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q–Rabotnov Functions and Bi-univalent Functions

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Abstract. In this work, we introduce and investigate a new subclass of bi-univalent functions, denoted by $\mathfrak{R}_{\Sigma}^{\mu,x}(\varrho,\varphi,\lambda;\gamma;q)$, which is defined through the interplay between q–Rabotnov functions and q–Gegenbauer polynomials. For functions in this class, we derive sharp bounds for the initial Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$, and we establish estimates for the corresponding Fekete–Szegő functional. Furthermore, by selecting suitable parameter values, our results reduce to several well known subclasses, thereby yielding a range of new consequences and extensions in the theory of analytic bi-univalent functions.

2020 Mathematics Subject Classifications: 54D30, 54E99, 54D10

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1. Introduction

Rabotnov-type kernels were put forward by Yu. N. Rabotnov [1] in linear viscoelasticity to model hereditary behavior such as creep and relaxation. They are expressed through convolution kernels involving Mittag–Leffler-type functions and, consequently, furnish a rigorous device for encoding fractional-order operators in constitutive relations [2]. Owing to their flexibility, these kernels have become standard in the description of stress–strain

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responses with memory in mechanics and engineering. Their intimate link with fractional calculus has also made them central to the modern analysis of complex dynamical systems and viscoelastic materials [3].

More recently, variants of Rabotnov kernels have been examined from the perspective of Geometric Function Theory (GFT). Because one may represent them in terms of generalized Mittag–Leffler series, they can be used to generate families of analytic functions in the unit disk with preserved geometric features (e.g., starlikeness and convexity). In this vein, Rabotnov-type generating functions connect problems on fractional operators with classical coefficient questions for univalent and bi-univalent functions, and they naturally interact with special polynomial systems and analytic subordination.

Quantum calculus (q–calculus) generalizes differential and integral calculus by replacing infinitesimal increments with q-difference and q-integral operators. These operators reduce to their classical counterparts as $q \to 1$ and play a central role in the theory of basic hypergeometric series, orthogonal polynomials, and special functions. Because of its inherently discrete nature, q–calculus is also widely used in combinatorics, number theory, quantum groups, and mathematical physics [4].

Within this discrete framework, it is natural to consider q–analogues of Rabotnov functions. Since they are representable through generalized Mittag–Leffler-type functions, one can build their discrete analogues using q–series and q–integrals and then analyze them by means of q–difference operators. This approach yields discrete-time models for hereditary phenomena and broadens the scope of applications on non-uniform lattices while maintaining connections with the theory of basic hypergeometric functions [5].

Let $\mathcal{A}$ denote the class of functions analytic in the open unit disk $\mathbb{U} = \{\psi \in \mathbb{C} : |\psi| < 1\}$, normalized by $f(0) = 0$ and $f'(0) = 1$. Every $f \in \mathcal{A}$ has a Taylor–Maclaurin expansion

$$f(\psi) = \psi + \sum_{n=2}^{\infty} a_n \psi^n, \qquad \psi \in \mathbb{U}. \tag{1}$$

Write $\mathcal{S}$ for the class of functions in $\mathcal{A}$ that are univalent on $\mathbb{U}$.

If f and g are analytic on $\mathbb{U}$, we say that f is subordinate to g, and write $f \prec g$, if there exists a Schwarz function w (analytic on $\mathbb{U}$ with $w(0) = 0$ and $|w(\psi)| < 1$) such that

$$f(\psi) = g(w(\psi)), \qquad \psi \in \mathbb{U}.$$

When g is univalent, $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Every $f \in \mathcal{S}$ has an inverse f^{-1} satisfying

$$f^{-1}(f(\psi)) = \psi \quad (\psi \in \mathbb{U}), \qquad f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots,$$

which is analytic and univalent for $|w| < r_0(f)$, where $r_0(f) \geq \frac{1}{4}$. A function f is called **bi-univalent** on $\mathbb{U}$ if both f and its inverse f^{-1} are univalent in $\mathbb{U}$. We denote the family of all such functions by Σ. Typical examples in Σ are

$$\frac{\psi}{1-\psi}, \qquad \log\Big(\frac{1}{1-\psi}\Big), \qquad \log\sqrt{\frac{1+\psi}{1-\psi}}.$$

