



Investigating Nonlinear Wave Stability and Dispersive Phenomena in Complex Systems: Dynamical Properties, Bifurcation Analysis, and Chaotic Behaviors

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Abstract. We present a systematic investigation of the time–fractional Kundu–Mukherjee–Naskar equation in $(2 + 1)$ –dimensions (FKMN), formulated with a truncated Mittag–Leffler kernel and a (β) –fractional derivative to incorporate memory effects and nonlocal temporal dynamics. This formulation captures dispersive–nonlinear wave propagation in heterogeneous media with fractional temporal responses, making it pertinent to applications in fluid mechanics, nonlinear optics, and plasma physics, including shallow-water dynamics, optical fiber transmission, and plasma instabilities.

A combined fractional-calculus and dynamical-systems framework is employed to characterize the model across parameter regimes. Bifurcation analysis delineates stability boundaries, while quasiperiodic and chaotic responses are identified and organized. Sensitivity analyses further quantify the impact of key coefficients. Exact traveling-wave solutions are derived using the Khater II (KII) method and an enhanced Kudryashov (EKud) approach, revealing new waveform geometries and dissipation patterns. These solutions are independently validated by the Adomian decomposition method, ensuring consistency between closed-form constructions and numerical approximations.

The findings highlight complex waveform structures, dissipative signatures, and parameter-dependent transitions, underscoring the FKMN model’s versatility in representing nonlinear phenomena across disciplines. Methodologically, the study advances a hybrid analytical–numerical pipeline that integrates fractional operators with rigorous dynamical analysis, thereby improving both solution fidelity and interpretability. These contributions establish the $(2 + 1)$ –dimensional FKMN equation as a paradigmatic test case for fractional evolution equations in applied mathematics and physics.

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1. Introduction

Nonlinear evolution equations (NLEEs) occupy a central position in mathematical physics as they encapsulate the interplay of dispersion, nonlinearity, and dissipation underlying wave phenomena in fluids, plasmas, and optical media [1]. While classical integer-order formulations provide useful models in many regimes, they often fail to capture hereditary effects and spatial heterogeneity that lead to anomalous transport and long-tailed relaxation [2]. Fractional differential equations address these limitations by incorporating memory and nonlocality through fractional operators, thereby offering a more faithful representation of complex dispersive–dissipative dynamics observed in natural and engineered media [3]. Within this broader framework, the FKMN class of equations has emerged as a versatile model for studying derivative-type nonlinearities and envelope dynamics, with applications ranging from nonlinear optics to shallow-water waves and plasma instabilities [4, 5].

Motivated by these developments, the present work investigates the $(2+1)$ –dimensional time–fractional FKMN equation, expressed as

$$p \frac{\partial^2 u}{\partial x \partial y} + i \mathcal{D}_t^\varepsilon u + i q u \left(u \frac{\partial v}{\partial x} - v \frac{\partial u}{\partial x} \right) = 0, \quad (1)$$

where $u(x, y, t)$ is the complex wave envelope, $v(x, y, t) = \bar{u}(x, y, t)$ its complex conjugate, (x, y) denote transverse and longitudinal coordinates, and t is time [6–9]. The cross-dispersion term u_{xy} describes anisotropic propagation in two spatial directions, while the time-fractional derivative $\mathcal{D}_t^\varepsilon$ with $(0 < \varepsilon \leq 1)$ introduces memory-driven temporal responses. In this study, $\mathcal{D}_t^\varepsilon$ is realized via a truncated Mittag–Leffler kernel and a β –fractional derivative, balancing physical interpretability with analytical tractability [10–14]. This formulation is particularly suitable for media exhibiting viscoelasticity, multiscale relaxation, or anomalous diffusion, such as optical fibers with noninstantaneous response, stratified shallow-water flows, and magnetized plasmas [10, 11].

To clarify the physical meaning of each term in (1), we summarize:

- **Dispersion:** $p u_{xy}$ represents mixed spatial dispersion, where p is the dispersion coefficient controlling wave spreading along both x and y .
- **Fractional temporal response:** $i \mathcal{D}_t^\varepsilon u$ encodes anomalous diffusion and nonlocal memory effects via a fractional derivative of order ε .
- **Nonlinear self-interaction:** $i q u (u v_x - v u_x)$ models wave self-modulation, with q governing the strength and nature of nonlinear interactions.

These components jointly establish the FKMN equation as a robust framework for simulating nonlinear dispersive media, especially in contexts where both nonlinearity and memory effects are prominent [15, 16].

The key coefficients p and q admit the following interpretations:

- **Dispersion coefficient p :** regulates the strength and directionality of spatial dispersion; its sign determines distinct dispersive regimes.
- **Nonlinearity coefficient q :** controls nonlinear interactions and their qualitative nature, with the sign of q distinguishing between focusing and defocusing behaviors.

Recent studies of the FKMN equation have advanced soliton theory, deriving solutions that characterize the balance of nonlinearity and dispersion [17]. Techniques such as the Kudryashov method, Adomian decomposition, and variational iteration have provided significant insight into solution structures and their stability properties. Nonetheless, open questions remain concerning the fractional and dissipative features of the model, which are fundamental for a deeper physical interpretation [18–21].

