



Submodular Lattice Isomorphisms Between Some Modules Over the Ring of Rational Functions

Gantina Rachmaputri^{1,*}, Pudji Astuti¹, Ahmad Muchlis¹, Hanni Garminia¹

¹ *Algebra Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Bandung, Jawa Barat, Indonesia*

Abstract. Fuhrmann's work introduced a lattice isomorphism between polynomial submodules and closed formal series submodules, which plays a crucial role in the study of discrete linear systems within the behavioral framework. However, existing studies primarily focus on causal discrete systems, leaving a gap in the analysis of anti-causal systems. This paper extends Willems's behavioral approach by establishing a lattice isomorphism between finitely generated submodules of the polynomial module and full-rank submodules of a free module over the ring of proper rational functions. The results provide a unifying algebraic structure that accommodates both causal and anti-causal systems. This generalization enhances the applicability of the behavioral framework and contributes to the ongoing development of algebraic system theory.

2020 Mathematics Subject Classifications: 16D80

Key Words and Phrases: Module over the ring of rational functions, bilinear form, lattice isomorphism

1. Introduction

Dynamical systems are commonly investigated through input and output frameworks. An approach in the study of dynamical systems, called behavioral framework, has been rapidly adopted since its introduction by Willems [1, 2]. Fuhrmann [3] borrowed the idea and introduced a behavior of a linear, discrete and time invariant system as a shift-invariant, complete subspace of the space of m -tuples of formal series functions. Fuhrmann established a duality relation between submodules in m -dimensional polynomial module and discrete dynamical systems in m -dimensional formal power series vanishing at infinity module. A detailed analysis of a related duality theory, in the context of multidimensional systems, is available in Oberst [4].

Behaviors of discrete and time invariant systems can be thought of as generalizations of Fuhrmann's rational models of the systems. One of the key results in Fuhrmann's

*Corresponding author.

DOI: <https://doi.org/10.29020/nybg.ejpam.v19i1.5441>

Email addresses: gantina.r@itb.ac.id (G. Rachmaputri), pudji.astuti@itb.ac.id (P. Astuti), muchlis@itb.ac.id (A. Muchlis), garminia@itb.ac.id (H. Garminia)

theory is a lattice isomorphism theorem between the collection of polynomial submodules and the collection of behaviors under investigation [3, 5, 6]. This key result has led to the development of behavior homomorphisms and investigation on their properties utilizing the existing results from the flourishing studies of linear systems via rational models.

The development of Fuhrmann's rational models are associated to causal discrete linear systems, which in turn are associated to backward shift operators. In the language of module theory, Fuhrmann's rational models are modules over a polynomial ring. Meanwhile, there are some situations that can be represented mathematically as dynamical systems in descriptor forms whose transfer functions are not causal for they contain polynomial parts. Realization theory of the anti-causal parts of such systems have been studied, see, for example, Conte and Perdon [7, 8] and Wimmer [9]. Those systems are associated with forward shift operators. In the language of module theory, they are modules over a ring of formal series or proper rational functions. The rational model of an anti-causal subsystem as a module over a ring of formal series functions and its dual submodule have also been developed [10] by adopting Fuhrmann's treatment of a causal discrete linear system [6]. The structural description of anticausal behaviors as finitely generated modules over formal series rings was previously established in [11]. The current work extends this framework by introducing a lattice-theoretic duality and establishing an isomorphism with full-rank submodules over the ring of proper rational functions. Moreover, the commutant lifting theorem of modules over a polynomial ring that plays an important role in the study of behavior homomorphisms [3] has also been obtained its parallel result in the context of modules over a ring of rational proper functions [12].

In many algebraic treatments of dynamical systems, polynomial matrices play a crucial role, particularly in system representation and transformation. A non-singular polynomial matrix is a square matrix with polynomial entries whose determinant is a nonzero polynomial. More formally, given a field F , a matrix $P(z) \in F[z]^{m \times m}$ is said to be non-singular if its determinant, $\det(P(z))$, is a nonzero polynomial in $F[z]$. This property ensures that $P(z)$ is invertible over the field of rational functions $F(z)$, meaning there exists another matrix $P^{-1}(z) \in F(z)^{m \times m}$ such that $P^{-1}(z)P(z) = I_m$ where I_m is the identity matrix of size $m \times m$. The use of non-singular polynomial matrices is particularly relevant in Fuhrmann's framework, where they are employed to establish isomorphic relations between different submodules. Their invertibility ensures that behavior homomorphisms remain well-defined and that the lattice structure under consideration is preserved. In this paper, non-singular polynomial matrices are used in the derivation of key results, particularly in the proof of 1, to ensure that the transformation between submodules remains consistent and mathematically rigorous.

Studying the above facts, it is of interest to obtain a parallel result of the above mentioned Fuhrmann's lattice isomorphism theorem in the context of lattices of submodules of modules over a ring of proper rational functions, and that is the aim of this paper.

