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# $\rho$ -Closed Sets in Topological Spaces

C. Devamanoharan<sup>1,\*</sup>, S.Pious Missier<sup>1</sup> and S. Jafari<sup>2</sup>

**Abstract.** In this paper, we introduce and study new classes of sets called  $\rho$ -closed sets and  $\rho_s$ -closed sets,  $\rho$ -open sets and  $\rho_s$ -open sets. Moreover, we present two new types of continuities called  $\rho$ -continuity and  $\rho_s$ -continuity and investigate some of their fundamental properties.

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**Key Words and Phrases:**  $\rho$ -closed,  $\rho_s$ -closed,  $\rho$ -open,  $\rho_s$ -open,  $\rho$ -continuity and  $\rho_s$ -continuity

#### 1. Introduction

The study of generalized closed sets in a topological space was initiated by Levine in [7] and the concept of  $T_{1/2}$  spaces was introduced. In 1996, H.Maki, J. Umehara and T. Noiri [9] introduced the class of pregeneralized closed sets and used them to obtain properties of pre- $T_{1/2}$  spaces. The modified forms of generalized closed sets and generalized continuity were studied by K. Balachandran, P. Sundaram and H. Maki [2]. In 2008, S. Jafari, T. Noiri, N. Rajesh and M.L. Thivagar [5] introduced the concept of  $\tilde{g}$ -closed sets and their properties. In this paper, we introduce new classes of sets called  $\rho$ -closed sets for topological spaces.

## 2. Preliminaries

Throughout this paper  $(X, \tau)$ ,  $(Y, \sigma)$  and  $(Z, \eta)$  will always denote topological spaces on which no separation axioms are assumed, unless otherwise mentioned. When A is a subset of  $(X, \tau)$ , Cl(A), Int(A) and D[A] denote the closure, the interior and the derived set of A, respectively. We recall some known definitions needed in this paper.

**Definition 1.** Let  $(X, \tau)$  be a topological space. A subset A of the space X is said to be

Email addresses: kanchidev@gmail.com (C.Devamanoharan), spmissier@yahoo.com (S. Missier), jafari@stofanet.dk (S.Jafari)

<sup>&</sup>lt;sup>1</sup> Post Graduate and Research Department of Mathematics, V.O.Chidambaram College, Thoothukudi 628 008, Tamil Nadu, India.

<sup>&</sup>lt;sup>2</sup> College of Vestsjaelland South, Herrestraede 11, 4200 Slagelse, Denmark.

<sup>\*</sup>Corresponding author.

- 1. Preopen [8] if  $A \subseteq int(cl(A))$  and preclosed if  $cl(int(A)) \subseteq A$ .
- 2. Semi-open [6] if  $A \subseteq cl(int(A))$  and semi-closed if  $int(cl(A)) \subseteq A$ .
- 3.  $\alpha$ -open [10] if  $A \subseteq int(cl(int(A)))$  and  $\alpha$ -closed if  $cl(int(cl(A))) \subseteq A$ .
- 4. Semi preopen [1] if  $A \subseteq cl(int(cl(A)))$  and semi preclosed if  $int(cl(int(A))) \subseteq A$ .
- 5. regular open [13] if A = int(cl(A)) and regular closed if A = cl(int(A)).
- 6.  $\pi$ -open [17] if it is the finite union of regular open sets.

**Definition 2** ([8]). Let  $(X, \tau)$  be a topological space and  $A \subseteq X$ .

- 1. The Pre-interior of A, denoted by pint(A), is the union of all preopen subsets of A.
- 2. The Pre-closure of A, denoted by Pcl(A), is the intersection of all Preclosed sets containing A.

**Lemma 1** ([1]). For any subset A of X, the following relations hold.

- 1.  $Scl(A) = A \cup int(cl(A))$ .
- 2.  $\alpha cl(A) = A \cup cl(int(cl(A)))$ .
- 3.  $Pcl(A) = A \cup cl(int(A))$ .
- 4.  $Spcl(A) = A \cup int(cl(int(A)))$ .

**Definition 3.** Let  $(X, \tau)$  be a topological space. A subset  $A \subseteq X$  is said to be

- 1. generalized closed (briely g-closed) [7] if  $cl(A) \subseteq U$  whenever  $A \subseteq U$  and U is open in X.
- 2. generalized preclosed (briely gp-closed) [11] if  $Pcl(A) \subseteq U$  whenever  $A \subseteq U$  and U is open in X.
- 3. generalized preregular closed (briely gpr-closed) [4] if  $Pcl(A) \subseteq U$  whenever  $A \subseteq U$  and U is regular open in X.
- 4. pregeneralized closed (briely pg-closed) [9] if  $Pcl(A) \subseteq U$  whenever  $A \subseteq U$  and U is preopen in X.
- 5.  $g^*$ -preclosed (briely  $g^*$ p-closed) [15] if  $Pcl(A) \subseteq U$  whenever  $A \subseteq U$  and U is g-open in X.
- 6. generalized semi-preclosed (briely gsp-closed) [3] if  $spcl(A) \subseteq U$  whenever  $A \subseteq U$  and U is open in X.
- 7. pre semi closed [16] if  $spcl(A) \subseteq U$  whenever  $A \subseteq U$  and U is g-open in X.
- 8.  $\pi gp$ -closed [12] if  $Pcl(A) \subseteq U$  whenever  $A \subseteq U$  and U is  $\pi$ -open in X.