The present study seeks to extend the theoretical understanding of the FKMN equation by integrating two powerful analytical techniques that named by the Khater II (KII) method and an enhanced Kudryashov (EKud) approach. These techniques supplemented by validation through the Adomian decomposition method. This hybrid methodology yields novel closed-form solutions and facilitates the exploration of dissipative dynamics inherent in fractional systems. Beyond solution construction, we examine a wave transformation,

$$u = \psi(\xi) e^{i\left(\frac{\kappa \Gamma(\varsigma+1) t^\varepsilon}{\varepsilon} + x\right)}, \quad v = \psi(\xi) e^{-i\left(\frac{\kappa \Gamma(\varsigma+1) t^\varepsilon}{\varepsilon} + x\right)}, \quad (2)$$

with $\xi = \frac{\lambda \Gamma(\varsigma+1) t^\varepsilon}{\varepsilon} + x + \kappa y$, applied to (1) under the truncated β -fractional derivative (see Appendix 6), where λ and κ are constants to be determined, we get

$$e^{i(\kappa t + x)} (-\kappa \psi + i \psi' (\lambda + \kappa p) + \kappa p \psi'' + 2 q \psi^3) = 0. \quad (3)$$

Separating the real and imaginary parts in Eq. (3), yields

$$\begin{cases} \text{Re.} & -\kappa \psi + \kappa p \psi'' + 2 q \psi^3 = 0, \\ \text{Im.} & \psi' (\lambda + \kappa p) = 0. \end{cases} \quad (4)$$

Thus, Eq. (1) converts into the next ordinary differential equation

$$-\kappa \psi + \kappa p \psi'' + 2 q \psi^3 = 0, \quad (5)$$

where $\lambda = -p \kappa$. Balancing the highest order derivative term and nonlinear term in Eq. (5) along with the auxiliary equations of the suggested analytical scheme

$$\begin{cases} \text{KII method's auxiliary equation} & \begin{cases} \phi'(\xi) \rightarrow -f(\xi)\phi(\xi), \\ f'(\xi) \rightarrow -\delta - f(\xi)^2, \end{cases} \\ \text{EKud method's auxiliary equation} & f''(\xi) \rightarrow f(\xi) - 2 \tau f(\xi)^3, \end{cases} \quad (6)$$

where δ, τ are arbitrary constants to be determined later, yield constructing the general solutions of Eq. (5) in the next form

$$\psi(\xi) = \begin{cases} \sum_{i=1}^n (a_i f(\xi)^i + b_i \phi(\xi) f(\xi)^{i-1}) + a_0, & \text{KII method's Sol.,} \\ \sum_{i=1}^n \left(a_i f(\xi)^i + b_i \left(\frac{f'(\xi)}{f(\xi)} \right)^i \right) + a_0, & \text{EKud method's Sol.,} \end{cases} \quad (7)$$

where $n = 1$, while a_i, b_i are arbitrary constants to be determined later.

In the broader setting of NLEEs, the FKMN equation is distinguished by its ability to capture fractional-order wave dynamics. Unlike classical models such as the nonlinear Schrödinger or Kadomtsev–Petviashvili equations, the FKMN framework incorporates nonlocal memory effects and long-range interactions, rendering it especially valuable for optical, fluid, and plasma systems where dissipative wave behaviors are prevalent [22–24]. While this study is primarily theoretical and computational, the results provide a foundation for future experimental exploration of fractional nonlinear dynamics.

Paper organization. Section 3 develops analytical and numerical solutions of the FKMN equation. Section 4 presents simulations illustrating solution dynamics and physical interpretation. Section 5 discusses theoretical implications and highlights novel contributions. Section 6 concludes with a summary of findings and future perspectives.

2. Dynamical Analysis of the FKMN Model

This investigation delves into the nonlinear FKMN model, focusing on its dynamical properties through bifurcation analysis, exploration of chaotic and quasi-periodic behaviors, and assessment of sensitivity to parameter variations. The goal is to uncover the fundamental mechanisms governing the system's evolution using advanced mathematical tools from nonlinear dynamical theory.

2.1. Bifurcation Analysis

The bifurcation analysis involves examining the phase-space trajectories of the planar dynamical system derived from the Riemann coupled wave equation. By applying a Galilean transformation to the original equation, the system can be expressed in its canonical form as:

$$\begin{cases} \frac{d\psi}{d\xi} = v, \\ \frac{dv}{d\xi} = \frac{\psi (\kappa - 2q\psi^2)}{\kappa p}. \end{cases} \quad (8)$$

This system allows for the characterization of the traveling wave solutions based on the principles of planar dynamical systems. The Hamiltonian function associated with the system is given by:

$$\mathcal{H}(\psi, v) = \frac{-\kappa \psi^2 + \kappa p v^2 + q \psi^4}{2 \kappa p} = \mathcal{G}, \quad (9)$$

where $\mathcal{H}(\psi, v)$ represents the total energy of the system, comprising the kinetic energy term $(\frac{v^2}{2})$ and the potential energy term $(\frac{q \psi^4 - \kappa \psi^2}{2 \kappa p})$.

2.1.1. Equilibrium Points

The equilibrium points of the system are determined by setting $(\frac{d\psi}{d\xi} = 0)$ and $(\frac{dv}{d\xi} = 0)$, which leads to:

$$\begin{cases} v = 0, \\ \psi (\kappa - 2 q \psi^2) = 0. \end{cases} \quad (10)$$

The solutions to (10) yield the equilibrium points $E_1 = (0, 0)$, $E_2 = (-\frac{\sqrt{\kappa}}{\sqrt{2}\sqrt{q}}, 0)$, and $E_3 = (\frac{\sqrt{\kappa}}{\sqrt{2}\sqrt{q}}, 0)$.

