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# Powers of Class $p$-$wA(s,t)$ Operators Associated with Generalized Aluthge Transformations

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**Abstract.** Let $T = U|T|$ be a polar decomposition of a bounded linear operator $T$ on a complex Hilbert space with $\ker U = \ker |T|$. $T$ is said to be class $p$-$wA(s,t)$ if

$$\left(|T^*|^t|T|^{2s}|T^*|^t\right)^{\frac{tp}{s+t}} \geq |T^*|^{2tp} \quad \text{and} \quad |T|^{2sp} \geq \left(|T|^s|T^*|^{2t}|T|^s\right)^{\frac{sp}{s+t}}$$

with $0 < p \leq 1$ and $0 < s, t, s+t \leq 1$. This is a generalization of $p$-hyponormal or class $A$ operators. In this paper, we shall show that if $T$ belongs to class $p$-$wA(s,t)$ operator for $0 < s,t \leq 1$ and $0 < p \leq 1$, then $T^n$ belongs to class $p_1$-$wA\left(\frac{s}{n}, \frac{t}{n}\right)$ for $0 < p_1 \leq p$ and for all positive integer $n$. As an immediate corollary of this result, we shall also show that if $T$ is a $p$-$w$-hyponormal operator, then $T^n$ is also $p_1$-$w$-hyponormal for $0 < p_1 \leq p$ and for all positive integer $n$.

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## 1. Introduction

Let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ and let $\ker(T)$, $ran(T)$ and $\sigma(T)$ denote the kernel, the range and the spectrum of $T \in B(\mathcal{H})$, respectively. Recall that an operator $T$ is said to be hyponormal if $T^*T \geq TT^*$. Aluthge [1] defined $p$-hyponormal operator as $(T^*T)^p \geq (TT^*)^p$ with $p \in (0,1]$, and he proved many interesting properties of $p$-hyponormal operators by using Furuta's inequality [2]. An invertible operator $T$ is said to be log-hyponormal if $\log(T^*T) \geq \log(TT^*)$. Moreover, by using Furuta's inequality, the class of $p$-hyponormal, log-hyponormal operators are extended to class $wA(s,t)$ operators with $0 < s,t$ as

$$\left(|T^*|^t|T|^{2s}|T^*|^t\right)^{\frac{t}{s+t}} \geq |T^*|^{2t} \tag{1}$$

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and

$$|T|^{2s} \geq \left(|T|^{s}|T^{*}|^{2t}|T|^{s}\right)^{\frac{s}{s+t}}. \tag{2}$$

However Ito and Yamazaki [3] proved that (1) implies (2). This is a striking result. An operator $T$ is said to be class $A(s,t)$ if $T$ satisfies (1). Hence Ito and Yamazaki proved that class $wA(s,t)$ coincides with class $A(s,t)$. It is known that every invertible $p$-hyponormal operator is log-hyponormal, every $p$-hyponormal, log-hyponormal operator is class $A(s,t)$ for all $0 < s,t$ and if $T$ is invertible and class $A(s,t)$ for all $0 < s,t$ then $T$ is log-hyponormal ([4], [5], [3], [6]).

Class p-wA(s,t) operators are known to have a variety of intriguing features as hyponormal operators, including the Putnam's inequality, the Fuglede-Putnam type theorem, the Weyl type theorem, subscalarity, and Weyl type theorem ([7],[8],[9], [10],[11], [12]). Despite the fact that there are numerous serious issues. One area of current interest in operator theory is the study of novel extensions of hyponormal operators, which are still open for hyponormal operators, such as the invariant subspace issue.

In this paper, we shall show that if $T$ belongs to class $p$-$wA(s,t)$ operator for $0 < s,t \leq 1$ and $0 < p \leq 1$, then $T^n$ belongs to class $p_1$-$wA(\frac{s}{n},\frac{t}{n})$ for $0 < p_1 \leq p$ and for all positive integer $n$. As an immediate corollary of this result, we shall also show that if $T$ is a $p$-$w$-hyponormal operator, then $T^n$ is also $p_1$-$w$-hyponormal for $0 < p_1 \leq p$ and for all positive integer $n$.

## 2. Preliminaries and Complementary Results

For $T \in B(\mathcal{H})$, set $|T| = (T^*T)^{\frac{1}{2}}$ as usual. By taking $U|T|x = Tx$ for $x \in \mathcal{H}$ and $Ux = 0$ for $x \in \ker|T|$, $T$ has a unique polar decomposition $T = U|T|$ with $\ker U = \ker|T|$. The following findings, which are very helpful for the study of non-normal operators, are introduced at the beginning of this section.