- 9.  $\hat{g}$ -closed [14] if  $cl(A) \subseteq U$  whenever  $A \subseteq U$  and U is semi open in X.
- 10. \*g-closed [5] if  $cl(A) \subseteq U$  whenever  $A \subseteq U$  and U is  $\hat{g}$  -open in X.
- 11. #g- semi closed (briely #gs-closed) [5] if  $scl(A) \subseteq U$  whenever  $A \subseteq U$  and U is \*g-open in X.
- 12.  $\tilde{g}$ -closed set [5] if  $cl(A) \subseteq U$  whenever  $A \subseteq U$  and U is #gs-open in X.

The complements of the above mentioned sets are called their respective open sets.

**Definition 4.** A function  $f:(X,\tau)\to (Y,\sigma)$  is said to be g-continuous [2] if  $f^{-1}(V)$  is g-closed in  $(X,\tau)$  for every closed set V of  $(Y,\sigma)$ .

# 3. Basic Properties of $\rho$ -Closed Sets

We introduce the following definition.

**Definition 5.** A subset A of a space  $(X, \tau)$  is said to be  $\rho$ -closed in  $(X, \tau)$  if  $Pcl(A) \subseteq Int(U)$  whenever  $A \subseteq U$  and U is  $\tilde{g}$ -open in  $(X, \tau)$ .

**Theorem 1.** Every open and preclosed subset of  $(X, \tau)$  is  $\rho$ -closed.

*Proof.* Let *A* be an open and Preclosed subset of  $(X, \tau)$ . Let  $A \subseteq U$  and *U* be  $\tilde{g}$ -open in *X*. Then  $Pcl(A) = A = Int(A) \subseteq Int(U)$ . Hence *A* is  $\rho$ -closed.

The converse of the above theorem need not be true as it is seen from the following example.

**Example 1.** Let  $X = \{a, b, c\}$  and  $\tau = \{\phi, \{a\}, \{c, a\}, X\}$ . Then the set  $A = \{a, b\}$  is  $\rho$ -closed but it is neither open set nor preclosed set in  $(X, \tau)$ .

**Theorem 2.** Every  $\rho$ -closed set is gp-closed.

*Proof.* Let *A* be any  $\rho$ -closed set in *X*. Let  $A \subseteq U$  and *U* be open in *X*. Every open set is  $\tilde{g}$ -open and thus *A* is  $\rho$ -closed. Therefore  $Pcl(A) \subseteq Int(U) = U$ . Hence *A* is gp-closed.

The converse of the above theorem need not be true as it is seen from the following example.

**Example 2.** Let  $X = \{a, b, c, d\}$  and  $\tau = \{\phi, \{c\}, \{a, b\}, \{a, b, c\}, X\}$ . Then the set  $A = \{a\}$  is gp-closed but not  $\rho$ -closed in  $(X, \tau)$ .

**Theorem 3.** Every  $\rho$ -closed set is gpr-closed.

*Proof.* Let A be any  $\rho$ -closed set. Let  $A \subseteq U$  and U be regular open. Observe that every regular open set is open and every open set is  $\tilde{g}$ -open and therefore A is  $\rho$ -closed. It follows that  $Pcl(A) \subseteq Int(U) = U$ . Hence A is gpr-closed.

The converse of the above theorem need not be true as it is seen from the following example.

**Example 3.** Let  $X = \{a, b, c\}$  and  $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ . Then the set  $A = \{a, b\}$  is gpr-closed but not  $\rho$ -closed in  $(X, \tau)$ .

**Theorem 4.** Every  $\rho$ -closed set is gsp-closed.

*Proof.* Let A be any  $\rho$ -closed set. Let  $A \subseteq U$  and U be open. Since every open set is  $\tilde{g}$ -open and thus A is  $\rho$ -closed . Therefore  $Pcl(A) \subseteq Int(U) = U$  and so  $Spcl(A) \subseteq U$ . Hence A is gsp-closed.

The converse of this theorem need not be true as it is seen from the following example.

**Example 4.** Let  $X = \{a, b, c, d, e\}$  and  $\tau = \{\phi, \{a, b\}, \{c, d\}, \{a, b, c, d\}, X\}$ . Then the set  $A = \{a, b\}$  is gsp- closed but not  $\rho$ -closed in  $(X, \tau)$ .

**Theorem 5.** Every  $\rho$ -closed set is  $\pi gp$ -closed.

*Proof.* Let A be any  $\rho$ -closed set. Let  $A \subseteq U$  and U be  $\pi$ -open. Since every  $\pi$ -open set is open and every open set is  $\tilde{g}$ -open and therefore A is  $\rho$ -closed. This means that  $Pcl(A) \subseteq Int(U) = U$ . Hence A is  $\pi gp$ -closed.