2.1.2. Stability Analysis

To assess the stability of the equilibrium points, we compute the Jacobian matrix $\mathcal{J}(\psi, v)$ for the system (8):

$$\mathcal{J}(\psi, v) = \begin{pmatrix} 0 & 1 \\ \frac{\kappa - 6 q \psi^2}{\kappa p} & 0 \end{pmatrix}. \quad (11)$$

The determinant of $\mathcal{J}(\psi, v)$ is:

$$\det(\mathcal{J}(\psi, v)) = \frac{6 q \psi^2 - \kappa}{\kappa p}. \quad (12)$$

The stability of the equilibrium points depends on the eigenvalues of $\mathcal{J}(\psi, v)$. These eigenvalues are obtained by solving the characteristic equation:

$$\det(\mathcal{J}(\psi, v) - \lambda I) = \det \left(\begin{pmatrix} 0 & 1 \\ \frac{\kappa - 6 q \psi^2}{\kappa p} & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = 0. \quad (13)$$

This simplifies to:

$$\det \begin{pmatrix} -\lambda & 1 \\ \frac{\kappa - 6 q \psi^2}{\kappa p} & -\lambda \end{pmatrix} = 0. \quad (14)$$

Simplifying this expression:

$$\lambda^2 - \frac{\kappa - 6q\psi^2}{\kappa p} = 0. \quad (15)$$

Solving this quadratic equation yields the eigenvalues of the matrix $\mathcal{J}(\psi, v)$.

The eigenvalues of this matrix indicate the system's stability and rate of change. Since the eigenvalues are real and have opposite signs, they suggest a saddle point, where the system exhibits both stable and unstable directions. The positive eigenvalue corresponds to an unstable direction, where trajectories move away from the equilibrium, while the negative eigenvalue corresponds to a stable direction, where trajectories converge toward the equilibrium. This behavior implies that the system is not oscillatory but evolves monotonically along certain directions. The magnitude of the eigenvalues indicates the rate of growth or decay in the system, with larger magnitudes suggesting faster dynamics. Thus, the eigenvalues provide crucial information about the system's stability and the speed at which it responds to perturbations.

The analysis reveals that the nature of the equilibrium points varies significantly with the parameter values (p, κ, q) , as summarized below:

- For $(p, \kappa, q) = (-2, 16, 4)$, E_1 is a center point, and E_2, E_3 are saddle points.
- For $(p, \kappa, q) = (3, -36, -9)$, E_1 is a saddle point, and E_2, E_3 are center points.

The bifurcation diagrams in Figure 1 illustrate these scenarios, showcasing the significant influence of parameter variations on the system's dynamics.

2.2. Chaotic and Quasi-Periodic Dynamics

To investigate the chaotic dynamics, a perturbation term of the form $\mathfrak{Z} \cos(\Upsilon \xi)$ is introduced into the system, yielding the modified equations:

$$\begin{cases} \frac{d\psi}{d\xi} = v, \\ \frac{dv}{d\xi} = \frac{\psi(\kappa - 2q\psi^2)}{\kappa p} + \mathfrak{Z} \cos(\Upsilon \xi). \end{cases} \quad (16)$$

Here, $\mathfrak{Z}$ denotes the amplitude of the perturbation, and Υ represents its frequency. Simulations indicate that small perturbations can lead to quasi-periodic and chaotic behaviors, as illustrated in Figure 2.

2.3. Sensitivity Analysis

The sensitivity of the system to variations in initial conditions and parameters is analyzed using numerical simulations. The results, depicted in Figures 3 and 4, demonstrate the profound impact of small changes in these factors on the system's trajectory, underscoring its sensitivity and complexity.

3. Exploring Waveform Evolution and Behavior

Building upon the outlined methodologies, this section delves into the analytical solutions of the investigated model, applying the proposed analytical schemes to thoroughly explore and derive relevant solution structures.

3.1. Solitary Wave Computations Via Khat II method

By employing the previously described analytical scheme, the solutions to Eq. (5), together with Eqs. (6) and (7), yield the following parameters:

Case i

$$a_0 \rightarrow 0, a_1 \rightarrow 0, \kappa \rightarrow b_1^2 q, p \rightarrow -\frac{1}{\delta}.$$

Case ii

$$a_0 \rightarrow 0, a_1 \rightarrow \frac{b_1}{\sqrt{\delta}}, \kappa \rightarrow -2b_1^2 q, p \rightarrow \frac{2}{\delta}.$$

Case iii

$$a_0 \rightarrow 0, b_1 \rightarrow 0, \kappa \rightarrow -2a_1^2 \delta q, p \rightarrow \frac{1}{2\delta}.$$

Consequently, the expressions that define the traveling wave solutions of the examined model are given for $\delta \neq 0$ as follows:

$$u_i(x, y, t) = b_1 e^{i\left(\frac{b_1^2 q \Gamma(\zeta+1) t^\varepsilon}{\varepsilon} + x\right)} \sec\left(\sqrt{\delta}\left(-\frac{\kappa p \Gamma(\zeta+1) t^\varepsilon}{\varepsilon} + x + \kappa y\right)\right), \quad (17)$$

$$u_i(x, y, t) = b_1 e^{i\left(\frac{b_1^2 q \Gamma(\zeta+1) t^\varepsilon}{\varepsilon} + x\right)} \csc\left(\sqrt{\delta}\left(-\frac{\kappa p \Gamma(\zeta+1) t^\varepsilon}{\varepsilon} + x + \kappa y\right)\right). \quad (18)$$