2. Bilinear Forms

Let \mathbb{F} be a field. As usual, \mathbb{F}^m denotes the space of all m -tuples with components in \mathbb{F} , and $\mathbb{F}[z]$ denotes the ring of all polynomials with coefficients in \mathbb{F} . Let $\mathbb{F}(z)$ denote the rational field of $\mathbb{F}[z]$ consisting of all functions in the form of $\frac{p}{q}$, where $p, q \in \mathbb{F}[z], q \neq 0$. A rational function $f = \frac{p}{q} \in \mathbb{F}(z)$, where $p, q \in \mathbb{F}[z], q \neq 0$, is called *proper* if $f = 0$ or $\deg(p) \leq \deg(q)$. Under the addition and multiplication operations in $\mathbb{F}(z)$, the set of all proper rational functions forms a discrete valuation domain with the unique prime element z^{-1} . We denote this domain with $\mathbb{F}_\infty(z)$. Clearly, the field $\mathbb{F}(z)$ can be decomposed into two parts, the polynomial part and the strictly proper part, as follows

$$\mathbb{F}(z) = \mathbb{F}[z] \oplus z^{-1}\mathbb{F}_\infty(z).$$

The above notations and decomposition extend naturally to the rational n -tuple space $\mathbb{F}(z)^n$ and to the rational matrix space $\mathbb{F}(z)^{m \times n}$.

Fuhrmann’s lattice isomorphism theorem was developed using a bilinear form on the truncated Laurent series space. Meanwhile, to obtain duality property, Wimmer [10] utilized a certain non-degenerate bilinear form on a Wimmer’s rational model. Hence, it is natural to suspect that a result parallel to Fuhrmann’s lattice isomorphism theorem can be obtained using a bilinear form that induces Wimmer’s bilinear form.

Through Lemma 1 and Lemma 2 below, we will show that Wimmer’s bilinear form can be derived from the following bilinear form on the space of rational functions:

$$\begin{aligned} \psi : \mathbb{F}(z)^m \times \mathbb{F}(z)^m &\longrightarrow \mathbb{F} \\ (\bar{x}, \bar{y}) &\longmapsto (\bar{y}^t \bar{x})_0, \end{aligned}$$

where \bar{y}^t is the transpose of $\bar{y} \in \mathbb{F}(z)^m$ and $(\bar{y}^t \bar{x})_0$ denotes the coefficient of z^0 in $\bar{y}^t \bar{x}$. Using this bilinear form, we construct a bilinear form on the space of all proper rational functions.

Lemma 1. *Consider $\mathbb{F}_\infty(z)^m$ as a vector space over \mathbb{F} and let $L \in \mathbb{F}[z]^{m \times m}$ be a non singular polynomial matrix. The mapping θ_L defined as*

$$\begin{aligned} \theta_L : \mathbb{F}_\infty(z)^m \times \mathbb{F}_\infty(z)^m &\longrightarrow \mathbb{F} \\ (\bar{x}, \bar{y}) &\longmapsto \psi(L^{-1}\bar{x}, \bar{y}) \end{aligned}$$

is a bilinear form on $\mathbb{F}_\infty(z)^m$ with the set of all degenerate elements of θ_L is

$$\mathbb{F}_\infty(z)^m \cap z^{-1}L\mathbb{F}_\infty(z)^m.$$

Proof. We see immediately that θ_L is a bilinear function. We will show that its degenerate subspace is

$$\mathbb{F}_\infty(z)^m \cap z^{-1}L\mathbb{F}_\infty(z)^m.$$

Let $\bar{x} = L\bar{f} \in \mathbb{F}_\infty(z)^m \cap z^{-1}L\mathbb{F}_\infty(z)^m$, for some $\bar{f} \in z^{-1}\mathbb{F}_\infty(z)^m$. It follows that $\theta_L(\bar{x}, \bar{y}) = 0$, for all $\bar{y} \in \mathbb{F}_\infty(z)^m$, hence \bar{x} is degenerate.

Conversely, let \bar{x} is degenerate. Write $L^{-1}\bar{x} = \bar{a} + z^{-1}\bar{b}$, where $\bar{a} \in \mathbb{F}[z]^m$ and $\bar{b} \in \mathbb{F}_\infty(z)^m$. Suppose

$$\bar{a} = (a_1 \ a_2 \ a_3 \ \dots \ a_m)^t \neq \bar{0},$$

for some $a_j \in \mathbb{F}[z]$ for $j = 1, 2, 3, \dots, m$. Let $a_i \neq 0$ for some $i \in \{1, 2, 3, \dots, m\}$. We can choose

$$\bar{y} = \left(0 \ \dots \ 0 \ \frac{1}{a_i} \ 0 \ \dots \ 0 \right)^t$$

in $\mathbb{F}_\infty(z)^m$, where $\frac{1}{a_i}$ is in the i -th position. It follows that $\theta_L(\bar{x}, \bar{y}) = 1$, a contradiction to the fact that \bar{x} is degenerate. Hence $a_j = 0$ for all $j \in \{1, 2, 3, \dots, m\}$ which imply that $x \in z^{-1}L\mathbb{F}_\infty(z)^m$. This establishes the fact that $\mathbb{F}_\infty(z)^m \cap z^{-1}L\mathbb{F}_\infty(z)^m$ is the set of all degenerate elements of bilinear function θ_L .

The decomposition $\mathbb{F}(z)^m = \mathbb{F}[z]^m \oplus z^{-1}\mathbb{F}_\infty(z)^m$ leads to the projection map π_+ from $\mathbb{F}(z)^m$ onto $\mathbb{F}[z]^m$. Next, we define a linear map

$$\begin{aligned} \rho^L : \mathbb{F}_\infty(z)^m &\longrightarrow \mathbb{F}[z]^m \\ \bar{y} &\longmapsto L\pi_+L^{-1}\bar{y}. \end{aligned}$$

Letting $U^L = \text{Im}(\rho^L)$ and observing that the kernel of ρ^L is $\mathbb{F}_\infty(z)^m \cap Lz^{-1}\mathbb{F}_\infty(z)^m$, we obtain that

$$\mathbb{F}_\infty(z)^m / (\mathbb{F}_\infty(z)^m \cap Lz^{-1}\mathbb{F}_\infty(z)^m)$$

is isomorphic to U^L . We use this isomorphic relation to define a new bilinear form. To each $(\bar{x}, \bar{y}) \in U^L \times U^{L^t}$ we assign $\varphi(\bar{x}, \bar{y}) = \theta_L(\bar{g}, \bar{h}) \in \mathbb{F}$, where $\bar{x} = \rho^L(\bar{g})$ and $\bar{y} = \rho^{L^t}(\bar{h})$. We then have

Lemma 2. φ is a non-degenerate bilinear form.