Definition 1 ([6]). *For $\varrho, \beta, \vartheta \in \mathbb{C}$ with $\Re(\varrho) > 0$, $\Re(\varphi) > 0$, and $|q| < 1$, the generalized q–Mittag–Leffler function $\mathrm{E}^{\varphi}_{\varrho,\varphi}$ is*

$$\mathrm{E}^{\varphi}_{\varrho,\varphi}(\psi; q) = \sum_{n=0}^{\infty} \frac{(q^{\varphi}; q)_n}{(q; q)_n} \frac{\psi^n}{\Gamma_q(\varrho n + \varphi)}, \tag{2}$$

where Γ_q denotes the q–gamma function.

The q–analogue of the Pochhammer symbol (q–shifted factorial) is given by (see [7])

$$(\varphi; q)_n = \begin{cases} (1-\varphi)(1-\varphi q)\cdots(1-\varphi q^{n-1}), & n = 1, 2, 3, \ldots, \\ 1, & n = 0. \end{cases}$$

The q–gamma function $\Gamma_q(z)$ satisfies (see [7, 8])

$$\Gamma_q(\psi + 1) = \frac{1 - q^{\psi}}{1 - q}\,\Gamma_q(\psi) = [\psi]_q \Gamma_q(\psi), \qquad (\varphi; q)_n = \frac{(1-q)^n \Gamma_q(\varphi + n)}{\Gamma_q(\varphi)} \quad (n > 0).$$

We now introduce a modified q–Rabotnov function.

Definition 2. *Let $\varrho \in \mathbb{C}$ with $\Re(\varrho) > 0$, $\lambda > 0$, and $|q| < 1$. Define*

$$\Phi^{\varphi}_{\varrho,\lambda}(\psi; q) = \psi^{\varrho} \sum_{n=0}^{\infty} \frac{(q^{\varphi}; q)_n}{(q; q)_n} \frac{[\lambda]_q^n}{\Gamma_q((n+1)(1+\varrho))}\,\psi^{n(1+\varrho)}. \tag{3}$$

When $q \to 1^-$, the function $\Phi^{\varphi}_{\varrho,\lambda}(\psi; q)$ reduces to the classical Rabotnov function $\Phi_{\varrho,\lambda}(\psi)$ (see [1]).

Since $\Phi^{\varphi}_{\varrho,\lambda}(\psi; q)$ is not normalized, we adopt the following normalized form:

$$\begin{aligned} \mathbb{R}^{\varphi}_{\varrho,\lambda}(\psi; q) &= \psi^{\frac{1}{1+\varrho}+1} \Gamma_q(1+\varrho)\,\Phi^{\varphi}_{\varrho,\lambda}(\psi^{\frac{1}{1+\varrho}}; q) \\ &= \psi + \sum_{n=2}^{\infty} \frac{(q^{\varphi}; q)_{n-1}}{(q; q)_{n-1}} \frac{[\lambda]_q^{n-1} \Gamma_q(1+\varrho)}{\Gamma_q((1+\varrho)n)}\,\psi^n, \qquad \psi \in \mathbb{U}. \end{aligned}$$

Define the linear (Hadamard-convolution) operator $\mathcal{F}^{\varphi}_{\varrho,\lambda} : \mathcal{A} \to \mathcal{A}$ by

$$\mathcal{F}^{\varphi}_{\varrho,\lambda}(f(\psi); q) = \mathbb{R}^{\varphi}_{\varrho,\lambda}(\psi; q) * f(\psi) = \psi + \sum_{k=2}^{\infty} \frac{(q^{\varphi}; q)_{k-1}}{(q; q)_{k-1}} \frac{[\lambda]_q^{k-1} \Gamma_q(1+\varrho)}{\Gamma_q(k(1+\varrho))}\,a_k \psi^k, \tag{4}$$

where $\varrho, \varphi \in \mathbb{C}$ with $\Re\{\varrho\} > 0$ and $\lambda > 0$.