$$u_{ii}(x, y, t) = b_1 e^{i\left(x - \frac{2b_1^2 q \Gamma(\zeta+1) t^\varepsilon}{\varepsilon}\right)} \left(\sec\left(\sqrt{\delta}\left(-\frac{\kappa p \Gamma(\zeta+1) t^\varepsilon}{\varepsilon} + x + \kappa y\right)\right) - \tan\left(\sqrt{\delta}\left(-\frac{\kappa p \Gamma(\zeta+1) t^\varepsilon}{\varepsilon} + x + \kappa y\right)\right) \right), \quad (19)$$

$$u_{ii}(x, y, t) = b_1 e^{i \left(x - \frac{2b_1^2 q \Gamma(\zeta+1) t^\varepsilon}{\varepsilon} \right)} \cot \left(\frac{1}{2} \sqrt{\delta} \left(-\frac{\kappa p \Gamma(\zeta+1) t^\varepsilon}{\varepsilon} + x + \kappa y \right) \right). \quad (20)$$

$$u_{iii}(x, y, t) = a_1 \sqrt{\delta} \left(-e^{i \left(x - \frac{2a_1^2 \delta q \Gamma(\zeta+1) t^\varepsilon}{\varepsilon} \right)} \right) \tan \left(\sqrt{\delta} \left(-\frac{\kappa p \Gamma(\zeta+1) t^\varepsilon}{\varepsilon} + x + \kappa y \right) \right), \quad (21)$$

$$u_{iii}(x, y, t) = a_1 \sqrt{\delta} e^{i \left(x - \frac{2a_1^2 \delta q \Gamma(\zeta+1) t^\varepsilon}{\varepsilon} \right)} \cot \left(\sqrt{\delta} \left(-\frac{\kappa p \Gamma(\zeta+1) t^\varepsilon}{\varepsilon} + x + \kappa y \right) \right). \quad (22)$$

3.2. Solitary Wave Computations Via EKud method

Utilizing the previously described analytical scheme, the solutions to Eq. (5), in conjunction with Eqs. (6) and (7), produce the specified parameters as follows:

Family I:

$$a_0 \rightarrow 0, a_1 \rightarrow 0, p \rightarrow -\frac{1}{2}, \kappa \rightarrow 2b_1^2 q.$$

Family II:

$$a_0 \rightarrow 0, a_1 \rightarrow ib_1 \sqrt{\tau}, p \rightarrow -2, \kappa \rightarrow 2b_1^2 q.$$

Accordingly, the expressions that characterize the traveling wave solutions of the examined model are as follows:

$$u_I(x, y, t) = e^{i \left(\frac{2b_1^2 q \Gamma(\zeta+1) t^\varepsilon}{\varepsilon} + x \right)} \left(b_1 \tanh \left(\frac{b_1^2 q (\Gamma(\zeta+1) t^\varepsilon + 2\varepsilon y)}{\varepsilon} + x \right) + b_2 \right), \quad (23)$$

$$u_I(x, y, t) = e^{i \left(\frac{2b_1^2 q \Gamma(\zeta+1) t^\varepsilon}{\varepsilon} + x \right)} \left(b_1 \left(\frac{2\tau}{4c^2 \exp \left(2 \left(\frac{b_1^2 q (\Gamma(\zeta+1) t^\varepsilon + 2\varepsilon y)}{\varepsilon} + x \right) \right) + \tau} - 1 \right) - b_2 \right). \quad (24)$$

$$u_{II}(x, y, t) = ib_1 e^{i \left(\frac{2b_1^2 q \Gamma(\zeta+1) t^\varepsilon}{\varepsilon} + x \right)} \left(\sqrt{\operatorname{sech}^2 \left(\frac{2b_1^2 q (2\Gamma(\zeta+1) t^\varepsilon + \varepsilon y)}{\varepsilon} + x \right)} + i \tanh \left(\frac{2b_1^2 q (2\Gamma(\zeta+1) t^\varepsilon + \varepsilon y)}{\varepsilon} + x \right) \right), \quad (25)$$

$$u_{II}(x, y, t) = - \frac{b_1 e^{i \left(\frac{2b_1^2 q \Gamma(\zeta+1) t^\varepsilon}{\varepsilon} + x \right)} \left(2ce^{\frac{2b_1^2 q (2\Gamma(\zeta+1) t^\varepsilon + \varepsilon y)}{\varepsilon} + x} - i\sqrt{\tau} \right)}{2ce^{\frac{2b_1^2 q (2\Gamma(\zeta+1) t^\varepsilon + \varepsilon y)}{\varepsilon} + x} + i\sqrt{\tau}}. \quad (26)$$

3.3. Numerical solutions

Here, we investigate the accuracy of the above-constructed solutions (Eqs. (17), (23)) via Khat II and EKud methods by implementing the Adomian decomposition scheme . Evaluating the requested conditions for the suggested numerical scheme from Eqs. (17), (23), leads to evaluate the analytical and numerical solutions as shown in Table 1, 2.

Table 1: This table presents a comparison between analytical and approximate solutions generated through the Adomian decomposition scheme for Eq. (17). The absolute error values demonstrate the high accuracy of the decomposition scheme, confirming its effectiveness in capturing the analytical solution's behavior with minimal deviation.