Proof. To begin with, we will show that the map φ is well defined. Let $\bar{g}, \bar{g}_1, \bar{h}$, and \bar{h}_1 are elements in $\mathbb{F}_\infty(z)^m$ satisfy $\rho^L(\bar{g}) = \rho^L(\bar{g}_1)$ and $\rho^{L^t}(\bar{h}) = \rho^{L^t}(\bar{h}_1)$. Then

$$\bar{g} - \bar{g}_1 \in \mathbb{F}_\infty(z)^m \cap Lz^{-1}\mathbb{F}_\infty(z)^m \text{ and } \bar{h} - \bar{h}_1 \in \mathbb{F}_\infty(z)^m \cap L^t z^{-1}\mathbb{F}_\infty(z)^m.$$

Hence,

$$\bar{g}_1 = \bar{g} + \bar{s} \quad \text{and} \quad \bar{h}_1 = \bar{h} + \bar{r},$$

for some $\bar{s} \in \mathbb{F}_\infty(z)^m \cap L^t z^{-1}\mathbb{F}_\infty(z)^m$ and $\bar{r} \in \mathbb{F}_\infty(z)^m \cap Lz^{-1}\mathbb{F}_\infty(z)^m$. As a result

$$\begin{aligned} \theta_L(\bar{g}_1, \bar{h}_1) &= \theta_L(\bar{g} + \bar{s}, \bar{h} + \bar{r}) \\ &= \theta_L(\bar{g}, \bar{h}) + \theta_L(\bar{g}, \bar{r}) + \theta_L(\bar{s}, \bar{h}) + \theta_L(\bar{s}, \bar{r}) \\ &= \theta_L(\bar{g}, \bar{h}). \end{aligned}$$

Thus the mapping φ is well defined.

It is a routine to show that φ is bilinear. Hence it remains to show that it is non-degenerate.

Let $\bar{x} \in U^L$ be a degenerate element. Suppose $\bar{x} = \rho^L(\bar{g})$ for some $\bar{g} \in \mathbb{F}_\infty(z)$. We will show that $\bar{x} = \bar{0}$.

We have $\langle \bar{x}, \bar{y} \rangle = 0$ for all $\bar{y} \in U^{L^t}$. Since any element $\bar{y} \in U^{L^t}$ is of the form $\bar{y} = \rho^{L^t}(\bar{h})$ for some $\bar{h} \in \mathbb{F}_\infty(z)^m$, we obtain $\theta_L(\bar{g}, \bar{h}) = \langle \bar{x}, \bar{y} \rangle = \bar{0}$ for all $\bar{h} \in \mathbb{F}_\infty(z)^m$. As a consequence $\bar{g} \in \mathbb{F}_\infty(z)^m \cap Lz^{-1}\mathbb{F}_\infty(z)^m$ and it clearly forces $\bar{x} = \bar{0}$. Thus there is no non-zero degenerate element and the proof is complete.

The bilinear form φ introduced in this paper coincides with the bilinear form $\langle \cdot, \cdot \rangle$ in [10], which is originally defined as a duality-preserving inner product on the module $\mathbb{F}_\infty(z)^m$. Specifically, in [10], the bilinear form is given by

$$\langle \bar{x}, \bar{y} \rangle = \theta_L(\bar{x}, \bar{y}).$$

In this work, we constructed φ via the same bilinear form θ_L , ensuring that φ inherits the same properties and serves as an extension of the formulation in [10]. Furthermore, this result aligns with the duality pairing established in Lemma 2.2 of [10], where the bilinear form is defined as $(u, w) = (y^T Lx)$, for $u \in U_L$ and $w \in U_L^T$ with corresponding pre-images $x = (p_L)^{-1}(u)$ and $y = (p_L^T)^{-1}(w)$. Theorem 2.2 in [10] further establishes that this bilinear form is non-degenerate and provides a natural pairing between the bases of U_L and U_L^T . This confirms that our construction of φ is consistent with prior work and extends the duality framework introduced in [10], thereby maintaining coherence with the established theory of polynomial matrices and module dualities.

3. Laurent Series Submodules

Throughout this and subsequent sections, we will consider the sets $\mathbb{F}(z)^m$, $\mathbb{F}[z]^m$, and $\mathbb{F}_\infty(z)^m$ as modules over $\mathbb{F}_\infty(z)$. We will define a class of annihilator submodules on $\mathbb{F}(z)^m$ and investigate some properties of this class to be used later.

