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A New Generalization of the Operator-Valued Poisson Kernel

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Abstract. The purpose of this paper is to give a new generalization of the operator-valued Poisson kernel and discuss its some applications.

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1. Introduction

Let $\mathscr{H}$ be a Hilbert space which will be always complex and let $\mathscr{L}(\mathscr{H})$ be the algebra of all bounded linear operators from $\mathscr{H}$ to $\mathscr{H}$. We write I for the identity operator on $\mathscr{H}$. For $T \in \mathscr{L}(\mathscr{H})$, we denote by $\sigma(T)$ the spectrum of T.

For two operators $S, T \in \mathscr{L}(\mathscr{H})$, we write $S \geq T$ to indicate that $S - T$ is positive, i.e., $\langle (S-T)x, x\rangle \geq 0$ for all $x \in \mathscr{H}$.

Let $A \in \mathscr{L}(\mathscr{H})$. For a complex valued function f analytic on a domain E of the complex plane containing the spectrum $\sigma(A)$ of A we denote $f(A)$ as Riesz-Dunford integral [2, p. 568], that is,

$$f(A) := \frac{1}{2\pi i} \int_C f(z)(zI - A)^{-1} dz, \tag{1}$$

where C is positively oriented simple closed rectifiable contour containing $\sigma(A)$.

Throughout the paper $\mathbb{D}$ will denote the open unit disc $\mathbb{D} = \{z : |z| < 1\}$ in the complex plane $\mathbb{C}$.

2. The (Scalar) Poisson Kernel and The Operator-valued Poisson Kernel

For $re^{it} \in \mathbb{D}$, the (*scalar*) *Poisson kernel* $P_{r,t}$ is defined by

$$P_{r,t}(e^{i\theta}) = \frac{1 - r^2}{\left(1 - re^{it}e^{-i\theta}\right)\left(1 - re^{-it}e^{i\theta}\right)} \tag{2}$$

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$$
\begin{aligned}
&= \frac{1}{1 - re^{it}e^{-i\theta}} + \frac{1}{1 - re^{-it}e^{i\theta}} - 1 \\
&= \sum_{n\geq 0} r^n e^{int} e^{-in\theta} + \sum_{n\geq 0} r^n e^{-int} e^{in\theta} - 1.
\end{aligned}
$$

It is the well-known property of the (scalar) Poisson kernel that the integral formula

$$\frac{1}{2\pi}\int_0^{2\pi} P_{r,t}(e^{i\theta})d\theta = 1$$

holds.

In [1], the author gave the definition of the *operator-valued Poisson kernel* $K_{r,t}(T) \in \mathscr{L}(\mathscr{H})$ for $T \in \mathscr{L}(\mathscr{H})$ such that $\sigma(T) \subset \overline{\mathbb{D}}$ and for $re^{it} \in \mathbb{D}$, in the following way:

$$K_{r,t}(T) = (I - re^{it}T^*)^{-1} + (I - re^{-it}T)^{-1} - I. \tag{3}$$

For an operator $T \in \mathscr{L}(\mathscr{H})$ and a polynomial $p(z) = \sum_{k=0}^{n} a_k z^k \in \mathbb{C}[z]_{|\overline{\mathbb{D}}}$, $p(T) \in \mathscr{L}(\mathscr{H})$ is defined by

$$p(T) = \sum_{k=0}^{n} a_k T^k.$$

Remark 1. *T^0 is defined to be the identity operator, whatever the operator T.*

Another way to define $p(rT)$ for $0 \leq r < 1$ is to use the operator-valued Poisson kernel.

Lemma 1 ([1])**.** *Let $T \in \mathscr{L}(\mathscr{H})$ such that $\sigma(T) \subset \overline{\mathbb{D}}$. For all $r \in [0,1)$, we have:*

$$p(rT) = \frac{1}{2\pi}\int_0^{2\pi} p(e^{it})K_{r,t}(T)dt\,, \qquad p \in \mathbb{C}[z]_{|\overline{\mathbb{D}}}. \tag{4}$$

Remark 2. *Note that in the case p identically equal to 1 we have*

$$\frac{1}{2\pi}\int_0^{2\pi} K_{r,t}(T)dt = I.$$

Remark 3. *Since the definition of the (scalar) Poisson kernel $P_{r,t}(e^{i\theta})$ in (2) is also valid for $|r| < 1$, the definition of the operator-valued Poisson kernel $K_{r,t}(T)$ in (3) is valid for $|r| < 1$ too. Thus we have the following definition and theorem.*

Definition 1. *Let $T \in \mathscr{L}(\mathscr{H})$ such that $\sigma(T) \subset \overline{\mathbb{D}}$. The operator-valued Poisson kernel is defined by*

$$K_{r,t}(T) = (I - re^{it}T^*)^{-1} + (I - re^{-it}T)^{-1} - I. \tag{5}$$

Here r is a real parameter satisfying $|r| < 1$.