The converse of the above theorem need not be true as it is seen from the following example.

**Example 5.** Let  $X = \{a, b, c\}$  and  $\tau = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$ . Then the set  $A = \{a, b\}$  is  $\pi gp$ -closed but not  $\rho$ -closed in  $(X, \tau)$ .

**Remark 1.**  $\rho$ -closedness and preclosedness are independent concepts as we illustrate by means of the following examples.

#### Example 6.

- 1. As in Example 1, the set  $A = \{a, b\}$  is  $\rho$ -closed but not preclosed in  $(X, \tau)$ .
- 2. Let  $X = \{a, b, c, d, e\}$  and  $\tau = \{\phi, \{a, b\}, \{a, b, c, \}, \{a, b, d\}, \{a, b, c, d\}, X\}$ . Then the set  $A = \{a\}$  is preclosed but not  $\rho$ -closed in  $(X, \tau)$ .

**Remark 2.**  $\rho$ -closed sets are independent concepts of semi-closed sets and semi-preclosed sets as we illustrate by means of the following example.

**Example 7.** Let  $X = \{a, b, c, d\}$  and  $\tau = \{\phi, \{b\}, \{c\}, \{b, c\}, \{b, c, d\}, X\}$ . Then the set  $A = \{a, b, c\}$  is  $\rho$ -closed but neither semi-closed nor semi-preclosed. The set  $B = \{c, d\}$  is both semi-closed and semi-preclosed but not  $\rho$ -closed.

**Remark 3.**  $\rho$ -closedness and presemiclosedness are independent concepts as we illustrate by means of the following examples.

## Example 8.

- 1. Let  $X = \{a, b, c\}$  and  $\tau = \{\phi, \{a\}, \{c\}, \{c, a\}, X\}$ . Then the set  $A = \{c\}$  is presemiclosed but not  $\rho$ -closed in  $(X, \tau)$ .
- 2. Let  $X = \{a, b, c\}$  and  $\tau = \{\phi, \{c\}, X\}$ . Then the set  $A = \{b, c\}$  is  $\rho$ -closed but not presemi*closed in*  $(X, \tau)$ .

**Remark 4.**  $\rho$ -closedness and g-closedness are independent concepts as we illustrate by means of the following examples.

## Example 9.

- 1. Let  $(X, \kappa)$  be a digital topology. Then the set  $A = \{4\}$  is closed in  $(X, \kappa)$  and thus g-closed. But the set A is not  $\rho$ -closed in  $(X, \kappa)$ , because there is a  $\tilde{g}$ -open set  $U = \{1, 2, 3, 4\}$  containing  $\{4\}$ , is not open in  $(X, \kappa)$  such that  $pcl(A) = \{4\} \nsubseteq int(U) = \{1, 2, 3\}$ .
- 2. Let  $X = \{a, b, c, d, e\}$  and  $\tau = \{\phi, \{a, b\}, \{a, b, d\}, \{a, b, c, d\}, \{a, b, d, e, \}, X\}$ . Then the set  $A = \{a, c, d\}$  is  $\rho$ -closed but not g-closed in  $(X, \tau)$ .

**Remark 5.**  $\rho$ -closedness and pg-closedness are independent concepts as we illustrate by means of the following example.

**Example 10.** Let  $X = \{a, b, c, d\} = Y$  and  $\tau = \{\phi, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}$  and  $\sigma = \{\phi, \{b\}, \{c\}, \{b, c\}, Y\}$ . Then the set  $A = \{c\}$  is pg-closed but not  $\rho$ -closed in  $(X, \tau)$  and the set  $B = \{a, b, c\}$  is  $\rho$ -closed but not pg-closed in  $(X, \tau)$ .

**Remark 6.**  $\rho$ -closedness and  $g^*p$ -closedness are independent concepts as we illustrate by means of the following example.

**Example 11.** Let  $X = \{a, b, c, d, e\}$  and  $\tau = \{\phi, \{b\}, \{d, e\}, \{b, d, e\}, \{a, c, d, e\}, X\}$ . Then the set  $A = \{a, b, d, e\}$  is  $\rho$ -closed but not  $g^*p$ -closed and the set  $B = \{d\}$  is  $g^*p$ -closed but not  $\rho$ -closed in  $(X, \tau)$ .

**Remark 7.**  $\rho$ -closedness and  $\alpha$ -closedness are independent concepts as we illustrate by means of the following examples.

## Example 12.

- 1. As in Example 9(1), the set  $A = \{4\}$  is  $\alpha$ -closed but not  $\rho$ -closed in  $(X, \kappa)$ .
- 2.  $X = \{a, b, c\}$  and  $\tau = \{\phi, \{c\}, \{b, c\}, Y\}$ . Then the set  $A = \{c, a\}$  is  $\rho$ -closed but not  $\alpha$ -closed in  $(X, \tau)$ .