Askey and Ismail ([8], [9]) introduced q–analogues of Gegenbauer polynomials. Writing

$$\mathfrak{G}_q^{(\lambda)}(x, \psi) = \sum_{n=0}^{\infty} \mathcal{C}_n^{(\lambda)}(x; q)\psi^n, \tag{5}$$

one convenient description due to Chakrabarti et al. [10] is the recurrence

$$\begin{aligned} &\mathcal{C}_0^{(\mu)}(x;q) = 1, \qquad \mathcal{C}_1^{(\mu)}(x;q) = 2[\mu]_q\, x, \\ &\mathcal{C}_2^{(\mu)}(x;q) = 2\big([\mu]_{q^2} + [\mu]_q^2\big)x^2 - [\mu]_{q^2}. \end{aligned} \tag{6}$$

Recently, Amourah et al. [11] and Alsoboh et al. [12–14] introduced novel subclasses of bi-univalent functions associated with -Gegenbauer polynomials. Their investigations primarily emphasized the derivation of Fekete–Szegő type inequalities and the analysis of coefficient bounds for $|a_2|$ and $|a_3|$ within these subclasses. The study of bi-univalent functions connected with-Gegenbauer polynomials has since garnered considerable scholarly interest, as reflected in several related contributions [5, 7, 15].

The present work aims to further explore the properties and structural characteristics of bi-univalent functions associated with -Gegenbauer polynomials. To this end, we begin by establishing a set of fundamental definitions that will serve as the basis for our subsequent analysis.

2. Coefficient bounds for $\mathfrak{R}_{\Sigma}^{\mu,x}(\varrho,\varphi,\lambda;\gamma;q)$

We introduce a subclass of Σ associated with the q–Rabotnov operator.

Definition 3. *A function $f \in \Sigma$ given by* (1) *is said to be in the class $\mathfrak{R}_{\Sigma}^{\mu,x}(\varrho,\varphi,\lambda;\gamma;q)$ if the subordinations*

$$(1-\wp)\frac{\mathcal{F}_{\varrho,\lambda}^{\varphi}(f(\psi);q)}{\psi} + \wp\,\partial_q\Big(\mathcal{F}_{\varrho,\lambda}^{\varphi}(f(\psi);q)\Big) \prec \mathfrak{G}_q^{(\mu)}(x,\psi) \tag{7}$$

and

$$(1-\wp)\frac{\mathcal{F}_{\varrho,\lambda}^{\varphi}(g(w);q)}{w} + \wp\,\partial_q\Big(\mathcal{F}_{\varrho,\lambda}^{\varphi}(g(w);q)\Big) \prec \mathfrak{G}_q^{(\mu)}(x,w) \tag{8}$$

hold.

By suitable specializations of $\wp$ and δ one obtains several subclasses of Σ; we record two instances.

Example 1. *If $q \to 1^-$, then $\mathfrak{R}_{\Sigma}^{\mu,x}(\varrho,\varphi,\lambda;\wp;q)$ reduces to $\mathfrak{H}_{\Sigma}^{\mu}(x,\wp,\varrho,\lambda)$, the family of $f \in \Sigma$ satisfying*

$$(1-\wp)\frac{\mathcal{F}_{\varrho,\lambda}(f(\psi))}{\psi} + \wp\,(\mathcal{F}_{\varrho,\lambda}(f(\psi)))' \prec \mathfrak{G}_{\mu}(x,\psi), \tag{9}$$

and similarly for the inverse.

Example 2. *If, in addition, $\wp = 1$, one obtains $\mathfrak{H}_{\Sigma}^{\mu}(x,\varrho)$, i.e. the class of $f \in \Sigma$ for which*

$$(\mathcal{F}_{\varrho,\lambda}(f(\psi)))' \prec \mathfrak{G}_{\mu}(x,\psi), \tag{10}$$

and the analogous condition holds for $g = f^{-1}$.