Value of ξ	Analy. Sol.	Approx. Sol.	—Error—	Value of ξ	Analy. Sol.	Approx. Sol.	—Error—
0	0	0	0	0.033	0.0659760524315067	0.0659760524313055	2.01256E-13
0.001	0.001999993333336	0.001999993333336	0	0.034	0.0679738094437799	0.0679738094435243	2.55559E-13
0.002	0.0039999946666752	0.0039999946666752	0	0.035	0.0699714306655589	0.0699714306652366	3.2227E-13
0.003	0.0059999820000648	0.0059999820000648	0	0.036	0.0719689121158597	0.0719689121154559	4.03746E-13
0.004	0.0079999573336064	0.0079999573336064	0	0.037	0.0739662498148143	0.0739662498143115	5.02723E-13
0.005	0.0099999166675000	0.0099999166675000	0	0.038	0.0759634397837024	0.0759634397830801	6.22294E-13
0.006	0.0119998560020736	0.0119998560020736	1.73472E-18	0.039	0.0779604780449833	0.0779604780442172	7.66054E-13
0.007	0.0139997713378151	0.0139997713378151	1.73472E-18	0.04	0.0799573606223271	0.0799573606213891	9.38083E-13
0.008	0.0159996586754046	0.0159996586754046	3.46945E-18	0.041	0.0819540835406468	0.0819540835395038	1.14302E-12
0.009	0.0179995140157459	0.0179995140157459	6.93889E-18	0.042	0.0839506428261294	0.0839506428247433	1.38609E-12
0.01	0.0199993333599989	0.0199993333599989	1.38778E-17	0.043	0.0859470345062676	0.0859470345045944	1.67323E-12
0.011	0.0219991127096115	0.0219991127096115	3.1225E-17	0.044	0.0879432546098917	0.0879432546078805	2.01118E-12
0.012	0.0239988480663513	0.0239988480663513	6.245E-17	0.045	0.0899392991672005	0.0899392991647931	2.40739E-12
0.013	0.0259985354323380	0.0259985354323379	1.17961E-16	0.046	0.0919351642097933	0.0919351642069230	2.8703E-12
0.014	0.0279981708100750	0.0279981708100748	2.11636E-16	0.047	0.0939308457707010	0.0939308457672917	3.40929E-12
0.015	0.0299977502024816	0.0299977502024812	3.67761E-16	0.048	0.0959263398844178	0.0959263398803829	4.03486E-12
0.016	0.0319972696129246	0.0319972696129240	6.17562E-16	0.049	0.0979216425869325	0.0979216425821739	4.75864E-12
0.017	0.0339967250452509	0.0339967250452499	9.99201E-16	0.05	0.0999167499157600	0.0999167499101664	5.59358E-12
0.018	0.0359961125038187	0.0359961125038172	1.57513E-15	0.051	0.1019116579099720	0.1019116579034180	6.55401E-12
0.019	0.0379954279935299	0.0379954279935275	2.42861E-15	0.052	0.1039063626102300	0.1039063626025740	7.65575E-12
0.02	0.0399946675198619	0.0399946675198582	3.66374E-15	0.053	0.1059008600588150	0.1059008600498980	8.91628E-12
0.021	0.0419938270888992	0.0419938270888938	5.4054E-15	0.054	0.1078951462996580	0.1078951462893030	1.03548E-11
0.022	0.0439929027073660	0.0439929027073582	7.84789E-15	0.055	0.1098892173783750	0.1098892173663820	1.19925E-11
0.023	0.0459918903826574	0.0459918903826462	1.11994E-14	0.056	0.1118830693422940	0.1118830693284410	1.38524E-11
0.024	0.0479907861228715	0.0479907861228557	1.57444E-14	0.057	0.1138766982404880	0.1138766982245280	1.59601E-11
0.025	0.0499895859368414	0.0499895859368196	2.18228E-14	0.058	0.1158701001238070	0.1158701001054640	1.83432E-11
0.026	0.0519882858341669	0.0519882858341370	2.98789E-14	0.059	0.1178632710449080	0.1178632710238760	2.1032E-11
0.027	0.0539868818252465	0.0539868818252061	4.03913E-14	0.06	0.1198562070582870	0.1198562070342270	2.40596E-11
0.028	0.0559853699213089	0.0559853699212548	5.4054E-14	0.061	0.1218489042203080	0.1218489041928460	2.74619E-11
0.029	0.0579837461344451	0.0579837461343736	7.15678E-14	0.062	0.1238413585892360	0.123841358579580	3.12781E-11
0.03	0.0599820064776403	0.0599820064775464	9.38694E-14	0.063	0.1258335662252680	0.1258335661897170	3.55504E-11
0.031	0.0619801469648051	0.0619801469646831	1.22034E-13	0.064	0.1278255231905630	0.1278255231502380	4.03249E-11

Table 2: This table compares analytical and approximate solutions for Eq. (23) and emphasizes the robustness of the Adomian decomposition scheme in generating solutions that closely match the analytical model. This table further validates the consistency and reliability of the scheme across varied model parameters.