**Definition 6.** A subset A of  $(X, \tau)$  is said to be  $\rho_s$ -closed in  $(X, \tau)$  if  $Pcl(A) \subseteq Int(cl(U))$  Whenever  $A \subseteq U$  and U is  $\tilde{g}$ -open in  $(X, \tau)$ .

**Theorem 6.** Every  $\rho$ -closed set is  $\rho_s$ -closed set.

*Proof.* Let *A* be any  $\rho$ -closed set. Let  $A \subseteq U$  and *U* be  $\tilde{g}$ -open in *X*. Since *A* is  $\rho$ -closed,  $Pcl(A) \subseteq Int(U) \subseteq Int(cl(U))$ . Hence *A* is  $\rho_s$ -closed.

The converse of the above theorem need not be true as it is seen from the following example.

**Example 13.** As in Example 2, the set  $A = \{a, c\}$  is  $\rho_s$ -closed but not  $\rho$ -closed in  $(X, \tau)$ .

**Remark 8.** From the above discussions and known results should be accompanied by a reference we have the following implications  $A \to B$  ( $A \leftrightarrow B$ ) represents A implies B but not conversely (A and B are independent of each other). See Figure 1.

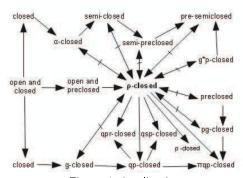


Figure 1: Implications.

## 4. Properties of $\rho$ -Closed Sets

**Definition 7.** The intersection of all  $\tilde{g}$ -open subsets of  $(X, \tau)$  containing A is called  $\tilde{g}$ -kernel of A and denoted by  $\tilde{g}$ -ker(A).

**Theorem 7.** If a subset A of  $(X, \tau)$  is  $\rho$ -closed then  $Pcl(A) \subseteq \tilde{g}-ker(A)$ .

*Proof.* Suppose that A is  $\rho$ -closed. Then  $Pcl(A) \subseteq Int(U)$  whenever  $A \subseteq U$  and U is  $\tilde{g}$ -open. Let  $x \in Pcl(A)$  and suppose  $x \notin \tilde{g}-ker(A)$ . Then there is a  $\tilde{g}$ -open set U containing A such that  $x \notin U$ . But U is a  $\tilde{g}$ -open set containing A. It follows that  $x \notin Pcl(A)$  and this is a contradiction.

The converse of the above theorem need not be true as it is seen from the following example.

**Example 14.** Let  $X = \{a, b, c, d, e\}$  and  $\tau = \{\phi, \{c\}, \{e\}, \{a, b\}, \{c, e\}, \{a, b, c\}, \{a, b, e\}, \{a, b, c, e\}, X\}$ . Then the set  $A = \{a\}$  satisfies  $Pcl(A) \subseteq \tilde{g} - ker(A)$ . But A is not  $\rho$ -closed  $(X, \tau)$ .

**Remark 9.** The union (intersection) of two  $\rho$ -closed sets need not be  $\rho$ -closed.

**Example 15.** Let  $X = \{a, b, c, d\}$  and  $\tau = \{\phi, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}$ .

- 1. Let  $A = \{a, b\}$  and  $B = \{a, c\}$ . Here A and B are  $\rho$ -closed sets. But  $A \cup B = \{a, b, c\}$  is not  $\rho$ -closed.
- 2. Let  $A = \{a, c\}$  and  $B = \{c, d\}$ . Here A and B are  $\rho$ -closed sets. But  $A \cap B = \{c\}$  is not  $\rho$ -closed.

**Theorem 8.** If a set A is  $\rho$ -closed, then Pcl(A) - A contains no nonempty closed set.

*Proof.* Let  $F \subseteq Pcl(A) - A$  be a nonempty closed set. Then  $F \subseteq Pcl(A)$  and  $A \subseteq X - F$ . Since X - F is  $\tilde{g}$ -open, then A is  $\rho$ -closed. Therefore  $Pcl(A) \subseteq Int(X - F) = X - cl(F)$ ,  $Cl(F) \subseteq X - Pcl(A)$ . And so  $F \subseteq X - Pcl(A)$ ,  $F \subseteq Pcl(A) \cap (X - Pcl(A)) = \{\emptyset\}$ . Hence Pcl(A) - A contains no nonempty closed set.

The converse of the above theorem need not be true as it is seen from the following example.

**Example 16.** Let  $X = \{a, b, c, d\}$  and  $\tau = \{\phi, \{c\}, \{a, b\}, \{a, b, c\}, X\}$ . Let  $A = \{a\}$ . Then Pcl(A) - A contains no nonempty closed set. But A is not  $\rho$ -closed in  $(X, \tau)$ .