**Lemma 1.** *(Furuta's Inequality [2]) If $A \geq B \geq 0$, then for each $r \geq 0$,*

*(i)* $(B^{r/2}A^pB^{r/2})^{1/q} \geq B^{\frac{r+p}{q}}$ *and*

*(ii)* $(A^{r/2}A^pA^{r/2})^{1/q} \geq (A^{r/2}B^pA^{r/2})^{1/q}$

*hold for $p \geq 0$ and $p \geq 1$ with $(1+r)q \geq p+r$.*

**Lemma 2.** *[13, Hansen's Inequality] If $A, B \in \mathcal{B}(\mathcal{H})$ satisfy $A \geq 0$ and $\|B\| \leq 1$, then*

$$(B^*AB)^{\alpha} \geq B^*A^{\alpha}B \qquad \text{for all } \alpha \in (0,1].$$

**Lemma 3.** *[14, Löwner-Heinz theorem] $A \geq B \geq 0$ ensure $A^{\alpha} \geq B^{\alpha}$ for any $\alpha \in [0,1]$.*

**Lemma 4** ([5])**.** *Let $A > 0$ and $B$ be an invertible operator. Then*

$$(BAB^*)^{\lambda} = BA^{1/2}(A^{1/2}B^*BA^{1/2})^{\lambda-1}A^{1/2}B^*$$

*holds for any real number $\lambda$.*

**Definition 1** ([15]). *An operator $T$ is said to be $p$-$w$-hyponormal for $0 < p \leq 1$ if*

$$(|T^*|^{1/2}|T||T^*|^{1/2})^{p/2} \geq |T^*|^p \text{ and } |T|^p \geq (|T|^{1/2}|T^*||T|^{1/2})^{p/2}$$

They also pointed out the following fact.

**Proposition 1.** *An operator $T$ is $p$-$w$-hyponormal for $0 < p \leq 1$ if and only if*

$$|\tilde{T}|^p \geq |T|^p \geq |(\tilde{T})^*|^p,$$

*where the polar decomposition of $T$ is $T = U|T|$ and $\tilde{T}$ is Aluthge transformation of $T$, i.e.,*

$$\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}.$$

**Definition 2** ([16]). *An operator $T$ is said to be $(s,p)$-$w$-hyponormal if*

$$(|T^*|^s|T|^{2s}|T^*|^s)^{\frac{p}{2}} \geq |T^*|^{2sp} \tag{3}$$

*and*

$$|T|^{2sp} \geq (|T|^s|T^*|^{2s}|T|^s)^{\frac{p}{2}} \tag{4}$$

*where $0 < p \leq 1$ and $0 < s$.*

Clearly, if $s = \frac{1}{2}$ an $(s,p)$-$w$-hyponormal operator is $p$-$w$-hyponormal. That is to say, the class of $(s,p)$-$w$-hyponormal operators contains the class of $p$-$w$-hyponormal operators. He also pointed out the following fact.

**Proposition 2** ([16]). *An operator $T$ is $(s,p)$-$w$-hyponormal for $0 < p \leq 1$ and $s > 0$ if and only if*

$$|\tilde{T}_{s,s}|^p \geq |T|^{2sp} \geq |(\tilde{T}_{s,s})^*|^p$$

**Definition 3** ([17]). *For $p > 0$, $r \geq 0$, and $q \geq 1$, an operator $T$ belongs to class $wF(p,r,q)$ if*

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{1}{q}} \geq |T|^{\frac{2(r+p)}{q}} \tag{5}$$

*and*

$$|T|^{2(r+p)(1-\frac{1}{q})} \geq (|T|^p|T^*|^{2r}|T|^p)^{1-\frac{1}{q}} \tag{6}$$

*denoting $(1-q^{-1})^{-1}$ by $q^*$ (when $q > 1$) because $q$ and $(1-q^{-1})^{-1}$ are a couple of conjugate exponents.*

**Remark 1.** *Put $\delta = \frac{p+r}{q} - r$ in (5) and (6) then $-r < \delta \leq p$. Moreover,*

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{r+\delta}{r+p}} \geq |T|^{2(r+\delta)} \tag{7}$$

*and*

$$|T|^{2(p-\delta)} \geq (|T|^p|T^*|^{2r}|T|^p)^{\frac{p-\delta}{p+r}} \tag{8}$$

*hold, an operator $T$ is a class $wA(p,r)$ operator if and only if $T$ is a class $wF(p,r,\frac{p+r}{r})$ operator.*

As a generalization of $(s,p)$-$w$-hyponormality, class $wA(s,t)$ and class $wF(p,r,q)$, Prasad and Tanahashi [18] introduced $p$-$wA(s,t)$ as follows.