Theorem 1. *Let $T \in \mathscr{L}(\mathscr{H})$ such that $\sigma(T) \subset \overline{\mathbb{D}}$. Then we have*

$$\frac{1}{2\pi}\int_0^{2\pi} K_{r,t}(T)dt = I, \tag{6}$$

where r is a real parameter satisfying $|r| < 1$.

The purpose of this paper is to give generalizations of (5) and (6).
Firstly, in the next Section we recall the generalization of the (scalar) Poisson kernel.

3. The Generalization of the (Scalar) Poisson Kernel

In [3], Haruki and Rassias gave the new generalizations of the Poisson kernel of the form

$$P(\theta, r) = \frac{1 - r^2}{\left(1 - re^{i\theta}\right)\left(1 - re^{-i\theta}\right)},$$

where r is a real parameter satisfying $|r| < 1$.

One of this generalizations which is taken into consideration by us as follows:

Definition 2. *Set*

$$Q(\theta; a, b) \stackrel{def}{=} \frac{1 - ab}{(1 - ae^{i\theta})(1 - be^{-i\theta})}, \tag{7}$$

where $a,\ b$ are complex parameters satisfying $|a| < 1$ and $|b| < 1$.

Then they proved the following integral formula for $Q(\theta; a, b)$.

Theorem 2.

$$\frac{1}{2\pi}\int_0^{2\pi} Q(\theta; a, b)\, d\theta = 1,$$

where $a,\ b$ are complex parameters satisfying $|a| < 1$ and $|b| < 1$.

Remark 4. *Note that we can express the generalization of the (scalar) Poisson kernel in (2) as*

$$Q_{a,b,t}\left(e^{i\theta}\right) = \frac{1 - ab}{(1 - ae^{it}e^{-i\theta})(1 - be^{-it}e^{i\theta})}.$$

4. A New Generalization of the Operator-valued Poisson Kernel

In this Section, we shall treat generalizations of (5) and (6).

Definition 3. *For $T \in \mathscr{L}(\mathscr{H})$ such that $\sigma(T) \subset \overline{\mathbb{D}}$, define the generalization of the operator-valued Poisson kernel $K_{r,t}(T)$ in the following way:*

$$Q_{a,b,t}(T) \overset{def}{=} (I - ae^{it}T^*)^{-1} + (I - be^{-it}T)^{-1} - I, \tag{8}$$

where a, b are complex parameters satisfying $|a| < 1$ and $|b| < 1$.

Remark 5. *Note that $Q_{a,b,t}(T) \in \mathscr{L}(\mathscr{H})$.*

Remark 6. *By taking $a = r$ and $b = r$ in (8), we find that (8) is a generalization of (5).*

Lemma 2. *We have the following equalities:*

$$\left(Q_{a,b,t}(T)\right)^* = Q_{\bar{b},\bar{a},t}(T) = Q_{\bar{a},\bar{b},-t}(T^*).$$

Lemma 3. *For $T \in \mathscr{L}(\mathscr{H})$ such that $\sigma(T) \subset \overline{\mathbb{D}}$, we have:*

$$\begin{aligned} Q_{a,b,t}(T) &= (I - ae^{it}T^*)^{-1}(I - abT^*T)(I - be^{-it}T)^{-1} && (9)\\ &= (I - be^{-it}T)^{-1}(I - abTT^*)(I - ae^{it}T^*)^{-1} && (10)\\ &= \sum_{n=0}^{\infty} a^n e^{int} T^{*n} + \sum_{n=0}^{\infty} b^n e^{-int} T^n - I. && (11) \end{aligned}$$

Proof. By (8), we get

$$\begin{aligned} Q_{a,b,t}(T) &= (I - ae^{it}T^*)^{-1} + (I - be^{-it}T)^{-1} - I \\ &= (I - ae^{it}T^*)^{-1}\left[I + (I - ae^{it}T^*)(I - be^{-it}T)^{-1} - (I - ae^{it}T^*)\right] \\ &= (I - ae^{it}T^*)^{-1}\left[(I - be^{-it}T) + (I - ae^{it}T^*) - (I - ae^{it}T^*)(I - be^{-it}T)\right] \\ &\quad (I - be^{-it}T)^{-1} \\ &= (I - ae^{it}T^*)^{-1}(I - abT^*T)(I - be^{-it}T)^{-1}. \end{aligned}$$

Thus we obtain (9).

Similarly, the equality

$$Q_{a,b,t}(T) = (I - be^{-it}T)^{-1} + (I - ae^{it}T^*)^{-1} - I$$

gives proof of (10).

On the other hand, since $\left\|ae^{it}T^*\right\| < 1$ and $\left\|be^{-it}T\right\| < 1$, we have

$$\sum_{n=0}^{\infty} a^n e^{int} T^{*n} = (I - ae^{it}T^*)^{-1}$$

and

$$\sum_{n=0}^{\infty} b^n e^{-int} T^n = (I - be^{-it}T)^{-1},$$

respectively [see 4, Theorem 7.10]. By the last two equalities above and (8), we get (11).