**Theorem 9.** If a set A is  $\rho$ -closed, then Pcl(A) - A contains no nonempty  $\tilde{g}$ -closed set.

*Proof.* Let F be a nonempty  $\tilde{g}$ -closed set such that  $F \subseteq Pcl(A) - A$ . Then  $F \subseteq Pcl(A)$  and  $A \subseteq X - F$ . We have  $Pcl(A) \subseteq Int(X - F)$ ,  $Pcl(A) \subseteq X - cl(F)$ ,  $cl(F) \subseteq X - Pcl(A)$ . Therefore  $F \subseteq Pcl(A) \cap (X - Pcl(A)) = \{\emptyset\}$ . Hence Pcl(A) - A contains no nonempty  $\tilde{g}$ -closed set.

The converse of the above theorem need not be true as it is seen from the following example.

**Example 17.** Let  $X = \{a, b, c, d\}$  and  $\tau = \{\phi, \{c\}, \{a, b\}, \{a, b, c\}, X\}$ . Let  $A = \{a\}$ . Then Pcl(A) - A contains no nonempty  $\tilde{g}$ -closed set. But A is not  $\rho$ -closed in  $(X, \tau)$ .

**Theorem 10.** If A is  $\rho$ -closed and  $A \subseteq B \subseteq Pcl(A)$ , then B is  $\rho$ -closed.

*Proof.* Let *U* be a  $\tilde{g}$ -open set of *X* such that  $B \subseteq U$ . Then  $A \subseteq U$  and since *A* is  $\rho$ -closed, we have  $Pcl(A) \subseteq Int(U)$ . Now  $Pcl(B) \subseteq Pcl(Pcl(A)) = Pcl(A) \subseteq Int(U)$ . Hence *B* is  $\rho$ -closed.

**Theorem 11.** If a subset A of  $(X, \tau)$  is  $\tilde{g}$ - open and  $\rho$ -closed, then A is preclosed in  $(X, \tau)$ .

*Proof.* If a subset A of  $(X, \tau)$  is  $\tilde{g}$ -open and  $\rho$ -closed. Then  $Pcl(A) \subseteq Int(A) \subseteq A$ . Hence A is Preclosed is  $(X, \tau)$ .

**Lemma 2** ([4]). *If A is regular open and gpr-closed, then A is preclosed.* 

**Theorem 12.** A regular open set of  $(X, \tau)$  is gpr-closed if and only if A is  $\rho$ -closed in  $(X, \tau)$ .

*Proof.* Let  $A \subseteq U$  and U be  $\tilde{g}$ -open in  $(X, \tau)$ . Since A is regular open and gpr-closed, A is preclosed by Lemma 2. Since every regular open is open, therefore A is open and preclosed. Hence by Theorem 1, A is  $\rho$ -closed. By Theorem 3, the converse is obvious.

**Theorem 13.** Let A be  $\rho$ -closed in  $(X, \tau)$  then A is preclosed if and only if Pcl(A) - A is  $\tilde{g}$ -closed.

*Proof.* **Necessity.** Let *A* be preclosed. Then Pcl(A) = A. Hence  $Pcl(A) - A = \{\emptyset\}$  which is  $\tilde{g}$ -closed.

**Sufficiency.** Suppose Pcl(A) - A is  $\tilde{g}$ -closed. Since A is  $\rho$ -closed and by Theorem 9,  $Pcl(A) - A = \{\emptyset\}$ . Then Pcl(A) = A. This means that A is preclosed.

**Theorem 14.** An open set A of  $(X, \tau)$  is gp-closed if and only if A is  $\rho$ -closed.

*Proof.* Let A be an open and gp-closed set. Let  $A \subseteq U$  and U be  $\tilde{g}$ -open in X. Since A is open,  $A = Int(A) \subseteq Int(U)$ . Observe that Int(U) is open and thus A is gp-closed. Hence  $Pcl(A) \subseteq Int(U)$  and A is an  $\rho$ -closed set. Conversely, By Theorem 2, every  $\rho$ -closed set is gp-closed.

**Theorem 15.** If a subset A of  $(X, \tau)$  is open and regular closed then A is  $\rho$ -closed.

*Proof.* Let *A* be an open and regular closed set. Since regular closed set is preclosed. Then *A* is open and preclosed. By Theorem 1, *A* is  $\rho$ -closed.

**Lemma 3** ( $\lceil 5 \rceil$ ). Every  $\tilde{g}$ -closed set is preclosed.

**Theorem 16.** If a subset A of  $(X, \tau)$  is open and  $\tilde{g}$ -closed, then A is  $\rho$ -closed.

*Proof.* Let *A* be open and  $\tilde{g}$ -closed. Then by Lemma 3, *A* is open and preclosed. Hence by Theorem 1, A is  $\rho$ -closed.

**Theorem 17.** In a topological space X, for each  $x \in X$ ,  $\{x\}$  is  $\tilde{g}$ -closed or its complement  $X - \{x\}$  is  $\rho$ -closed in  $(X, \tau)$ .

*Proof.* Suppose that  $\{x\}$  is not  $\tilde{g}$  closed in  $(X,\tau)$ . Then  $X - \{x\}$  is not  $\tilde{g}$ -open and the only  $\tilde{g}$ -open set containing  $X - \{x\}$  is X. Therefore  $Pcl(X - \{x\}) \subseteq X$  and Int(X) = X and  $Pcl(X - \{x\}) \subseteq Int(X)$ . Hence  $X - \{x\}$  is  $\rho$ -closed in  $(X,\tau)$ .