Notice that the ring of proper rational functions $\mathbb{F}_\infty(z)$ is a principal ideal domain and $\mathbb{F}_\infty(z)^m$ is a free module over $\mathbb{F}_\infty(z)$. Also, $\mathbb{F}(z)^m$ is an $\mathbb{F}_\infty(z)$ -module. The additive group $\mathbb{F}[z]^m$ becomes an $\mathbb{F}_\infty(z)$ -module by defining the following action of $\mathbb{F}_\infty(z)$ on $\mathbb{F}[z]^m$: $\alpha \cdot \bar{x} = \pi_+(\alpha \bar{x})$, for every $\alpha \in \mathbb{F}_\infty(z)$ and $\bar{x} \in \mathbb{F}[z]^m$. The following lemma is just a special case of a property of finitely generated free modules over a principal ideal domain.

Lemma 3. *A subset $A \subseteq \mathbb{F}_\infty(z)^m$ is a submodule of $\mathbb{F}_\infty(z)^m$ if and only if $A = D\mathbb{F}_\infty(z)^m$, for some matrix $D \in \mathbb{F}_\infty(z)^{m \times m}$.*

Similar characterization holds for finitely generated submodules of $\mathbb{F}(z)^m$ as shown in the next lemma.

Lemma 4. *Let K be a submodule of $\mathbb{F}(z)^m$. Then K is finitely generated if and only if $K = B\mathbb{F}_\infty(z)^m$, for some matrix $B \in \mathbb{F}(z)^{m \times m}$.*

Proof. For any matrix $B \in \mathbb{F}(z)^{m \times m}$, it is obvious that the set $B\mathbb{F}_\infty(z)^m$ is a submodule of $\mathbb{F}(z)^m$ finitely generated by the set of columns of B .

Let K be a finitely generated submodule of $\mathbb{F}(z)^m$ with $S = \{\bar{a}_1, \dots, \bar{a}_k\} \subset K$ generates K . Write $\bar{a}_i = \bar{b}_i + z^{-1}\bar{c}_i$ for some $\bar{a}_i \in \mathbb{F}[z]^m$, $\bar{c}_i \in \mathbb{F}_\infty(z)^m$, $i = 1, \dots, k$ and let

$$n_0 = \max\{\deg(\bar{a}_i) : i = 1, \dots, k\}.$$

We obtain $z^{-n_0}\bar{a}_i \in \mathbb{F}_\infty(z)^m$ for all $i = 1, \dots, k$. Hence $z^{-n_0}K$ is a submodule of $\mathbb{F}_\infty(z)^m$. Applying Lemma 1 we obtain $z^{-n_0}K = B\mathbb{F}_\infty(z)^m$ for some $B \in \mathbb{F}_\infty(z)^{m \times m}$. Hence $K = \bar{B}\mathbb{F}_\infty(z)^m$ for some $\bar{B} = z^{n_0}B \in \mathbb{F}(z)^{m \times m}$.

For any subset S of $\mathbb{F}(z)^m$, the set $\{\bar{g} \in \mathbb{F}(z)^m \mid \psi(\bar{f}, \bar{g}) = 0, \text{ for all } \bar{f} \in M\}$ is called the *annihilator* of S , denoted with S^\perp . Notice that the annihilator of any subset of $\mathbb{F}(z)^m$ is a subspace of $\mathbb{F}(z)^m$.

The following two propositions are direct consequences of the definition and properties of the rational space $\mathbb{F}(z)$.

Proposition 1. *Let S and T be any subsets of $\mathbb{F}(z)^m$. Then:*

- (a) S^\perp is a subspace of $\mathbb{F}(z)^m$,
- (b) if $S \subseteq T$, then $T^\perp \subseteq S^\perp$, and
- (c) if S and T are subspaces of $\mathbb{F}(z)^m$, then

$$\begin{aligned} (S \cap T)^\perp &\supseteq S^\perp + T^\perp \text{ and} \\ (S + T)^\perp &= S^\perp \cap T^\perp. \end{aligned}$$

Proposition 2. *The annihilator of the subspace $\mathbb{F}_\infty(z)^m$ is $z^{-1}\mathbb{F}_\infty(z)^m$, that is,*

$$(\mathbb{F}_\infty(z)^m)^\perp = z^{-1}\mathbb{F}_\infty(z)^m.$$

Before proceeding further, we say that a $\mathbb{F}_\infty(z)$ -submodule K of $\mathbb{F}(z)^m$ is of *full rank* if $\text{rank}(K) = m$.

Lemma 5. *Let K be a full rank submodule of $\mathbb{F}_\infty(z)^m$. Then*

$$K^\perp = z^{-1} (C^t)^{-1} \mathbb{F}_\infty(z)^m,$$

for a full rank matrix $C \in \mathbb{F}_\infty(z)^{m \times m}$ satisfying $K = C\mathbb{F}_\infty(z)^m$.

Proof. Let K be a full rank submodule of $\mathbb{F}_\infty(z)^m$. By Lemma 3, $K = C\mathbb{F}_\infty(z)^m$ for a full rank matrix C in $\mathbb{F}_\infty(z)^{m \times m}$. By direct calculation,

$$z^{-1} (C^t)^{-1} \mathbb{F}_\infty(z)^m \subseteq K^\perp.$$

To prove that $K^\perp = z^{-1} (C^t)^{-1} \mathbb{F}_\infty(z)^m$ we show $K^\perp \subseteq z^{-1} (C^t)^{-1} \mathbb{F}_\infty(z)^m$.