**Definition 4** ([18])**.** *An operator $T$ belongs to class $p$-$wA(s,t)$ if*

$$(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{tp}{s+t}} \geq |T^*|^{2tp} \tag{9}$$

*and*

$$|T|^{2sp} \geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{sp}{s+t}} \tag{10}$$

*where $0 < p \leq 1$ and $0 < s,t,s+t \leq 1$.*

**Remark 2.** *(1) In Definition 3, if we take $p_1 = \min\{\frac{p+r}{qr}, \frac{(p+r)(q-1)}{pq}\}$, then $T$ is class $p_1$-$wA(p,r)$. Hence, class $p_1$-$wA(p,r)$ is a generalization of class $wF(p,r,q)$.*
*(2)Clearly, if $s=t$, a class $p$-$wA(s,t)$ operator is $(s,p)$-$w$-hyponormal operator.*

For the sake of conveniens, we call class $p$-$wA(1,1)$ class $p$-$wA$ for short. They pointed out the following fact which states that class $p$-$wA(s,t)$ can be expressed via generalized Aluthge transformation.

**Proposition 3** ([18])**.** *An operator $T$ belongs to class $p$-$wA(s,t)$ for $s,t>0$ and $0<p\leq 1$ if and only if*

$$|\tilde{T}_{s,t}|^{\frac{2tp}{t+s}} \geq |T|^{2tp} \text{ and } |T|^{2sp} \geq |(\tilde{T}_{s,t})^*|^{\frac{2sp}{s+t}},$$

*where the polar decomposition of $T$ is $T=U|T|$ and $\tilde{T}_{s,t}$ is generalized Aluthge transformation of $T$, i.e.,*

$$\tilde{T}_{s,t} = |T|^sU|T|^t.$$

In [7], the authors proved that a set of class $p$-$wA(s,t)$ operators are increasing for $0<s,t$ and decreasing for $0<p\leq 1$.

**Theorem 1.** *([7]) If $T \in B(\mathcal{H})$ is class $p$-$wA(s,t)$ and $0 < s \leq \alpha, 0 < t \leq \beta, 0 < p_1 \leq p \leq 1$, then $T$ is class $p_1$-$wA(\alpha,\beta)$.*

Ito and Yamazaki [3] proved that (1) implies (2). However it is not known that whether (9) implies (10) or not. Class $A(1,1)$ is said to be class $A$ and class $A(\frac{1}{2},\frac{1}{2})$ is said to be $w$-hyponormal (see [3–5, 19]).
We need the following lemma in the sequel.

**Lemma 5** ([5])**.** *Let $A>0$ and $T=U|T|$ be the polar decomposition of $T$. Then for each $\alpha>0$ and $\beta>0$, the following assertions hold:*

*(i)* $U^*U(|T|^\beta A|T|^\beta)^\alpha = (|T|^\beta A|T|^\beta)^\alpha$.

*(ii)* $UU^*(|T^*|^\beta A|T^*|^\beta)^\alpha = (|T^*|^\beta A|T^*|^\beta)^\alpha$.

*(iii)* $(U|T|^\beta A|T|^\beta U^*)^\alpha = U(|T|^\beta A|T|^\beta)^\alpha U^*$.

*(iv)* $(U^*|T^*|^\beta A|T^*|^\beta U)^\alpha = U^*(|T^*|^\beta A|T^*|^\beta)^\alpha U$.

**Proposition 4.** *The following assertions hold:*

*(i) An operator $T$ is $p$-$w$-hyponormal if and only if $T$ belongs to class $p$-$wA(\frac{1}{2}, \frac{1}{2})$.*

*(ii) An operator $T$ belongs to class $p$-$wA$ for $0 < p \leq 1$ if and only if*

$$|T^2|^p \geq |T|^{2p} \tag{11}$$

*and*

$$|T^*|^{2p} \geq |T^{2*}|^p. \tag{12}$$

*Proof.* (i) We have only to put $s = t = \frac{1}{2}$ in Proposition 3.
(ii) proof of (11): $|T^2|^p \geq |T|^{2p}$ is equivalent to $|\tilde{T}_{1,1}|^p \geq |T|^{2p}$. We easily obtain $|T^2|^p = (T^*T^*TT)^{p/2} = (|T|U^*|T|^2U|T|)^{p/2} = |\tilde{T}_{1,1}|^p$, so that the proof is complete.

proof of (12): $|T^*|^{2p}$ is equivalent to $|T|^{2p} \geq |\tilde{T}_{1,1}|^p$. Suppose that

$$|T^*|^{2p} \geq |T^{*2}|^p = (TTT^*T^*)^{p/2} = (U|T||T^*|^2|T|U^*)^{p/2}. \tag{13}$$

By (iii) of Lemma 5, (13) holds if and only if

$$U|T|^{2p}U^* \geq U(|T|U|T|^2U^*|T|)^{p/2}U^*. \tag{14}$$

(14) ensures the following (15) by (i) of Lemma 5

$$|T|^{2p} \geq (|T|U|T|^2U^*|T|)^{p/2}. \tag{15}$$

(14) follows from (15), so that the proof is complete. Finally $|\tilde{T}_{1,1}|^p \geq |T|^{2p}$ and $|T|^{2p} \geq |(\tilde{T}_{1,1})^*|^p$ if and only if $T$ is class $p$-$wA(1, 1)$ by Proposition 3.