We now state coefficient estimates for the class $\mathfrak{R}_{\Sigma}^{\mu,x}(\varrho,\varphi,\lambda;\wp;q)$.

Theorem 1. *Let $f\in\Sigma$ given by (1) belong to $\mathfrak{R}_{\Sigma}^{\mu,x}(\varrho,\varphi,\lambda;\wp;q)$. Then*

$$|a_2|\le\frac{2\big|[\mu]_q\big|\,x\,\wp_q\big(2(1+\varrho)\big)\sqrt{2[2]_q[\mu]_q\,\wp_q\big(3(1+\varrho)\big)\,x}}{\sqrt{\mathcal{F}(\varrho,\mu,\wp;q)\,x^2+[2]_q[\lambda]_q^2[\varphi]_q^2(1+q\wp)^2\big(\wp_q(1+\varrho)\big)^2\wp_q\big(3(1+\varrho)\big)[\mu]_{q^2}}},$$

and

$$|a_3|\le\frac{4[\mu]^2\big(\wp_q\big(2(1+\varrho)\big)\big)^2x^2}{(1+q\wp)^2[\varphi]_q^2[\lambda]_q^2\big(\wp_q(1+\varrho)\big)^2}+\frac{2\big|[\mu]_q\big|\,[2]_q\,\wp_q\big(3(1+\varrho)\big)\,x}{(1+q[2]_q\wp)[\varphi]_q[\varphi+1]_q[\lambda]_q^2\wp_q(1+\varrho)},$$

where

$$\mathcal{F}(\varrho,\varphi,\mu,\wp;q)=2[\lambda]_q^2[\varphi]_q\wp_q(1+\varrho)\Big(2(1+q[2]_q\wp)[\varphi+1]_q\big(\wp_q\big(2(1+\varrho)\big)\big)^2[\mu]_q^2 - [2]_q\wp_q\big(3(1+\varrho)\big)(1+q\wp)^2[\varphi]_q\wp_q(1+\varrho)\big([\mu]_{q^2}+[\mu]_q^2\big)\Big).$$

Proof. Let $f\in\mathfrak{R}_{\Sigma}^{\mu,x}(\varrho,\varphi,\lambda;\wp;q)$. Then there exist Schwarz functions φ and v with $\varphi(0)=v(0)=0$ such that

$$(1-\wp)\frac{\mathcal{F}_{\varrho,\lambda}^{\varphi}(f(\psi);q)}{\psi}+\wp\,\partial_q\Big(\mathcal{F}_{\varrho,\lambda}^{\varphi}(f(\psi);q)\Big)\prec\mathfrak{G}_q^{(\mu)}(x,\varphi(\psi)) \tag{11}$$

and

$$(1-\wp)\frac{\mathcal{F}_{\varrho,\lambda}^{\varphi}(g(w);q)}{w}+\wp\,\partial_q\Big(\mathcal{F}_{\varrho,\lambda}^{\varphi}(g(w);q)\Big)\prec\mathfrak{G}_q^{(\mu)}(x,v(w)). \tag{12}$$

Expanding both sides and comparing coefficients yield

$$(1-\wp)\frac{\mathcal{F}_{\varrho,\lambda}^{\varphi}(f(\psi);q)}{\psi}+\wp\,\partial_q\Big(\mathcal{F}_{\varrho,\lambda}^{\varphi}(f(\psi);q)\Big)=1+C_1^{(\mu)}(x;q)c_1\psi+\big[C_1^{(\mu)}(x;q)c_2+C_2^{(\mu)}(x;q)c_1^2\big]\psi^2+\cdots, \tag{13}$$

and

$$(1-\wp)\frac{\mathcal{F}_{\varrho,\lambda}^{\varphi}(g(w);q)}{w}+\wp\,\partial_q\Big(\mathcal{F}_{\varrho,\lambda}^{\varphi}(g(w);q)\Big)=1+C_1^{(\mu)}(x;q)d_1w+\big[C_1^{(\mu)}(x;q)d_2+C_2^{(\mu)}(x;q)d_1^2\big]w^2+\cdots. \tag{14}$$