Let $\bar{x} \in K^\perp$ and $C^t\bar{x} = \bar{a} + z^{-1}\bar{b}$, where $\bar{a} \in \mathbb{F}[z]^m$ and $\bar{b} \in \mathbb{F}_\infty(z)^m$. Suppose $\bar{a} = (a_1 \ a_2 \ a_3 \ \dots \ a_m)^t \neq \bar{0}$, where $a_k \in \mathbb{F}[z]$ for $k = 1, 2, 3, \dots, m$. Letting $a_i \neq 0$ for some $i \in \{1, 2, 3, \dots, m\}$, we choose $\bar{h} = \left(0 \ \dots \ 0 \ \frac{1}{a_i} \ 0 \ \dots \ 0\right)^t$ in $\mathbb{F}_\infty(z)^m$, whose non-zero component is in the i -th position. Then $C\bar{h} \in K$ which results in $\psi(\bar{x}, C\bar{h}) = 1$, contradicting the fact that $\bar{x} \in K^\perp$. Therefore, $a_k = 0$ for all $k \in \{1, 2, 3, \dots, m\}$ and consequently, $C^t\bar{x} \in z^{-1}\mathbb{F}_\infty(z)^m$. We thus get $\bar{x} \in z^{-1} (C^t)^{-1} \mathbb{F}_\infty(z)^m$, hence $K^\perp \subseteq z^{-1} (C^t)^{-1} \mathbb{F}_\infty(z)^m$, this completes the proof.

By applying Lemma 5, Lemma 4, Proposition 1, and Proposition 2 we immediately obtain the following corollary.

Corollary 1. *If K is a full rank submodule of $\mathbb{F}_\infty(z)^m$, then K^\perp is a finitely generated submodule of $\mathbb{F}(z)^m$ and $z^{-1}\mathbb{F}_\infty(z)^m \subseteq K^\perp$.*

For the rest of this section we will show that any full rank submodule of $\mathbb{F}_\infty(z)^m$ is closed in the sense that it is equal to the double annihilator of itself.

Lemma 6. *Let K be a finitely generated submodule of $\mathbb{F}(z)^m$. If K contains $z^{-1}\mathbb{F}_\infty(z)^m$, then K^\perp is a full rank submodule of $\mathbb{F}_\infty(z)^m$.*

Proof. The annihilator K^\perp is a subspace. Hence, to show K^\perp is submodule of $\mathbb{F}_\infty(z)^m$, it is sufficient to show that the ring action is closed in K^\perp and which is true since K is a submodule. Let $\alpha \in \mathbb{F}_\infty(z)$ and $\bar{x} \in K^\perp$. For any $\bar{a} \in K$, we obtain $\psi(\alpha\bar{x}, \bar{a}) = \psi(\bar{x}, \alpha\bar{a}) = 0$.

Using similar approach used in the proof of Lemma 4, the inclusion

$$z^{-1}\mathbb{F}_\infty(z)^m \subseteq K \subseteq z^k\mathbb{F}_\infty(z)^m$$

holds for some nonnegative integer k . By Proposition 1 and Proposition 2, it follows that

$$z^{-k-1}\mathbb{F}_\infty(z)^m \subseteq K^\perp \subseteq \mathbb{F}_\infty(z)^m$$

and consequently K^\perp has a full rank. Thus K^\perp is a full rank submodule of $\mathbb{F}_\infty(z)^m$ as required.

Lemma 7. *If K is a full rank submodule of $\mathbb{F}_\infty(z)^m$, then $(K^\perp)^\perp = K$.*

Proof. Let K be a full rank submodule of $\mathbb{F}_\infty(z)^m$. It is obvious that $K \subseteq (K^\perp)^\perp$. Hence it remains to show that $(K^\perp)^\perp \subseteq K$.

According to Lemma 3 and Lemma 5,

$$K = D\mathbb{F}_\infty(z)^m \quad \text{and} \quad K^\perp = z^{-1}(D^t)^{-1}\mathbb{F}_\infty(z)^m$$

for a full rank matrix D in $\mathbb{F}_\infty(z)^{m \times m}$. Let $\bar{x} \in (z^{-1}(D^t)^{-1}\mathbb{F}_\infty(z)^m)^\perp$. Write

$$z^{-1}D^{-1}\bar{x} = \bar{a} + z^{-1}\bar{b} \in \mathbb{F}(z)^m$$

for some $\bar{a} \in \mathbb{F}[z]^m$ and $\bar{b} \in \mathbb{F}_\infty(z)^m$. Suppose $\bar{a} = (a_1 \ a_2 \ a_3 \ \dots \ a_m)^t \neq \bar{0}$, where $a_i \in \mathbb{F}[z]$ for $i = 1, 2, 3, \dots, m$. Let $a_j \neq 0$ for some j in $\{1, 2, 3, \dots, m\}$. Let $\bar{g} = \left(0 \ \dots \ 0 \ \frac{1}{a_j} \ 0 \ \dots \ 0\right)^t \in \mathbb{F}_\infty(z)^m$ whose non-zero component is in the j -th position. By choosing $\bar{f} = z^{-1}(D^t)^{-1}\bar{g} \in z^{-1}(D^t)^{-1}\mathbb{F}_\infty(z)^m$, we have $\psi(\bar{x}, \bar{f}) = 1$. This contradicts the fact that $\bar{x} \in (z^{-1}(D^t)^{-1}\mathbb{F}_\infty(z)^m)^\perp$. Hence $\bar{a} = \bar{0}$. Thus $z^{-1}D^{-1}\bar{x} \in z^{-1}\mathbb{F}_\infty(z)^m$ and, consequently, $\bar{x} \in D\mathbb{F}_\infty(z)^m$. Therefore, $(K^\perp)^\perp \subseteq K$, and the lemma follows.