**Theorem 2** ([20]). *Let $0 < p \leq 1$ and let $A$ and $B$ be positive operators such that*

$$A^{q\alpha_0} \geq (A^{\alpha_0/2}B^{\beta_0}A^{\alpha_0/2})^{\frac{q\alpha_0}{\alpha_0+\beta_0}} \tag{16}$$

*and*

$$(B^{\beta_0/2}A^{\alpha_0}B^{\beta_0/2})^{\frac{q\beta_0}{\alpha_0+\beta_0}} \geq B^{q\beta_0} \tag{17}$$

*hold for fixed $\alpha_0 > 0$ and $\beta_0 > 0$. Then the following inequalities hold:*

$$A^{q_1\alpha} \geq (A^{\alpha/2}B^{\beta}A^{\alpha/2})^{\frac{q_1\alpha}{\alpha+\beta}} \tag{18}$$

*and*

$$(B^{\beta/2}A^{\alpha}B^{\beta/2})^{\frac{q_1\beta}{\alpha+\beta}} \geq B^{q_1\beta} \tag{19}$$

*for all $\alpha \geq \alpha_0$, $\beta \geq \beta_0$ and $0 < q_1 \leq q$.*

We shall give simplified proof of Theorem 1.

**Corollary 1.** *If $T \in B(\mathcal{H})$ is class $p$-$wA(s,t)$ and $0 < s \leq \alpha, 0 < t \leq \beta, 0 < p_1 \leq p \leq 1$, then $T$ is class $p_1$-$wA(\alpha, \beta)$.*

*Proof.* Suppose that $T$ is class $p$-$wA(s,t)$ for $s > 0$, $t > 0$ and $0 < p \leq 1$, i.e., the following (20) and (21) hold.

$$(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{tp}{s+t}} \geq |T^*|^{2tp}. \tag{20}$$

$$|T|^{2sp} \geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{sp}{s+t}}. \tag{21}$$

By Theorem 2, we have

$$(|T^*|^\beta|T|^{2\alpha}|T^*|^\beta)^{\frac{p_1\beta}{\alpha+\beta}} \geq |T^*|^{2p_1\beta} \text{ and } |T|^{2p_1\alpha} \geq (|T|^\alpha|T^*|^{2\beta}|T|^\alpha)^{\frac{p_1\alpha}{\alpha+\beta}}$$

for any $\alpha \geq s$, $\beta \geq t$ and $0 < p_1 \leq p$. Therefore $T$ is class $p_1$-$wA(\alpha,\beta)$ for any $\alpha \geq s$, $\beta \geq t$ and $0 < p_1 \leq p$.

## 3. Main Results

In order to give a proof of our main result, we prepare the following results.

**Proposition 5** ([20]). *Let $A$ and $B$ be positive operators. Then the following assertions hold:*

*(i) If $(B^{\frac{\beta_0}{2}}A^{\alpha_0}B^{\frac{\beta_0}{2}})^{\frac{\beta_0 p}{\alpha_0+\beta_0}} \geq B^{\beta_0 p}$ holds for fixed $\alpha_0 > 0$ and $\beta_0 > 0$ and $0 < p_1 \leq p \leq 1$, then*

$$(B^{\frac{\beta}{2}}A^{\alpha_0}B^{\frac{\beta}{2}})^{\frac{\beta p_1}{\alpha_0+\beta}} \geq B^{\beta p_1} \tag{22}$$

*holds for any $\beta \geq \beta_0$. Moreover, for each fixed $\gamma \geq -\alpha_0$,*

$$f_{\alpha_0,\gamma}(\beta) = (A^{\frac{\alpha_0}{2}}B^\beta A^{\frac{\alpha_0}{2}})^{\frac{(\alpha_0+\gamma)p_1}{\alpha_0+\beta}}$$

