$$

$$+\frac{1}{M^2(\varphi(b),\varphi(x))}$$

$$\times\Big[g^{n+1}(1)(1-g(1))^{\alpha-n-1}f(\varphi(b)+g(1)M(\varphi(b),\varphi(x)))$$

$$-g^{n+1}(0)(1-g(0))^{\alpha-n-1}f(\varphi(b)+g(0)M(\varphi(b),\varphi(x)))\Big]$$

$$+\frac{1}{M^{\alpha+2}(\varphi(b),\varphi(x))}$$

$$\times\left[(n+1)\int_{\varphi(b)+g(0)M(\varphi(b),\varphi(x))}^{\varphi(b)+g(1)M(\varphi(b),\varphi(x))}(t-\varphi(b))^n\right.$$

$$\times(\varphi(b)+M(\varphi(b),\varphi(x))-t)^{\alpha-n-1}f(t)dt$$

$$-(\alpha-n-1)\int_{\varphi(b)+g(0)M(\varphi(b),\varphi(x))}^{\varphi(b)+g(1)M(\varphi(b),\varphi(x))}(t-\varphi(b))^{n+1}$$

$$\left.\left.\left.\times(\varphi(b)+M(\varphi(b),\varphi(x))-t)^{\alpha-n-2}f(t)dt\right]\right\}\right|$$

$$\leq\left(\frac{1}{2}\right)^{\frac{1}{rq}}\frac{\delta^{\frac{1}{p}}(g(t);p,\alpha,n)}{M(\varphi(a),\varphi(b))}$$

$$\times\left\{M^{\alpha+2}(\varphi(a),\varphi(x))\left[B^r_{g(1)}\left(1-\frac{1}{2r},1+\frac{1}{2r}\right)f''(\varphi(a))^{rq}\right.\right.$$

$$\left.+B^r_{g(1)}\left(1+\frac{1}{2r},1-\frac{1}{2r}\right)f''(\varphi(x))^{rq}\right]^{\frac{1}{rq}}$$

$$+M^{\alpha+2}(\varphi(b),\varphi(x))\left[B^r_{g(1)}\left(1-\frac{1}{2r},1+\frac{1}{2r}\right)f''(\varphi(b))^{rq}\right.$$

$$\left.\left.+B^r_{g(1)}\left(1+\frac{1}{2r},1-\frac{1}{2r}\right)f''(\varphi(x))^{rq}\right]^{\frac{1}{rq}}\right\}, \tag{9}$$

$$|I_{f,g,M(\cdot,\cdot),\varphi}(x;\alpha,n,1,a,b)|\leq\left(\frac{1}{2}\right)^{\frac{1}{rq}}\frac{H^{1-\frac{1}{q}}}{M(\varphi(a),\varphi(b))}$$

$$\times\left\{M^{\alpha+2}(\varphi(a),\varphi(x))\right.$$

$$\times\left[\left(\beta(n+2,\alpha-n)B_{g(1)}\left(1-\frac{1}{2r},1+\frac{1}{2r}\right)-D(g(t);\alpha,n,r)\right)^r f''(\varphi(a))^{rq}\right.$$

$$+\left(\beta(n+2,\alpha-n)B_{g(1)}\left(1+\frac{1}{2r},1-\frac{1}{2r}\right)-C(g(t);\alpha,n,r)\right)^{r}f''(\varphi(x))^{rq}\Bigg]^{\frac{1}{rq}}$$

$$+M^{\alpha+2}(\varphi(b),\varphi(x))$$

$$\times\Bigg[\left(\beta(n+2,\alpha-n)B_{g(1)}\left(1-\frac{1}{2r},1+\frac{1}{2r}\right)-D(g(t);\alpha,n,r)\right)^{r}f''(\varphi(b))^{rq}$$

$$+\left(\beta(n+2,\alpha-n)B_{g(1)}\left(1+\frac{1}{2r},1-\frac{1}{2r}\right)-C(g(t);\alpha,n,r)\right)^{r}f''(\varphi(x))^{rq}\Bigg]^{\frac{1}{rq}}\Bigg\}. \quad (10)$$

Letting $M(\varphi(x),\varphi(y)) = A, G, H, P_r, I, L, L_p, M_p, \forall x, y \in I$ in (9) and (10), we get the inequalities involving means for a particular choices of a nonnegative twice differentiable $MT_{(r;g,1,\varphi)}$-preinvex function f. The details are left to the interested reader.

5. Conclusions

In this paper, we proved some new integral inequalities for the left-hand side of Gauss-Jacobi type quadrature formula involving $MT_{(r;g,m,\varphi)}$-preinvex functions. Also, we established some new Hermite-Hadamard type integral inequalities for $MT_{(r;g,m,\varphi)}$-preinvex functions via conformable fractional integrals. These general inequalities give us some new estimates for Hermite-Hadamard type conformable fractional integral and fractional integral inequalities.

Motivated by this new interesting class of $MT_{(r;g,m,\varphi)}$-preinvex functions we can indeed see to be vital for fellow researchers and scientists working in the same domain.

We conclude that our methods considered here may be a stimulant for further investigations concerning Hermite-Hadamard type integral inequalities for various kinds of preinvex functions involving classical integrals, Riemann-Liouville fractional integrals, k-fractional integrals, local fractional integrals, fractional integral operators, q-calculus, (p,q)-calculus, time scale calculus and conformable fractional integrals.

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