Using a similar approach as in the proof of Lemma 6, we obtain the following lemma.

Lemma 8. *Let K be a finitely generated submodule of $\mathbb{F}(z)^m$. If K contains $z^{-1}\mathbb{F}_\infty(z)^m$, then $(K^\perp)^\perp = K$.*

In the next section, we will use the last four lemmas above to construct some lattice isomorphisms of submodules over the ring of proper rational functions.

4. Lattice Isomorphisms

We consider the collection \mathbb{A} of all finitely generated submodules of $\mathbb{F}(z)^m$ containing $z^{-1}\mathbb{F}_\infty(z)^m$ as a partially ordered set (poset) under inclusion. In this poset, we define two binary operations $\vee_{\mathbb{A}}$ and $\wedge_{\mathbb{A}}$ by taking, for every $G, H \in \mathbb{A}$, the smallest submodule in the poset that contains both G and H as $G \vee_{\mathbb{A}} H$ and the largest submodule in the poset that is contained in both G and H as $G \wedge_{\mathbb{A}} H$. It is clear that $G \vee_{\mathbb{A}} H = G + H$ and $G \wedge_{\mathbb{A}} H = G \cap H$. The system $(\mathbb{A}; \vee_{\mathbb{A}}, \wedge_{\mathbb{A}})$ is then a lattice.

In similar fashion, we construct another lattice $(\mathbb{B}; \vee_{\mathbb{B}}, \wedge_{\mathbb{B}})$, where \mathbb{B} is the collection of all finitely generated $\mathbb{F}_\infty(z)$ -submodules of $\mathbb{F}[z]^m$.

The third lattice has the collection \mathbb{C} of all full rank submodules of $\mathbb{F}_\infty(z)^m$ with containment (\supseteq) as its underlying partial order. Since the order in \mathbb{C} is a reverse of the order in \mathbb{A} and \mathbb{B} , we interchange the two lattice operations. We define $G \wedge_{\mathbb{C}} H$ to be the largest submodule in the poset that is contained in both G and H , and $G \vee_{\mathbb{C}} H$ to be the smallest submodule in the poset that contains both G and H . We again have $G \vee_{\mathbb{C}} H = G \cap H$ and $G \wedge_{\mathbb{C}} H = G + H$. Notice that the intersection and the sum of two full rank submodules of $\mathbb{F}_\infty(z)^m$ are again full rank submodules of $\mathbb{F}_\infty(z)^m$. This follows from the fact that $\mathbb{F}_\infty(z)$ is a discrete valuation domain with a unique prime element and from the Smith-McMillan decomposition theorem.

We will show that the three lattices \mathbb{A}, \mathbb{B} and \mathbb{C} are isomorphic. First, we show that the lattices \mathbb{A} and \mathbb{C} are isomorphic.

Theorem 1. *The mapping $\tau : \mathbb{C} \longrightarrow \mathbb{A}$ defined by $\tau(G) = G^\perp$, for every $G \in \mathbb{C}$, is a lattice isomorphism.*

Proof. By Corollary 1, τ is a well-defined mapping. Following Lemma 7, it is injective. It follows from Lemma 6 and Lemma 8 that τ is surjective. Hence it remains to show that τ is a lattice homomorphism.

Let $G, H \subseteq \mathbb{F}_\infty(z)^m$ be full rank submodules. Since $G \supseteq G \vee_{\mathbb{C}} H$ and $H \supseteq G \vee_{\mathbb{C}} H$, we have $G^\perp \vee_{\mathbb{A}} H^\perp \subseteq (G \vee_{\mathbb{C}} H)^\perp$. By surjectivity of τ , take a full rank submodule $E \subseteq \mathbb{F}_\infty(z)^m$ satisfying $E^\perp = G^\perp \vee_{\mathbb{A}} H^\perp$. Applying Lemma 8 we have

$$G \vee_{\mathbb{C}} H = ((G \vee_{\mathbb{C}} H)^\perp)^\perp \supseteq (E^\perp)^\perp = E.$$

On the other hand, $E^\perp \subseteq G^\perp$ and $E^\perp \subseteq H^\perp$ yield

$$E = (E^\perp)^\perp \supseteq (G^\perp)^\perp \vee_{\mathbb{C}} (H^\perp)^\perp = G \vee_{\mathbb{C}} H.$$

Thus we have $E = G \vee_{\mathbb{C}} H$ and so

$$(G \vee_{\mathbb{C}} H)^\perp = G^\perp \vee_{\mathbb{A}} H^\perp.$$

Using similar approach it can be shown that

$$(G \wedge_{\mathbb{C}} H)^\perp = G^\perp \wedge_{\mathbb{A}} H^\perp.$$

We conclude that τ is a lattice isomorphism.

Next, recall that in the decomposition $\mathbb{F}(z)^m = \mathbb{F}[z]^m \oplus z^{-1}\mathbb{F}_\infty(z)^m$, π_+ is the projection of $\mathbb{F}(z)^m$ onto $\mathbb{F}[z]^m$. Hence, it is a surjective module homomorphism with kernel $z^{-1}\mathbb{F}_\infty(z)^m$. This leads us to the following proposition.