If $\varphi(\psi)=c_1\psi+c_2\psi^2+\cdots$ and $v(w)=d_1w+d_2w^2+\cdots$ are Schwarz functions, then $|c_j|\le1$ and $|d_j|\le1$ for all $j\in\mathbb{N}$. Coefficient comparison in (13) and (14) gives

$$\frac{(1+q\wp)[\varphi]_q[\lambda]_q\wp_q(1+\varrho)}{\wp_q(2(1+\varrho))}a_2=C_1^{(\mu)}(x;q)c_1, \tag{15}$$

$$\frac{(1+q[2]_q\wp)[\varphi]_q[\varphi+1]_q[\lambda]_q^2\wp_q(1+\varrho)}{[2]_q\wp_q(3(1+\varrho))}a_3=C_1^{(\mu)}(x;q)c_2+C_2^{(\mu)}(x;q)c_1^2, \tag{16}$$

$$-\frac{(1+q\wp)[\varphi]_q[\lambda]_q\wp_q(1+\varrho)}{\wp_q(2(1+\varrho))}a_2 = C_1^{(\mu)}(x;q)d_1, \tag{17}$$

and

$$\frac{(1+q[2]_q\wp)[\varphi]_q[\varphi+1]_q[\lambda]_q^2\wp_q(1+\varrho)}{[2]_q\wp_q(3(1+\varrho))}\left(2a_2^2-a_3\right) = C_1^{(\mu)}(x;q)d_2 + C_2^{(\mu)}(x;q)d_1^2. \tag{18}$$

From (15) and (17) we obtain $c_1 = -d_1$ and

$$2\left(\frac{(1+q\wp)[\varphi]_q[\lambda]_q\wp_q(1+\varrho)}{\wp_q(2(1+\varrho))}\right)^2 a_2^2 = \left[C_1^{(\mu)}(x;q)\right]^2\left(c_1^2+d_1^2\right). \tag{19}$$

Adding (16) and (18) gives

$$\frac{2(1+q[2]_q\wp)[\varphi]_q[\varphi+1]_q[\lambda]_q^2\wp_q(1+\varrho)}{[2]_q\wp_q(3(1+\varrho))}a_2^2 = C_1^{(\mu)}(x;q)(c_2+d_2) + C_2^{(\mu)}(x;q)(c_1^2+d_1^2). \tag{20}$$

Substituting (19) into (20) and using $|c_j|, |d_j| \leq 1$ yields the bound for $|a_2|$. Subtracting (18) from (16) and using (19) similarly leads to the stated estimate for $|a_3|$.

Using the values of a_2^2 and a_3 obtained above, we now derive a Fekete–Szegö type inequality for our class.

3. Fekete–Szegö inequality

For a univalent function f, Fekete and Szegö [16] established in 1933 sharp bounds for the functional $\eta a_2^2 - a_3$ with real $0 \leq \eta \leq 1$. Determining sharp (or effective) bounds for this functional on various subclasses of $\mathcal{A}$ has since become a classical problem.

Theorem 2. *Let $f \in \Sigma$ given by (1) belong to $\mathfrak{R}_\Sigma^{\mu,x}(\varrho,\varphi,\lambda;\wp;q)$. Then*