*is a decreasing function for $\beta \geq \max\{\beta_0, \gamma\}$. Hence the inequality*

$$(A^{\frac{\alpha_0}{2}}B^{\beta_1}A^{\frac{\alpha_0}{2}})^{p_1} \geq (A^{\frac{\alpha_0}{2}}B^{\beta_2}A^{\frac{\alpha_0}{2}})^{\frac{p_1(\alpha_0+\beta_1)}{\alpha_0+\beta_2}} \tag{23}$$

*holds for any $\beta_1$ and $\beta_2$ such that $\beta_2 \geq \beta_1 \geq \beta_0$ and $0 < p_1 \leq p$.*

*(ii) If $A^{\alpha_0 p} \geq (A^{\frac{\alpha_0}{2}}B^{\beta_0}A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0 p}{\alpha_0+\beta_0}}$ holds for fixed $\alpha_0 > 0$ and $\beta_0 > 0$ and $0 < p_1 \leq p \leq 1$, then*

$$A^{\alpha p_1} \geq (A^{\frac{\alpha}{2}}B^{\beta_0}A^{\frac{\alpha}{2}})^{\frac{\alpha p_1}{\alpha+\beta_0}} \tag{24}$$

*holds for any $\alpha \geq \alpha_0$. Moreover, for each fixed $\delta \geq -\beta_0$,*

$$g_{\beta_0,\delta}(\alpha) = (B^{\frac{\beta_0}{2}}A^\alpha B^{\frac{\beta_0}{2}})^{\frac{(\delta+\beta_0)p_1}{\alpha+\beta_0}}$$

*is an increasing function for $\alpha \geq \max\{\alpha_0, \delta\}$. Hence the inequality*

$$(B^{\frac{\beta_0}{2}}A^{\alpha_2}B^{\frac{\beta_0}{2}})^{\frac{p_1(\alpha_1+\beta_0)}{\alpha_2+\beta_0}} \geq (B^{\frac{\beta_0}{2}}A^{\alpha_2}B^{\frac{\beta_0}{2}})^{p_1} \tag{25}$$

*holds for any $\alpha_1$ and $\alpha_2$ such that $\alpha_2 \geq \alpha_1 \geq \alpha_0$ and $0 < p_1 \leq p$.*

We also prepare the following lemma in order to give a proof of Theorem 3.

**Lemma 6** ([9])**.** *Let $A, B$ and $C$ be positive operators. Then the following assertions hold for each $p \geq 0$, $r \in [0,1]$ and $0 < q \leq 1$:*

*(i) If $(B^{r/2}A^pB^{r/2})^{\frac{rq}{p+r}} \geq B^{rq}$ and $B \geq C$, then $(C^{r/2}A^pC^{r/2})^{\frac{rq}{p+r}} \geq C^{rq}$.*

*(ii) If $A \geq B$, $B^{rq} \geq (B^{r/2}C^pB^{r/2})^{\frac{rq}{p+r}}$ and the condition*

$$\begin{aligned} \textit{if} \lim_{n\longrightarrow\infty} B^{1/2}x_n &= 0 \textit{ and } \lim_{n\longrightarrow\infty} A^{1/2}x_n \textit{ exists,} \\ \textit{then} \lim_{n\longrightarrow\infty} A^{1/2}x_n &= 0 \textit{ for any sequence of vectors} \{x_n\} \end{aligned} \tag{26}$$

*hold, then $A^{rq} \geq (A^{r/2}C^pA^{r/2})^{\frac{rq}{p+r}}$.*

The following lemma is very important in the sequel.

**Lemma 7** ([19])**.** *Let $S$ be a positive operator and $0 < q \leq 1$. If $\lim_{n\longrightarrow\infty} Sx_n = 0$ and $\lim_{n\longrightarrow\infty} S^qx_n$ exists, then $\lim_{n\longrightarrow\infty} S^qx_n = 0$ for any sequence of vectors $\{x_n\}$.*

Firstly, we show the following result which is an extension of Theorem 1 of [19].

**Theorem 3.** *Let $T$ be a class $p$-$wA$ operator. Then the following assertions hold:*

*(i) $|T^{n+1}|^{\frac{2np}{n+1}} \geq |T^n|^{2p}$ and $|T^{n*}|^{2p} \geq |T^{n+1*}|^{\frac{2np}{n+1}}$ hold for all positive integer $n$.*

*(ii) $|T^{2n}|^p \geq |T^n|^{2p}$ and $|T^{n*}|^{2p} \geq |T^{2n*}|^p$ hold, i.e., $T^n$ also belongs to class $p$-$wA$ for all positive integer $n$.*

*(iii) $|T^n|^{\frac{2p}{n}} \geq \cdots \geq |T^n|^p \geq |T|^{2p}$ and $|T^*|^{2p} \geq |T^{2*}|^p \geq \cdots \geq |T^{n*}|^{\frac{2p}{n}}$.*