Proposition 3. *As $\mathbb{F}_\infty(z)$ -modules, the quotient module $\mathbb{F}(z)^m/z^{-1}\mathbb{F}_\infty(z)^m$ is isomorphic to $\mathbb{F}[z]^m$.*

As a consequence to the Proposition 3, there exists a one to one correspondence between the collection of submodules of $\mathbb{F}(z)^m/z^{-1}\mathbb{F}_\infty(z)^m$ and the collection of submodules of $\mathbb{F}[z]^m$. On the other hand, there is a one to one correspondence between the collection of submodules of $\mathbb{F}(z)^m/z^{-1}\mathbb{F}_\infty(z)^m$ and the collection of submodules of $\mathbb{F}(z)^m$ that contains $z^{-1}\mathbb{F}_\infty(z)^m$. Since $z^{-1}\mathbb{F}_\infty(z)^m$ is a finitely generated $\mathbb{F}_\infty(z)$ -module, there is a one to one correspondence between the collection of finitely generated submodules of $\mathbb{F}(z)^m/z^{-1}\mathbb{F}_\infty(z)^m$ and the collection of finitely generated submodules of $\mathbb{F}(z)^m$ that contains $z^{-1}\mathbb{F}_\infty(z)^m$. The following theorem then follows.

Theorem 2. *The lattices \mathbb{A} and \mathbb{B} are isomorphic.*

These Theorem 1 and Theorem 2 assert a one-to-one correspondence between the collection \mathbb{C} of full rank submodules of $\mathbb{F}_\infty(z)$ -module $\mathbb{F}_\infty(z)^m$ and the collection \mathbb{B} of finitely generated submodules of $\mathbb{F}_\infty(z)$ -module $\mathbb{F}[z]^m$.

Example 1. *Numerical Example: Transformation of a Submodule*

To illustrate Theorem 1, consider a submodule M within the polynomial module $\mathbb{F}[z]^2$, which is generated by the following polynomial matrix

$$L = \begin{bmatrix} z & 1 \\ 0 & z^2 \end{bmatrix}.$$

This submodule consists of all elements obtained by multiplying L with any vector from $\mathbb{F}[z]^2$, meaning

$$M = \left\{ L \begin{bmatrix} f_1(z) \\ f_2(z) \end{bmatrix} \mid f_1(z), f_2(z) \in \mathbb{F}[z] \right\}.$$

By applying the Lattice Isomorphism Theorem, we obtain the corresponding submodule in the module of proper rational functions $\mathbb{F}_\infty(z)^2$, given by $M' = L \cdot \mathbb{F}_\infty(z)^2$. To demonstrate this transformation with a concrete example, consider selecting $f_1(z) = z^{-1}$ and $f_2(z) = 1$. Substituting these values into the equation above results in

$$L \begin{bmatrix} z^{-1} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 + z^{-1} \\ z^2 \end{bmatrix}.$$

This shows that the submodule M' in $\mathbb{F}_\infty(z)^2$ consists of rational function linear combinations of elements such as

$$\begin{bmatrix} 1 + z^{-1} \\ z^2 \end{bmatrix}.$$

This example illustrates how a polynomial submodule in $\mathbb{F}[z]^m$ transforms into a submodule in $\mathbb{F}_\infty(z)^m$ under the lattice isomorphism, preserving the algebraic structure while extending its domain to proper rational functions.

Example 2. *Application in Linear System Representation*

Theorem 2 has direct applications in linear system representation and control theory, particularly in behavioral systems where submodules correspond to input-output relationships. To demonstrate this, consider a discrete-time linear system represented in state-space form

$$\begin{bmatrix} x_1(z) \\ x_2(z) \end{bmatrix} = \begin{bmatrix} z & 1 \\ 0 & z^2 \end{bmatrix} \begin{bmatrix} u_1(z) \\ u_2(z) \end{bmatrix}.$$

This system is naturally associated with a submodule in $\mathbb{F}[z]^2$, given by

$$M = \{X = LU \mid U \in \mathbb{F}[z]^2\}.$$

By applying Theorem 1, this system can be equivalently represented in the domain of proper rational functions as

$$M' = \{X = LU \mid U \in \mathbb{F}_\infty(z)^2\}.$$

This means that the system can be analyzed within the space of proper rational functions, which allows for anti-causal components to be incorporated naturally. Beyond just transforming system representations, Theorem 2 enables the identification of dual submodules, which play a crucial role in observability and controllability analysis. The dual submodule associated with the system is defined as

$$M^\perp = \{Y \mid Y^T X = 0, \forall X \in M\}.$$

In control theory, this dual submodule is significant because it represents the observable subspace of the system. Ensuring that M^\perp remains structurally isomorphic to M confirms that the system's fundamental properties remain invariant under lattice isomorphism, making this result valuable for studying stability and realization theory.

5. Concluding Remarks

This paper introduces an important lattice isomorphism between two distinct collections: the collection of finitely generated submodules of a polynomial module and the collection of full rank submodules of a free module over the ring of proper rational functions. Furthermore, it explores the concept of behaviors in the context of system theory. An autonomous behavior, as demonstrated by Fuhrmann (Proposition 3.4 [3]), can be represented as a set of trajectories of an autonomous system of the form $\mathcal{B} = C(zI - A)^{-1}\xi \mid \xi \in \mathbb{F}^n$. Additionally, Sylviani et al. [13] have shown that the set of trajectories of any time-invariant discrete linear system constitutes a behavior.

These findings prompt the question of whether Fuhrmann's theory generalizes the trajectory sets of time-invariant discrete linear systems, which form complete $\mathbb{F}[z]$ -submodules

of formal series functions. If this is indeed the case, and considering that the set of trajectories of an anti-causal system, denoted as $C(zN - I)^{-1}\xi | \xi \in \mathbb{F}^n$ for a nilpotent matrix N [10], forms a finitely generated $\mathbb{F}_\infty(z)$ -submodule of the polynomial modules $\mathbb{F}[z]^m$, we can define a behavior of an anti-causal discrete system as a finitely generated $\mathbb{F}_\infty(z)$ -submodule of the polynomial module $\mathbb{F}[z]^m$. Consequently, these results provide a valuable tool to extend Fuhrmann's study to anti-causal discrete systems.