#### Definition 8.

- 1. The union of all  $\rho$ -open sets, each contained in a set A in a topological space X is called the  $\rho$ -interior of A and is denoted by  $\rho$  int(A).
- 2. The intersection of all  $\rho$ -closed sets each containing a set A in topological space X is called the  $\rho$ -closure of A and is denoted by  $\rho$  cl(A).

**Lemma 4.** If a subset A of  $(X, \tau)$  is  $\rho$ -closed then  $A = \rho - cl(A)$ .

The converse of the above lemma need not be true as it is seen from the following example.

**Example 18.** Let  $X = \{a, b, c\}$  and  $\tau = \{\phi, \{c\}, X\}$ . If we consider  $A = \{c\}$  then  $A = \rho - cl(A)$  But A is not  $\rho$ -closed in  $(X, \tau)$ .

#### Definition 9.

- 1. Let  $(X, \tau)$  be a topological space,  $A \subseteq X$  and  $x \in X$ . Then x is said to be a pre-limit point of A if every preopen set containing x contains a point of A different from x.
- 2. Let  $(X, \tau)$  be a topological space and  $A \subseteq X$ . The set of all prelimit points of A is said to be a pre-derived set of A and is denoted by  $D_p[A]$ .

**Theorem 18.** If  $D[A] \subseteq D_p[A]$  for each subset A of a space  $(X, \tau)$ , then the union of two  $\rho$ -closed sets is  $\rho$ -closed.

*Proof.* Let A and B be  $\rho$ -closed subsets of X and U be a  $\tilde{g}$ -open set such that  $A \cup B \subseteq U$ . Then  $Pcl(A) \subseteq Int(U)$  and  $Pcl(B) \subseteq Int(U)$ . For each subset A of X we have  $D_p[A] \subseteq D[A]$ . Thus cl(A) = Pcl(A) and cl(B) = Pcl(B). Therefore  $cl(A \cup B) = cl(A) \cup cl(B) = Pcl(A) \cup Pcl(B) \subseteq Int(U)$ . But  $Pcl(A \cup B) \subseteq cl(A \cup B)$ . So  $Pcl(A \cup B) \subseteq Int(U)$  and hence  $A \cup B$  is  $\rho$ -closed.

**Theorem 19.** A subset A of  $(X, \tau)$  is both open and  $\rho$ -closed, then A is regular open.

*Proof.* Suppose *A* is open and  $\rho$ -closed. Then *A* is  $\tilde{g}$ -open and  $\rho$ -closed. By Theorem 16, *A* is preclosed. Hence pcl(A) = A. Since *A* is open, cl(A) = A. Therefore int(A) = int(cl(A)) implies A = int(cl(A)) and hence *A* is regular open.

The converse of the above theorem need not be true as it is seen from the following example.

**Example 19.** As in Example 8(1), the set  $A = \{a\}$  is regular open but not  $\rho$ -closed in  $(X, \tau)$ .

It should be noticed that in a locally indiscrete space openness and  $\rho$ -closedness are equivalent. Moreover, it is well-known that if  $(X, \tau)$  is locally indiscrete and T0, then  $(X, \tau)$  must be discrete.

**Definition 10.** A space X is called a  $p\tilde{g}$ -space if the intersection of a preclosed set with a  $\tilde{g}$ -closed set is  $\tilde{g}$ -closed.

**Example 20.** Let  $X = \{a, b, c, d\}$  and  $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$ . Then  $(X, \tau)$  is a  $p\tilde{g}$ -space. But, let  $\tau = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, X\}$  then  $(X, \tau)$  is not a  $p\tilde{g}$ -space.

**Theorem 20.** For a subset A of a  $p\tilde{g}$ -space  $(X, \tau)$  the following are equivalent.

1. A is  $\rho$ -closed.

- 2.  $Cl(\{x\}) \cap A \neq \phi$  for each  $x \in Pcl(A)$ .
- 3. Pcl(A) A contains no nonempty  $\tilde{g}$ -closed sets.

Proof.