Lemma 4. *Let $T \in \mathscr{L}(\mathscr{H})$ such that $\sigma(T) \subset \overline{\mathbb{D}}$. Then*

$$\|T\| \leq 1 \Longleftrightarrow Q_{a,\bar{a},t}(T) \geq 0.$$

Proof. The proof is same as proof of the Lemma 2.4 in [1].

Now we give a similar result to Lemma 1 by means of (11).

Lemma 5. *Let $T \in \mathscr{L}(\mathscr{H})$ such that $\sigma(T) \subset \overline{\mathbb{D}}$. For $q(z) \in \mathbb{C}[z]_{|\overline{\mathbb{D}}}$, we have*

$$q(bT) = \frac{1}{2\pi}\int_0^{2\pi} q(e^{it})Q_{a,b,t}(T)dt,$$

where a, b are complex parameters satisfying $|a| < 1$ and $|b| < 1$.

Proof. Let $q(z) = \sum_{k=0}^{N} a_k z^k$. Using (11) and considering the equality $\int_0^{2\pi} e^{imt}dt = 0$ for $m \in \mathbb{Z}\setminus\{0\}$, we obtain

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} q(e^{it})Q_{a,b,t}(T)dt &= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{k=0}^{N} a_k b^k T^k\right)dt \\ &= \sum_{k=0}^{N} a_k b^k T^k \\ &= q(bT).\end{aligned}$$

Corollary 1. *Note that in the case q identically equal to 1 we have*

$$\frac{1}{2\pi}\int_0^{2\pi} Q_{a,b,t}(T)dt = I. \tag{12}$$

Now we give another proof of (12) independently a polynomial. For this purpose we will use the Riesz-Dunford integral.

Theorem 3. *For $T \in \mathscr{L}(\mathscr{H})$ such that $\sigma(T) \subset \overline{\mathbb{D}}$, we have*

$$\frac{1}{2\pi}\int_0^{2\pi} Q_{a,b,t}(T)dt = I, \tag{13}$$

where a, b are complex parameters satisfying $|a| < 1$ and $|b| < 1$.

Proof. By (8), we have

$$\frac{1}{2\pi}\int_0^{2\pi} Q_{a,b,t}(T)dt = \frac{1}{2\pi}\int_0^{2\pi} \left[(I - ae^{it}T^*)^{-1} + (I - be^{-it}T)^{-1} - I\right] dt. \tag{14}$$

We set

$$I_1 = \frac{1}{2\pi}\int_0^{2\pi} (I - ae^{it}T^*)^{-1}dt, \tag{15}$$

$$I_2 = \frac{1}{2\pi}\int_0^{2\pi} (I - be^{-it}T)^{-1}dt \tag{16}$$

and

$$I_3 = \frac{1}{2\pi}\int_0^{2\pi} Idt. \tag{17}$$

So, by (15), (16) and (17), (14) is of the form

$$\frac{1}{2\pi}\int_0^{2\pi} Q_{a,b,t}(T)dt = I_1 + I_2 - I_3. \tag{18}$$

It is clear that

$$I_3 = I. \tag{19}$$

Next we shall calculate I_1 and I_2.

Firstly, we have

$$I_1 = \frac{1}{2\pi}\int_0^{2\pi} (I - ae^{it}T^*)^{-1}dt = \frac{1}{2\pi}\int_0^{2\pi} e^{-it}(e^{-it}I - aT^*)^{-1}dt.$$

Making substitution $z = e^{-it}$ in the last integral, we find

$$I_1 = -\frac{1}{2\pi i}\int_{|z|=1} (zI - aT^*)^{-1}dz,$$

where the integral along the $|z| = 1$ is in the negative direction. Hence, by the Riesz-Dunford integral (1), we have

$$I_1 = I. \tag{20}$$

Similarly, we get

$$I_2 = \frac{1}{2\pi}\int_0^{2\pi} (I - be^{-it}T)^{-1} dt = \frac{1}{2\pi}\int_0^{2\pi} e^{it}(e^{it}I - bT)^{-1} dt.$$

If we set $z = e^{it}$ then the last integral is of the form

$$I_2 = \frac{1}{2\pi i}\int_{|z|=1} (zI - bT)^{-1} dz,$$

where the integral along the $|z| = 1$ is in the positive direction. So, by the Riesz-Dunford integral (1), we obtain

$$I_2 = I. \tag{21}$$

Therefore, by (18), (19), (20) and (21) we get (13).

Remark 7. *By taking $a = r$ and $b = r$ in (13), we find that (13) is a generalization of (6).*

Corollary 2. *If we set $a = r$ and $b = r$ in Theorem 3 then we obtain Theorem 1. Hence Theorem 3 gives another proof of Theorem 1.*

Remark 8. *Note that $Q_{a,b,t}(T)$ in (8) is an operator-valued form of $Q_{a,b,t}\left(e^{i\theta}\right)$ in Remark 4.*

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