Rosenthal's work establishes the duality between coding theory and system theory by introducing convolutional codes as submodules of the free module $\mathbb{F}^n[z]$ [14]. Building upon this foundation, we have further demonstrated a lattice isomorphism between the aforementioned submodules and the full rank submodules of the free module $\mathbb{F}_\infty(z)$. This extension allows the coefficients of the polynomials in convolutional codes to be Laurent series, encompassing both positive and negative powers of z . Consequently, the coefficients can be selected from the field of formal Laurent series $\mathbb{F}_\infty(z)$ over a field \mathbb{F} .

Introducing $\mathbb{F}_\infty(z)$ -modules for convolutional codes expands the range of coefficients that can be employed, providing increased flexibility in the algebraic structure of the codes. This approach holds potential implications for analyzing, representing, and understanding convolutional codes within the framework of Laurent series arithmetic.

6. Future Work and Research Directions

Our study establishes a lattice isomorphism between polynomial submodules and proper rational function modules, laying a foundation for further exploration in generalized algebraic structures. Several promising directions emerge from our findings.

(i) Extension to M-Hazy Rings

M-Hazy rings, which introduce fuzzified convexity in ideal theory [15], offer a broader framework for studying submodular structures. Investigating lattice isomorphisms under M-hazy ring homomorphisms [16] could extend our results beyond classical module theory.

(ii) Applications in Control Theory

Investigating the implications of lattice isomorphisms in realization theory and system identification would provide insights into the structural preservation of system models. The ability to map polynomial-based systems into proper rational function representations can aid in control design and stability analysis.

(iii) Extensions to Multidimensional Systems

The current framework focuses on one-dimensional systems, but an important extension would be to explore higher-dimensional lattice structures. Generalizing our results to multidimensional systems could reveal new connections between algebraic system theory and signal processing.

(iv) Connections to Coding Theory

Lattice isomorphisms play a crucial role in the structural analysis of convolutional codes. Future work could investigate how our framework relates to convolutional coding theory, particularly in minimal encoder representations and dual code analysis.

By exploring these directions, we can broaden the applicability of lattice isomorphisms to non-classical algebraic structures, soft computing, control systems, multidimensional models, and coding theory, making our framework more robust for future mathematical and engineering applications.

Acknowledgements

The authors gratefully acknowledge the funding from the ITB research grant under Program PPMI FMIPA 2025, Grant No. FMIPA.PPMI-KK-PN-08-2025. The authors also express their sincere appreciation to H. K. Wimmer for his valuable ideas at the early stage of this research.

References

- [1] J.C. Willems. System theoretic models for the analysis of physical systems. *Ricerche di Automatica*, 10:71–106, 1979.
- [2] J.C. Willems and J.W. Polderman. *Introduction to Mathematical Systems Theory: A Behavioral Approach*. Springer, Berlin, 1997.
- [3] P.A. Fuhrmann. A study of behavior. *Linear Algebra and Its Applications*, 351-352:303–380, 2002.
- [4] U. Oberst. Multidimensional constant linear system. *Acta Appl. Math.*, 20:1–175, 1990.
- [5] P.A. Fuhrmann. Algebraic system theory: An analyst’s point of view. *J. of Franklin Inst.*, 301:521–540, 1976.
- [6] P.A. Fuhrmann. Duality in polynomial models with applications to geometric control theory. *IEEE Transactions on Automatic Control*, AC-26:284–295, 1981.
- [7] G. Conte and A. Perdon. Generalized state space realizations of non-proper rational transfer functions. *Systems and Control Letters*, 1:270–276, 1982.
- [8] G. Conte and A. Perdon. Infinite zero module and infinite pole module. *Lecture Notes in Control and Information Sciences*, 62:302–315, 1984.
- [9] H.K. Wimmer. The structure of nonsingular polynomial matrices. *Mathematical System Theory*, 14:367–379, 1981.
- [10] H.K. Wimmer. Polynomial matrices and dualities. *System and Control Letters*, 1:200–203, 1981.
- [11] Gantina Rachmaputri. Algebraic structure of the anti-causal system. *Journal of Physics: Conference Series*, 890(1):012122, sep 2017.
- [12] P. Astuti and H.K. Wimmer. Homomorphism of modules associated with polynomial matrices with infinite elementary divisors. *Systems and Control Letters*, 44:333–337, 2001.

- [13] A. Sylviani, H. Garminia, and P. Astuti. Behavior for time invariant finite dimensional discrete linear system. *J. Math. Fund. Sci.*, 45:1:39–52, 2013.
- [14] J. Rosenthal, J. M. Schumacher, and E. V. York. On behaviors and convolutional codes. *IEEE Transactions on Information Theory*, 42:1881–1891, 1996.
- [15] F. Mehmood, F. Shi, and K. Hayat. A new approach to the fuzzification of rings. *Journal of Nonlinear and Convex Analysis*, 21:2637–2646, 2020.
- [16] F. Mehmood, F. Shi, K. Hayat, and X. Yang. The homomorphism theorems of m-hazy rings and their induced fuzzifying convexities. *Mathematics*, 8:411, 2020.