Value of x	Analy. Sol.	Approx. Sol.	—Error—	Value of x	Analy. Sol.	Approx. Sol.	—Error—
0	3	3	0	0.033	2.96701197378425	2.99332152445293	0.0263095506686883
0.001	2.9990000033333	2.99902399176250	0.0000239914291704	0.034	2.96601309527811	2.99396297414490	0.0279198788667900
0.002	2.99800000266666	2.99809593752812	0.0000959348614593	0.035	2.96501428466722	2.99465637201124	0.0296420873440195
0.003	2.99700000899997	2.99721580070953	0.0002157917095573	0.036	2.96401554394207	2.99540200016609	0.0313864562240189
0.004	2.99600002133320	2.99638355494877	0.0003835336155711	0.037	2.96301687509259	2.99620014982837	0.0331832747357792
0.005	2.99500004166625	2.99559918407842	0.0005991424121690	0.038	2.96201828010815	2.99705112129257	0.0350628411844192
0.006	2.99400007199896	2.99486268208301	0.0008626100840483	0.039	2.96101976097751	2.99795522389974	0.0369354629222349
0.007	2.99300011433109	2.99417405306080	0.0011739387297101	0.04	2.96002131988884	2.99891277600886	0.03892514563200200
0.008	2.99200017066230	2.99353331118585	0.0015331405235490	0.041	2.95902295822968	2.99992410496832	0.0409011467386473
0.009	2.99100024299213	2.99294048067037	0.0019402376782436	0.042	2.95802467858694	3.00098954708786	0.04290486835009245
0.01	2.9900003332000	2.99239559572746	0.0023952624074610	0.043	2.95702648274687	3.00210944761057	0.0450829648637066
0.011	2.98900044364519	2.99189870053406	0.0028982568888658	0.044	2.95602837269505	3.00328416068534	0.0472557879902822
0.012	2.98800057596682	2.99144984919426	0.0034492732274334	0.045	2.95503035041640	3.00451404933941	0.0494836989230136
0.013	2.98700073228383	2.99104910570289	0.0040483734190633	0.046	2.95403241739510	3.00579948545134	0.0517870675562396
0.014	2.98600091459496	2.99069654390946	0.0046956293145022	0.047	2.95303457711465	3.00714084972409	0.0541062726094439
0.015	2.98500112489876	2.99039224748232	0.0053911225835628	0.048	2.95203683005779	3.00853853165847	0.0565867016006797
0.016	2.98400136519354	2.99013630987318	0.0061349446796402	0.049	2.95103917870653	3.00999292952678	0.0589537508202485
0.017	2.98300163747737	2.98992883428191	0.0069271968045306	0.05	2.95004162504212	3.01150445034676	0.0614628253046372
0.018	2.98200194374809	2.98976993362164	0.0077679898735474	0.051	2.94904417104501	3.01307350985573	0.0640293388107169
0.019	2.98100228600324	2.98965973048416	0.0086574444809244	0.052	2.94804681869488	3.01470053248507	0.0666537137901875
0.02	2.98000266624007	2.98959835710560	0.0095956908655270	0.053	2.94704956997059	3.01638595133487	0.0693363813642756
0.021	2.97900308645555	2.98958595533240	0.0105828688768450	0.054	2.94605242685017	3.01813020814886	0.0720777812986930
0.022	2.97800354864632	2.98962267658759	0.0116191279412772	0.055	2.94505539131081	3.01993375328965	0.0748783619788354
0.023	2.97700405480867	2.98970868183739	0.0127046270287168	0.056	2.94405846532885	3.02179704571409	0.0777385803852382
0.024	2.97600460693856	2.98984414155798	0.0138393346194149	0.057	2.94306165087976	3.02372055294903	0.0806589020692727
0.025	2.97500520703158	2.99002923570271	0.0150240286711338	0.058	2.94206494993810	3.02570475106719	0.0836398011290980
0.026	2.97400585708292	2.99026415366951	0.0162582965865914	0.059	2.94106836447755	3.02775012466340	0.0866817601858521
0.027	2.97300655908738	2.99054909426856	0.0175425351811875	0.06	2.94007189647086	3.02985716683095	0.0897552703600939
0.028	2.97200731503935	2.99088426569036	0.0188769506510140	0.061	2.93907554788985	3.03202637913832	0.0929508312484773
0.029	2.97100812693278	2.99126988547392	0.0202617585411446	0.062	2.93807932070538	3.03425827160607	0.0961789509006863
0.03	2.97000899676118	2.99170618047540	0.0216971837142159	0.063	2.93708321688737	3.03655336268396	0.0994701457965910
0.031	2.96900992651760	2.99219338683687	0.0231834603192742	0.064	2.93608723840472	3.03891217922838	0.1028349408236590

4. Solution's graphical representations

This section presents a detailed graphical and tabular analysis of the analytical and numerical solutions to the nonlinear fractional model. The figures illustrate solution profiles obtained via the Khater II (KII) and enhanced Kudryashov (EKud) methods, focusing on oscillatory behavior, amplitude stability, phase dynamics, and energy distribution. Complementary tables provide a quantitative comparison between analytical and approximate solutions obtained through the Adomian decomposition method (ADM), thereby validating both the fidelity and robustness of the derived results.