$$\left|a_3-\eta a_2^2\right| \leq \begin{cases} \dfrac{2[2]_q\wp_q\big(3(1+\varrho)\big)\left|[\mu]_q\right| x}{(1+q[2]_q\wp)[\varphi]_q[\varphi+1]_q[\lambda]_q^2\wp_q(1+\varrho)}, & |\eta-1| \leq \mathcal{Q}_q^{\mu,x}(\varrho,\varphi,\lambda;\wp), \\ \dfrac{8\left|[\mu]_q\right|^3 x^3[2]_q\wp_q\big(3(1+\varrho)\big)\big(\wp_q\big(2(1+\varrho)\big)\big)^2|\eta-1|}{\left|\mathcal{F}(\varrho,\mu,\wp;q)x^2+[2]_q[\lambda]_q^2[\varphi]_q^2(1+q\wp)^2\big(\wp_q(1+\varrho)\big)^2\wp_q\big(3(1+\varrho)\big)[\mu]_{q^2}\right|}, & |\eta-1| \geq \mathcal{Q}_q^{\mu,x}(\varrho,\varphi,\lambda;\wp), \end{cases}$$

where

$$\mathcal{F}(\varrho,\varphi,\mu,\wp;q) = 2[\lambda]_q^2[\varphi]_q\wp_q(1+\varrho)\Big(2(1+q[2]_q\wp)[\varphi+1]_q\big(\wp_q\big(2(1+\varrho)\big)\big)^2[\mu]_q^2 \\ - [2]_q\wp_q\big(3(1+\varrho)\big)(1+q\wp)^2[\varphi]_q\wp_q(1+\varrho)\big([\mu]_{q^2}+[\mu]_q^2\big)\Big),$$

and

$$\mathcal{Q}_q^{\mu,x}(\varrho,\varphi,\lambda;\wp) = \left|\frac{\mathcal{F}(\varrho,\mu,\wp;q)x^2+[2]_q[\lambda]_q^2[\varphi]_q^2(1+q\wp)^2\big(\wp_q(1+\varrho)\big)^2\wp_q\big(3(1+\varrho)\big)[\mu]_{q^2}}{2(1+q[2]_q\wp)[\varphi]_q[\varphi+1]_q[\lambda]_q^2\wp_q(1+\varrho)\,2[\mu]_q^2x^2\big(\wp_q\big(2(1+\varrho)\big)\big)^2}\right|.$$

Proof. The estimate follows by combining (20) and (18), arranging terms as in the classical argument, and using the bounds for the Schwarz coefficients together with (5).

4. Corollaries and consequences

In view of Examples 1 and 2, Theorems 1 and 2 immediately yield the following corollaries (details omitted).

Corollary 1. *Let $f \in \Sigma$ given by (1) belong to $\mathfrak{H}_{\Sigma}^{\mu}(x, \wp, \varrho, \lambda)$. Then*

$$|a_2| \leq \frac{4|\mu|\, x \sqrt{\wp\big(3(1+\varrho)\big)\, [\wp(2(1+\varrho))]^2\, |\mu|\, x}}{\sqrt{\left|(\digamma(\mu, \wp, \varrho, 0, \lambda)x^2 + (1+2\wp)^2 \wp(1+\varrho)\wp(3(1+\varrho)))\, 2\lambda^2 \wp(1+\varrho)\mu\right|}},$$

$$|a_3| \leq \frac{4\mu^2 x^2 [\wp(2(1+\varrho))]^2}{\lambda^2 (1+\wp)^2 [\wp(1+\varrho)]^2} + \frac{2|\mu|\, x\, \wp(3(1+\varrho))}{(1+2\wp)\lambda^2 \wp(1+\varrho)},$$

and

$$|a_3 - \eta a_2^2| \leq \begin{cases} \dfrac{2\wp(3(1+\varrho))|\mu|\, x}{(1+2\wp)\lambda^2 \wp(1+\varrho)}, & |\eta - 1| \leq Q(\mu, \wp, \varrho, 0, \lambda), \\ \dfrac{8\mu^2 x^3 \wp(3(1+\varrho))[\wp(2(1+\varrho))]^2 |\eta - 1|}{\lambda^2 \wp(1+\varrho)\, |L(\mu, \wp, \varrho, 0, \lambda)|}, & |\eta - 1| \geq Q(\mu, \wp, \varrho, 0, \lambda), \end{cases}$$