*Proof.* (i) Put $A_n = |T^n|^{2/n}$ and $B_n = |T^{n*}|^{2/n}$ for each positive integer $n$. By definition (9) and (10), $T$ belongs to class $p$-$wA$ if and only if $(|T^*||T|^2|T^*|)^{p/2} \geq |T^*|^{2p}$ and $|T|^{2p} \geq (|T||T^*|^2|T|)^{p/2}$, that is,

$$(B_1^{1/2}A_1B_1^{1/2})^{p/2} \geq B_1^p \tag{27}$$

and

$$A_1^p \geq (A_1^{1/2}B_1A_1^{1/2})^{p/2}. \tag{28}$$

We shall prove

$$A_{n+1}^{np} = |T^{n+1}|^{\frac{2np}{n+1}} \geq |T^n|^{2p} = A_n^{np} \tag{29}$$

and

$$B_n^{np} = |T^{n*}|^{2p} \geq |T^{n+1*}|^{\frac{2np}{n+1}} = B_{n+1}^{np} \tag{30}$$

hold for all positive integer $n$ by induction.
By (11) and (12) in (ii) of Proposition 4, (29) and (30) hold for $n = 1$.

Assume that (29) and (30) hold for $n = 1, 2, \cdots, k-1$. Firstly, we shall prove that (29) holds for $n = k$. By (29) and Löwner-Heinz theorem for $p/n \in [0, 1]$, $A_{n+1} \geq A_n$ holds for $n = 1, 2, \cdots, k-1$, so that we have

$$A_k \geq A_{k-1} \geq \cdots \geq A_2 \geq A_1. \tag{31}$$

We remark that $A_1$ and $A_k$ satisfy the condition: if $\lim\limits_{n\longrightarrow\infty} A_1^{p/2}x_n = 0$ and $\lim\limits_{n\longrightarrow\infty} A_k^{p/2}x_n$ exists, then $\lim\limits_{n\longrightarrow\infty} A_k^{p/2}x_n = 0$ for any sequence of vectors $\{x_n\}$. Since

$$\begin{aligned}
\lim_{n\longrightarrow\infty} A_1^{p/2}x_n = 0 &\Leftrightarrow \lim_{n\longrightarrow\infty} |T|^p x_n = 0 \\
&\Leftrightarrow \lim_{n\longrightarrow\infty} T^p x_n = 0 \text{ (by Lemma 7)} \\
&\Rightarrow \lim_{n\longrightarrow\infty} T^{kp} x_n = T^{pk-p}(\lim_{n\longrightarrow\infty} T^p x_n) = 0 \\
&\Leftrightarrow \lim_{n\longrightarrow\infty} |T^k|^p x_n = 0 \text{ (by Lemma 7)} \\
&\Leftrightarrow \lim_{n\longrightarrow\infty} A_k^{kp/2} x_n = 0 \\
&\Rightarrow \lim_{n\longrightarrow\infty} A_k^{p/2} x_n = 0 \text{ (by Lemma 7).}
\end{aligned}$$

By applying (ii) of Lemma 6 to (28) and (31), we have

$$A_k^p \geq (A_k^{1/2} B_1 A_k^{1/2})^{p/2}. \tag{32}$$

By applying (ii) of Proposition 5 to (32),

$$(B_1^{1/2} A_k^{\alpha_2} B_1^{1/2})^{\frac{(\alpha_1+1)p}{\alpha_2+1}} \geq (B_1^{1/2} A_k^{\alpha_1} B_1^{1/2})^p \tag{33}$$

holds for any $\alpha_1$ and $\alpha_2$ such that $\alpha_2 \geq \alpha_1 \geq 1$, so that we have

$$(B_1^{1/2} A_k^k B_1^{1/2})^{\frac{pk}{k+1}} \geq (B_1^{1/2} A_k^{k-1} B_1^{1/2})^p \geq (B_1^{1/2} A_{k-1}^{k-1} B_1^{1/2})^p \tag{34}$$

since the first inequality is obtained by putting $\alpha_1 = k-1$ and $\alpha_2 = k$ in (33), and the second inequality holds since (29) holds for $n = k-1$ by the inductive assumption. (34) yields the following (35):

$$(|T^*||T^k|^2|T^*|)^{\frac{kp}{k+1}} \geq (|T^*||T^{k-1}|^2|T^*|)^p. \tag{35}$$