- **Fig. 1:** Phase-space trajectories of the FKMN system across different parameter sets. The diagrams capture the influence of parameter variation on dynamical behavior and highlight transitions from stable to unstable equilibria through bifurcation structures.
- **Fig. 2:** Phase portraits illustrating quasiperiodic and chaotic responses. The intricate patterns formed under perturbations demonstrate the system's sensitivity to parameter shifts and emphasize the complex topology of solution trajectories.
- **Fig. 3:** Sensitivity of solution dynamics to small variations in initial conditions and parameters. Results show how minor perturbations strongly affect trajectory evolution, confirming the system's complexity.
- **Fig. 4:** Effects of distinct initial positions on system evolution. The trajectories reveal that initial conditions play a decisive role in determining long-term behavior.
- **Fig. 5:** Graphical representation of solution (17) obtained using the KII method. The real component (subfigures a, d, g) shows persistent oscillatory profiles characteristic of soliton-like dynamics. The imaginary component (b, e, h) exhibits periodic variations consistent with balanced energy flow. The absolute value plots (c, f, i) confirm amplitude stability, indicating robust, non-dispersive propagation.
- **Fig. 6:** Graphical solution of (21) under the KII method for a distinct parameter regime. The real part displays increased amplitude and waveform broadening. Imaginary parts reveal periodicity in energy transfer, while absolute value plots demonstrate preserved amplitude, reinforcing soliton-like stability despite parameter changes.
- **Fig. 7:** Waveform behavior of solution (23) derived via the EKud method. Real parts (a, d, g) highlight oscillatory modulation; imaginary parts (b, e, h) capture phase shifts essential for energy distribution; absolute values (c, f, i) confirm stable amplitude propagation, demonstrating the EKud method's reliability in capturing nonlinear wave features.
- **Fig. 8:** Additional EKud solution (25) with alternative parameters. Real parts show amplitude variations, imaginary components emphasize phase stability, and absolute

values illustrate robust energy dispersion, underscoring the method's effectiveness in describing fractional nonlinear wave dynamics.

- **Table 1:** Comparison of analytical solution (17) with ADM approximations. The table lists values of ξ , exact and approximate solutions, and absolute errors, which remain consistently small, demonstrating the high accuracy of ADM.
- **Table 2:** Comparison of solution (23) with ADM approximations under varied parameters. As with Table 1, the close correspondence between analytical and approximate values confirms the reliability and precision of ADM in replicating fractional nonlinear dynamics.

In summary, the graphical analyses reveal soliton-like oscillatory stability, amplitude preservation, and robust phase dynamics, while the tables confirm the high accuracy of ADM in approximating exact solutions. Together, these results establish the effectiveness of the KII and EKud approaches for capturing intricate wave phenomena in nonlinear fractional systems and demonstrate the consistency of the hybrid analytical–numerical framework.

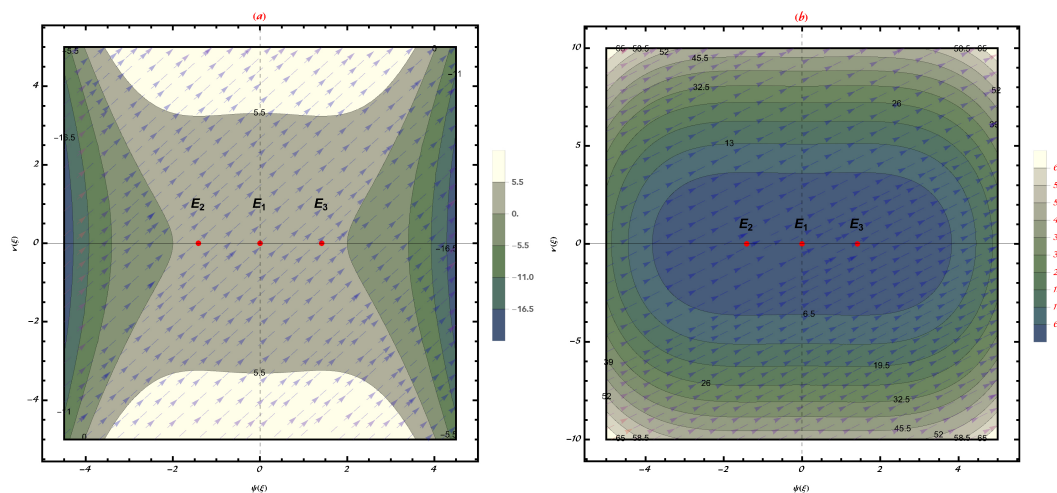


Figure 1: Bifurcation diagrams illustrating the phase-space trajectories of the FKMN model. The diagrams show the equilibrium points and their stability characteristics for different parameter values. For $(p, \kappa, q) = (-2, 16, 4)$, E_1 is a center point, and E_2, E_3 are saddle points. For $(p, \kappa, q) = (3, -36, -9)$, E_1 is a saddle point, and E_2, E_3 are center points. These diagrams highlight the significant influence of parameter variations on the system's dynamics.