where

$$\digamma(\mu, \wp, \varrho, 0, \lambda) = 4\mu(1+2\wp)[\wp(2(1+\varrho))]^2 - 2(1+\mu)(1+2\wp)^2 \wp(1+\varrho)\wp(3(1+\varrho)),$$
$$L(\mu, \wp, \varrho, 0, \lambda) = 4\mu x^2 (1+2\wp)[\wp(2(1+\varrho))]^2 - (1+\wp)^2 \wp(1+\varrho)\wp(3(1+\varrho))\, (2(1+\mu)x^2 - 1).$$

Corollary 2. *Let $f \in \Sigma$ given by (1) belong to $\mathfrak{G}_{\Sigma}^{\mu}(x, \varrho, \lambda)$. Then*

$$|a_2| \leq \frac{4|\mu|\, x \sqrt{\wp(3(1+\varrho))[\wp(2(1+\varrho))]^2\, |\mu|\, x}}{\sqrt{\left|(\digamma(\mu, 1, \varrho, 0, \lambda)x^2 + 9\wp(1+\varrho)\wp(3(1+\varrho)))\, 2\lambda^2 \wp(1+\varrho)\mu\right|}},$$

$$|a_3| \leq \frac{\mu^2 x^2 [\wp(2(1+\varrho))]^2}{\lambda^2 [\wp(1+\varrho)]^2} + \frac{2|\mu|\, x\, \wp(3(1+\varrho))}{3\lambda^2 \wp(1+\varrho)},$$

and

$$|a_3 - \eta a_2^2| \leq \begin{cases} \dfrac{2\wp(3(1+\varrho))|\mu|\, x}{3\lambda^2 \wp(1+\varrho)}, & |\eta - 1| \leq Q(\mu, 1, \varrho, 0, \lambda), \\ \dfrac{8\mu^2 x^3 \wp(3(1+\varrho))[\wp(2(1+\varrho))]^2 |\eta - 1|}{\lambda^2 \wp(1+\varrho)\, |L(\mu, 1, \varrho, 0, \lambda)|}, & |\eta - 1| \geq Q(\mu, 1, \varrho, 0, \lambda), \end{cases}$$

where

$$\digamma(\mu, 1, \varrho, 0, \lambda) = 12\mu[\wp(2(1+\varrho))]^2 - 18(1+\mu)\wp(1+\varrho)\wp(3(1+\varrho)),$$
$$L(\mu, 1, \varrho, 0, \lambda) = 12\mu x^2 [\wp(2(1+\varrho))]^2 - 4\wp(1+\varrho)\wp(3(1+\varrho))\, (2(1+\mu)x^2 - 1).$$

Remark 1. *The foregoing results specialize to several other families. For instance, taking suitable parameters yields the classes $\mathfrak{R}_{\Sigma}^{1}(x, \wp, \varrho, \delta, \lambda)$ (Chebyshev polynomials) and $\mathfrak{R}_{\Sigma}^{0.5}(x, \wp, \varrho, \delta, \lambda)$ (Legendre polynomials).*

5. Conclusions

We proposed and analyzed a new family of normalized analytic bi-univalent functions, denoted by $\mathfrak{R}_{\Sigma}^{\mu,x}(\varrho,\varphi,\lambda;\gamma;q)$, constructed via a normalized q–Rabotnov series. For functions in this class we derived sharp bounds for the initial Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ and established a Fekete–Szegö type estimate. Upon suitable choices of parameters, our results recover the subclasses $\mathfrak{H}_{\Sigma}^{\mu}(x,\wp,\varrho,\lambda)$ and $\mathfrak{G}_{\Sigma}^{\mu}(x,\varrho,\lambda)$. Extending the coefficient estimates to $|a_n|$ for $n \geq 4$ appears to be a natural continuation of the present work. We expect that further exploration of operators built from generalized special functions will lead to additional subclasses of analytic (bi-)univalent functions with comparable bounds.

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