Let $T = U|T|$ be the polar decomposition of $T$, then $T^* = U^*|T^*|$ is the polar decomposition of $T^*$. Hence we have

$$\begin{aligned}
|T^{k+1}|^{\frac{2kp}{k+1}} &= (T^*|T^k|^2 T)^{\frac{kp}{k+1}} \\
&= (U^*|T^*||T^k|^2|T^*|U)^{\frac{kp}{k+1}} \\
&= U^*(|T^*||T^k|^2|T^*|)^{\frac{kp}{k+1}} U
\end{aligned}$$

$$
\begin{aligned}
&\geq U^*(|T^*||T^{k-1}|^2|T^*|)^p U \ \text{ (by 35)}\\
&= (U^*|T^*||T^{k-1}|^2|T^*|U)^p\\
&= (T^*|T^{k-1}|^2 T)^p\\
&= |T^k|^{2p},
\end{aligned}
$$

so that it is proved that (29) holds for $n = k$.
Secondly, we shall prove that (30) holds for $n = k$. By (30) and Löwner-Heinz theorem for $p/n \in [0,1]$, $B_n^p \geq B_{n+1}^p$ holds for $n = 1, 2, \cdots, k-1$, so that we have

$$B_1^p \geq B_2^p \geq \cdots \geq B_{k-1}^p \geq B_k^p. \tag{36}$$

By applying (i) of Lemma 6 to (27) and (36), we have

$$(B_k^{1/2} A_1 B_k^{1/2})^{p/2} \geq B_k^p. \tag{37}$$

By applying (i) of Proposition 5 to (37),

$$(A_1^{1/2} B_k^{\beta_1} A_1^{1/2})^p \geq (A_1^{1/2} B_k^{\beta_2} A_1^{1/2})^{\frac{(1+\beta_1)p}{1+\beta_2}} \tag{38}$$

holds for any $\beta_1$ and $\beta_2$ such that $\beta_2 \geq \beta_1 \geq 1$ and $0 < p \leq 1$, so that we have

$$(A_1^{1/2} B_{k-1}^{k-1} A_1^{1/2})^p \geq (A_1^{1/2} B_k^{k-1} A_1^{1/2})^p \geq (A_1^{1/2} B_k^{k} A_1^{1/2})^{\frac{pk}{k+1}} \tag{39}$$

since the first holds since (30) holds for $n = k-1$ by the inductive assumption, and the second is obtained by putting $\beta_1 = k-1$ and $\beta_2 = k$ in (38). (39) yields the following (40)

$$(|T||T^{k-1*}|^2|T|)^p \geq (|T||T^{k*}|^2|T|)^{\frac{pk}{k+1}}. \tag{40}$$

Let $T = U|T|$ be the polar decomposition of $T$, then we have

$$
\begin{aligned}
|T^{k+1*}|^{\frac{2kp}{k+1}} &= (T|T^{k*}|^2 T)^{\frac{kp}{k+1}}\\
&= (U|T||T^{k*}|^2|T|U^*)^{\frac{kp}{k+1}}\\
&= U(|T||T^{k*}|^2|T|)^{\frac{kp}{k+1}} U^*\\
&\leq U(|T||T^{k-1*}|^2|T|)^p U^* \ \text{ (by (40)}\\
&= (U|T||T^{k-1*}|^2|T|U^*)^p\\
&= (T|T^{k-1*}|^2 T^*)^p\\
&= |T^{k*}|^{2p},
\end{aligned}
$$

so that it is proved that (30) holds for $n = k$. Consequently the proof of (i) is complete.

Proof of (ii). By (i) and Löwner-Heinz theorem for $\frac{np}{k} \in [0,1]$, $|T^{k+1}|^{\frac{2np}{k+1}} \geq |T^k|^{\frac{2np}{k}}$ and $|T^{k*}|^{\frac{2np}{k}} \geq |T^{k+1*}|^{\frac{2np}{k+1}}$ holds for any positive integer $k$ such that $k \geq n$, so that we have

$$|T^{2n}|^p \ = \ |T^{2n}|^{\frac{2np}{2n}} \geq |T^{2n-1}|^{\frac{2np}{2n-1}} \geq |T^{2n-2}|^{\frac{2np}{2n-2}}$$

$$\geq \cdots \geq |T^{n+1}|^{\frac{2np}{n+1}} \geq |T^n|^{\frac{2np}{n}} = |T^n|^{2p}$$

and

$$\begin{aligned} |T^{n*}|^{2p} &= |T^{n*}|^{\frac{2np}{n}} \geq |T^{n+1*}|^{\frac{2np}{n+1}} \geq |T^{n+2*}|^{\frac{2np}{n+2}} \\ &\geq \cdots |T^{2n-1*}|^{\frac{2np}{2n-1}} \geq |T^{2n*}|^{\frac{2np}{2n}} = |T^{2n*}|^{p}. \end{aligned}$$

Proof of (iii). (iii) has been already proved by (32) and (36) in the proof of (i).