- **1** ⇒ **2**: Let *A* be  $\rho$ -closed and  $x \in Pcl(A)$ . If  $cl(\{x\}) \cap A = \phi$ , then  $A \subseteq X cl(\{x\})$ . Since  $cl(\{x\})$  is closed,  $X cl(\{x\})$  is open and so  $\tilde{g}$ -open and thus *A* is  $\rho$ -closed.  $Pcl(A) \subseteq Int(X cl(\{x\}))$ ,  $x \notin Pcl(A)$  which is a contradiction. Therefore  $cl(\{x\}) \cap A \neq \phi$ .
- **2** ⇒ **3**: Let  $cl(\{x\}) \cap A \neq \phi$  for each  $x \in Pcl(A)$  and  $K \subseteq Pcl(A) A$  be a non empty  $\S$ -closed set, then  $K \subseteq Pcl(A)$  and  $A \subseteq X K$ . If there exist an  $x \in K$  then by (2)  $cl(\{x\}) \cap A \neq \phi$ .  $cl(\{x\}) \cap A \subseteq K \cap A \subseteq (Pcl(A) A) \cap A$  which is a contradiction. Hence Pcl(A) A contains no nonempty  $\S$ -closed sets.
- **3** ⇒ **1:** Let Pcl(A) A contains no nonempty  $\tilde{g}$ -closed set and  $A \subseteq U$  and U be  $\tilde{g}$  open in X. If  $Pcl(A) \nsubseteq Int(U)$ , then  $Pcl(A) \cap (Int(U))^c \neq \phi$ . Since the space is a  $p\tilde{g}$ -space,  $Pcl(A)] \cap (Int(U))^c$  is a nonempty  $\tilde{g}$ -closed subset of Pcl(A) A which is a contradiction. Hence A is  $\rho$ -closed.

## 5. $\rho$ -Open Sets and $\rho_s$ -Open Sets

#### Definition 11.

- 1. A subset A of  $(X, \tau)$  is said to be  $\rho$ -open in  $(X, \tau)$  if its complement X A is  $\rho$ -closed in  $(X, \tau)$ .
- 2. A subset A of  $(X, \tau)$  is said to be  $\rho_s$ -open in  $(X, \tau)$  if its complement X A is  $\rho_s$ -closed in  $(X, \tau)$ .

**Theorem 21.** Let  $(X, \tau)$  be a topological space and  $A \subseteq X$ .

- 1. A is an  $\rho$ -open set if and only if  $cl(K) \subseteq pint(A)$  whenever  $K \subseteq A$  and K is  $\tilde{g}$ -closed.
- 2. A is an  $\rho_s$ -open set if and only if  $cl(int(K)) \subseteq pint(A)$  whenever  $K \subseteq A$  and K is  $\tilde{g}$ -closed.
- 3. If A is  $\rho$ -open, then A is  $\rho_s$ -open.

Proof.

1. **Necessity.** Let A be an  $\rho$ -open set in  $(X, \tau)$ . Let  $K \subseteq A$  and K be  $\tilde{g}$ -closed. Then X - A is  $\rho$ -closed and it is contained in the  $\tilde{g}$ -open set X - K. Therefore  $Pcl(X - A) \subseteq Int(X - K)$ ,  $X - pint(A) \subseteq X - cl(K)$ , Hence  $cl(K) \subseteq pint(A)$ .

**Sufficiency.** If K is  $\tilde{g}$ -closed set such that  $cl(K) \subseteq pint(A)$  whenever  $K \subseteq A$ . It follows that  $X - A \subseteq X - K$  and  $X - pint(A) \subseteq X - cl(K)$ . Therefore  $Pcl(X - A) \subseteq Int(X - K)$ . Hence X - A is  $\rho$ -closed and A becomes an  $\rho$ -open set.

- 2. **Necessity:** Let A be an  $\rho_s$ -open set in  $(X, \tau)$ . Let  $K \subseteq A$  and K be  $\tilde{g}$ -closed. Then X A is  $\rho_s$ -closed and is contained in the  $\tilde{g}$ -open set X K. Therefore  $Pcl(X A) \subseteq Int(cl(X K))$  and so  $X pint(A) \subseteq int(X int(K)) = X cl(int(K))$ . Hence  $cl(int(K) \subseteq pint(A))$ . **Sufficiency:** If K is  $\tilde{g}$ -closed set such that  $cl(int(K)) \subseteq pint(A)$  whenever  $K \subseteq A$ , it follows that  $X A \subseteq X K$ ,  $X pint(A) \subseteq X cl(int(K))$  and  $Pcl(X A) \subseteq X cl(int(K)) = Int(cl(X K))$ . Hence X A is  $\rho_s$ -closed and A becomes an  $\rho_s$ -open set.
- 3. Let *A* be an  $\rho$ -open. Let also  $K \subseteq A$  and *K* be  $\tilde{g}$ -closed. Since *A* is  $\rho$ -open we have  $cl(K) \subseteq pint(A)$ . Therefore  $cl(int(K) \subseteq pint(A))$ . Hence by 2, *A* is  $\rho_s$ -open.

**Theorem 22.** *If*  $pint(A) \subseteq B \subseteq A$  *and* A *is*  $\rho$ -open then B *is*  $\rho$ -open.

*Proof.* If  $pint(A) \subseteq B \subseteq A$ , then  $X - A \subseteq X - B \subseteq X - pint(A)$  that is  $X - A \subseteq X - B \subseteq Pcl(X - A)$ . Observe that X - A is  $\rho$ -closed and by Theorem 10 X - B is  $\rho$ -closed and hence B is  $\rho$ -open.