**Theorem 4.** *Let $T$ be a class $p$-$wA(s,t)$ operator for $s \in (0,1]$, $t \in (0,1]$ and $0 < p \leq 1$. Then $T^n$ belongs to class $p_1$-$wA(\frac{s}{n}, \frac{t}{n})$ for all positive integer $n$ and $0 < p_1 \leq p$.*

*Proof.* Assume that $T$ belongs to class $p$-$wA(s,t)$ for $0 < s,t \leq 1$ and $0 < p \leq 1$. Put $A_n = |T^n|^{\frac{2}{n}}$ and $B_n = |T^{n*}|^{\frac{2}{n}}$ for each positive integer $n$, then $T$ belongs to class $p$-$wA(s,t)$ if and only if

$$(B_1^{t/2} A_1^s B_1^{t/2})^{\frac{pt}{t+s}} \geq B_1^{tp} \tag{41}$$

and

$$A_1^{sp} \geq (A_1^{s/2} B_1^t A_1^{s/2})^{\frac{sp}{s+t}} \tag{42}$$

by the definition (9) and (10). And $T$ belongs to class $p$-$wA(1,1)$ by Theorem 1, so that by (iii) of Theorem 3,

$$A_n \geq A_1 \tag{43}$$

and

$$B_1 \geq B_n \tag{44}$$

hold for all positive integer $n$. By applying (i) of Lemma 6 to (41) and (44) we have

$$(B_n^{t/2} A_1^s B_n^{t/2})^{\frac{tp}{s+t}} \geq B_n^{tp} \tag{45}$$

and by applying (ii) of Lemma 6 to (42) and (43), we have

$$A_n^s \geq (A_n^{s/2} B_1^t A_n^{s/2})^{\frac{sp}{s+t}}. \tag{46}$$

Since $A_1$ and $A_n$ satisfy the condition: If $\lim\limits_{n\longrightarrow\infty} A_1^{p/2} x_n = 0$ and $\lim\limits_{n\longrightarrow\infty} A_k^{p/2} x_n$ exists, then $\lim\limits_{n\longrightarrow\infty} A_k^{p/2} x_n = 0$ for any sequence of vectors $\{x_n\}$ as mentioned in the proof of Theorem 3. Then we have

$$(B_n^{t/2} A_n^s B_n^{t/2})^{\frac{tp}{s+t}} \geq (B_n^{t/2} A_1^s B_n^{t/2})^{\frac{tp}{s+t}} \geq B_n^{tp} \tag{47}$$

since the first inequality holds by (43) and Löwner-Heinz theorem and the second is (45), and we have

$$A_n^s \geq (A_n^{s/2} B_1^t A_n^{s/2})^{\frac{sp}{s+t}} \geq (A_n^{s/2} B_n^t A_n^{s/2})^{\frac{sp}{s+t}} \tag{48}$$

since the first inequality is (46) itself and the second holds by (44) and Löwner-Heinz theorem. (47) and (48) yield the following (49) and (50), respectively:

$$(|T^{n*}|^{\frac{t}{n}}|T^n|^{2\frac{s}{n}}|T^{n*}|^{\frac{t}{n}})^{\frac{(\frac{t}{n})p_1}{\frac{t}{n}+\frac{s}{n}}} \geq |T^{n*}|^{p_1\frac{t}{n}} \tag{49}$$

and

$$|T^n|^{\frac{2sp_1}{n}} \geq (|T^n|^{\frac{s}{n}}|T^{n*}|^{(2\frac{t}{n})}|T^n|^{\frac{s}{n}})^{\frac{(\frac{s}{n})p_1}{\frac{s}{n}+\frac{t}{n}}} \tag{50}$$

so that $T^n$ belongs to class $p_1$-$wA(\frac{s}{n}, \frac{t}{n})$ by the Definition (9) and (10).

**Corollary 2.** *Let $T$ be a $p$-$w$-hyponormal operator for $0 < p \leq 1$. Then $T^n$ is also a $p_1$-$w$-hyponormal for all positive integer $n$ and $p_1$ such that $0 < p_1 \leq p$.*

*Proof.* If $T$ is a $p$-$w$-hyponormal, then it follows by (i) of Proposition 4 that $T$ belongs to class $p$-$wA(\frac{1}{2}, \frac{1}{2})$ and hence $T^n$ belongs to class $p_1$-$wA(\frac{1}{2n}, \frac{1}{2n})$ for all positive integer $n$ by Theorem 4, so that $T^n$ belongs to class $p_1$-$wA(\frac{1}{2}, \frac{1}{2})$ by Theorem 1. Hence the proof is complete.

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