**Theorem 23.** If  $A \subseteq K$  is  $\rho$ -closed then Pcl(A) - A is  $\rho$ -open.

*Proof.* Let *A* be an  $\rho$ -closed. Then by Theorem 9, Pcl(A) - A contains no nonempty  $\tilde{g}$ -closed set. Therefore  $\phi = K \subseteq Pcl(A) - A$  and  $\phi = K$  is  $\tilde{g}$ -closed. Clearly  $cl(K) \subseteq pint(Pcl(A) - A)$ . Hence by Theorem 21, Pcl(A) - A is  $\rho$ -open.

# 6. $\rho$ -Continuity and $\rho_s$ -Continuity

Let  $f:(X,\tau)\to (Y,\sigma)$  be a function from a topological space  $(X,\tau)$  into a topological space  $(Y,\sigma)$ .

#### Definition 12.

- 1. A function  $f:(X,\tau)\to (Y,\sigma)$  is said to be  $\rho$ -continuous (resp.  $\rho_s$ -continuous) if  $f^{-1}(V)$  is  $\rho$ -closed (resp.  $\rho_s$ -closed) in  $(X,\tau)$  for every closed set V of  $(Y,\sigma)$ .
- 2. A function  $f:(X,\tau) \to (Y,\sigma)$  is said to be  $\rho$ -irresolute (resp.  $\rho_s$ -irresolute) if  $f^{-1}(V)$  is  $\rho$ -closed (resp.  $\rho_s$ -closed) in  $(X,\tau)$  for every  $\rho$ -closed (resp.  $\rho_s$ -closed) set V of  $(Y,\sigma)$ .

#### Example 21.

- 1. Let  $X = \{a, b, c\}$ ,  $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$  and  $\sigma = \{\phi, \{c\}, \{b, c\}, X\}$ . Define  $f: (X, \tau) \to (X, \sigma)$  by f(a) = f(b) = b and f(c) = a, then f is  $\rho$ -continuous.
- 2. Let  $X = \{a, b, c\}$ ,  $\tau = \{\phi, \{c\}, \{a, b\}, X\}$  and  $\sigma = \{\phi, \{a, b\}, X\}$ . Define  $f: (X, \tau) \to (X, \sigma)$  by f(a) = c, f(b) = b and f(c) = a. Then the inverse image of every  $\rho$ -closed set is  $\rho$ -closed under f. Hence f is  $\rho$ -irresolute.

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The composition of two  $\rho$ -continuous functions need not be $\rho$ -continuous as it is shown by the following example.

**Example 22.** Let  $X = \{a, b, c\}$ ,  $\tau = \{\phi, \{b\}, X\}$ ,  $\sigma = \{\phi, \{a, b\}, X\}$  and  $\eta = \{\phi, \{a\}, \{a, b\}, X\}$ . Define  $f: (X, \tau) \to (X, \sigma)$  by f(a) = c, f(b) = a, f(c) = b and define  $g: (X, \sigma) \to (X, \eta)$  by g(a) = c, g(b) = a and g(c) = b. Then f and g are  $\rho$ -continuous but  $g \circ f$  is not  $\rho$ -continuous. Since  $\{c\}$  is closed in  $(X, \eta)$   $(g \circ f)^{-1}(\{c\}) = f^{-1}(g^{-1}(\{c\})) = f^{-1}(\{a\}) = \{b\}$  which is not  $\rho$ -closed in  $(X, \tau)$ .

**Theorem 24.** Let  $f:(X,\tau)\to (Y,\sigma)$  and  $g:(Y,\sigma)\to (Z,\eta)$  be two functions. Then

- 1.  $g \circ f$  is  $\rho$ -continuous if g is continuous and f is  $\rho$ -continuous.
- 2.  $g \circ f$  is  $\rho$ -irresolute if g is  $\rho$ -irresolute and f is  $\rho$ -irresolute.
- 3.  $g \circ f$  is  $\rho$ -continuous if g is  $\rho$ -continuous and f is  $\rho$ -irresolute.

Proof.

- 1. Let V be closed in  $(Z, \eta)$ . Since g is continuous,  $g^{-1}(V)$  is closed in  $(Y, \sigma)$ . As f is  $\rho$ -continuous,  $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$  is  $\rho$ -closed in  $(X, \tau)$ . Hence  $g \circ f$  is  $\rho$ -continuous.
- 2. Let V be  $\rho$ -closed in  $(Z, \eta)$ . Since g is  $\rho$ -irresolute,  $g^{-1}(V)$  is  $\rho$ -closed in  $(Y, \sigma)$ . As f is  $\rho$ -irresolute,  $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$  is  $\rho$ -closed in  $(X, \tau)$  Therefore  $g \circ f$  is  $\rho$ -irresolute.
- 3. Let V be closed in  $(Z, \eta)$ . since g is  $\rho$ -continuous,  $g^{-1}(V)$  is  $\rho$ -closed in  $(Y, \sigma)$ . As f is  $\rho$ -irresolute,  $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$  is  $\rho$ -closed in  $(X, \tau)$ . Therefore  $g \circ f$  is  $\rho$ -continuous.

## 7. References

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