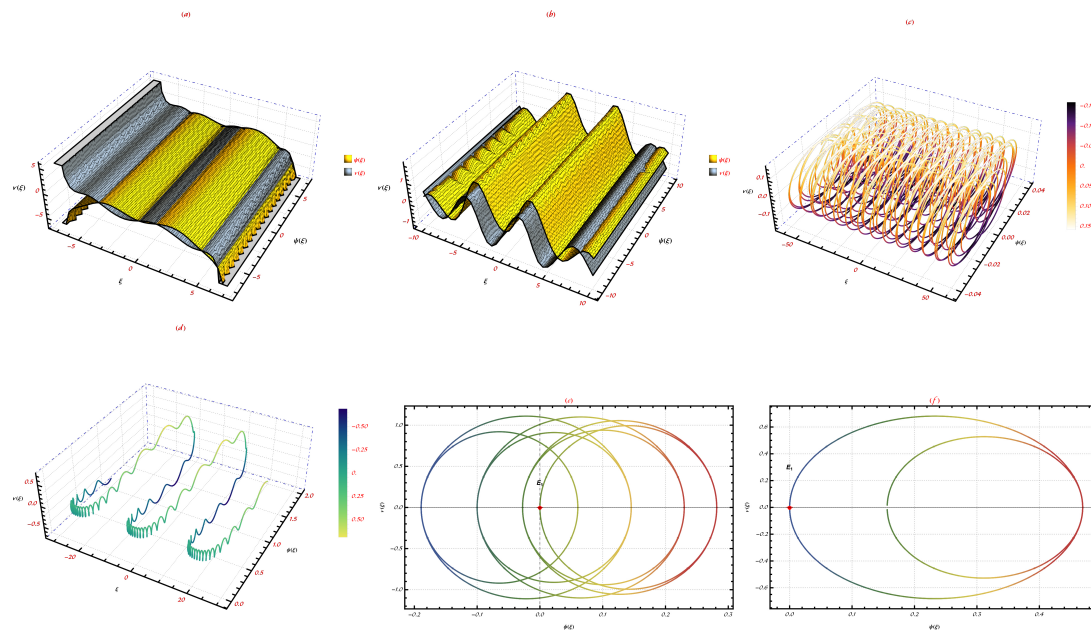


Figure 2: Phase portraits illustrating the chaotic and quasi-periodic behaviors of the FKMN model under perturbation. The introduction of a perturbation term $3 \cos(\Upsilon \xi)$ leads to complex dynamics, including quasi-periodic and chaotic behaviors. The amplitude 3 and frequency Υ of the perturbation significantly influence the system's trajectories, as depicted in the simulations.

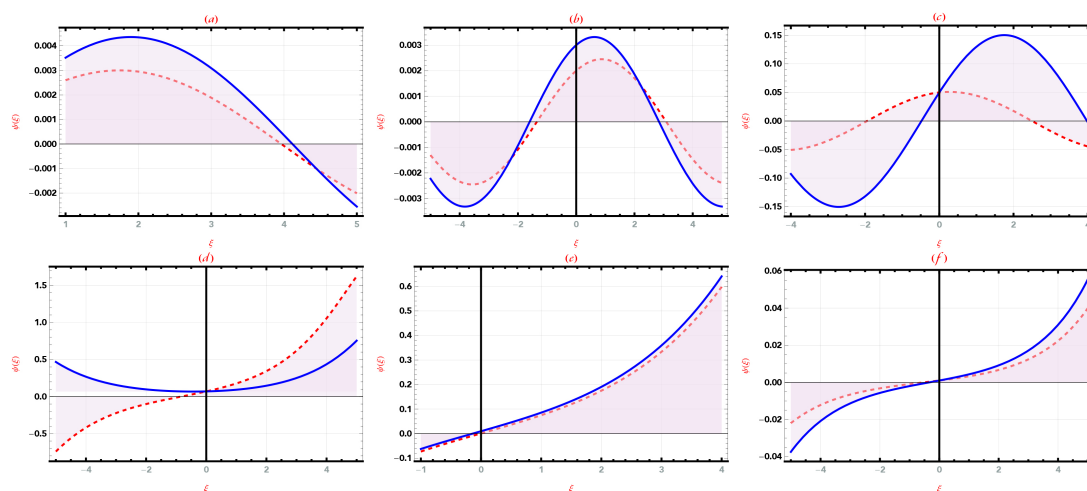


Figure 3: Sensitivity analysis of the FKMN model to variations in initial conditions. Numerical simulations demonstrate the profound impact of small changes in initial conditions on the system's trajectory. The results highlight the system's sensitivity and complexity, underscoring the importance of precise initial conditions in predicting the system's behavior.

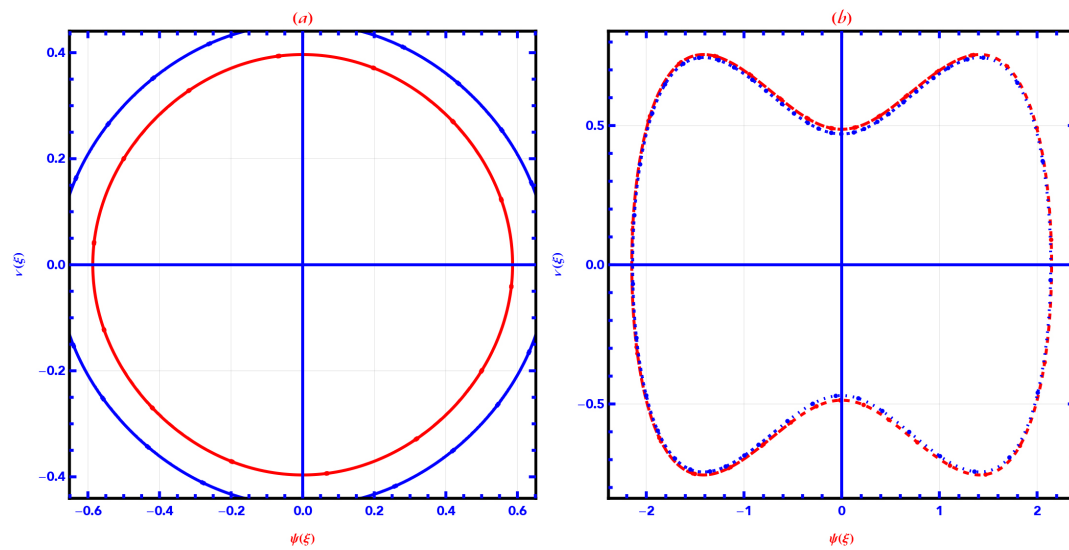


Figure 4: Sensitivity analysis of the FKMN model to variations in parameters. Numerical simulations demonstrate the profound impact of small changes in parameters on the system's trajectory. The results highlight the system's sensitivity and complexity, underscoring the importance of precise parameter values in predicting the system's behavior.

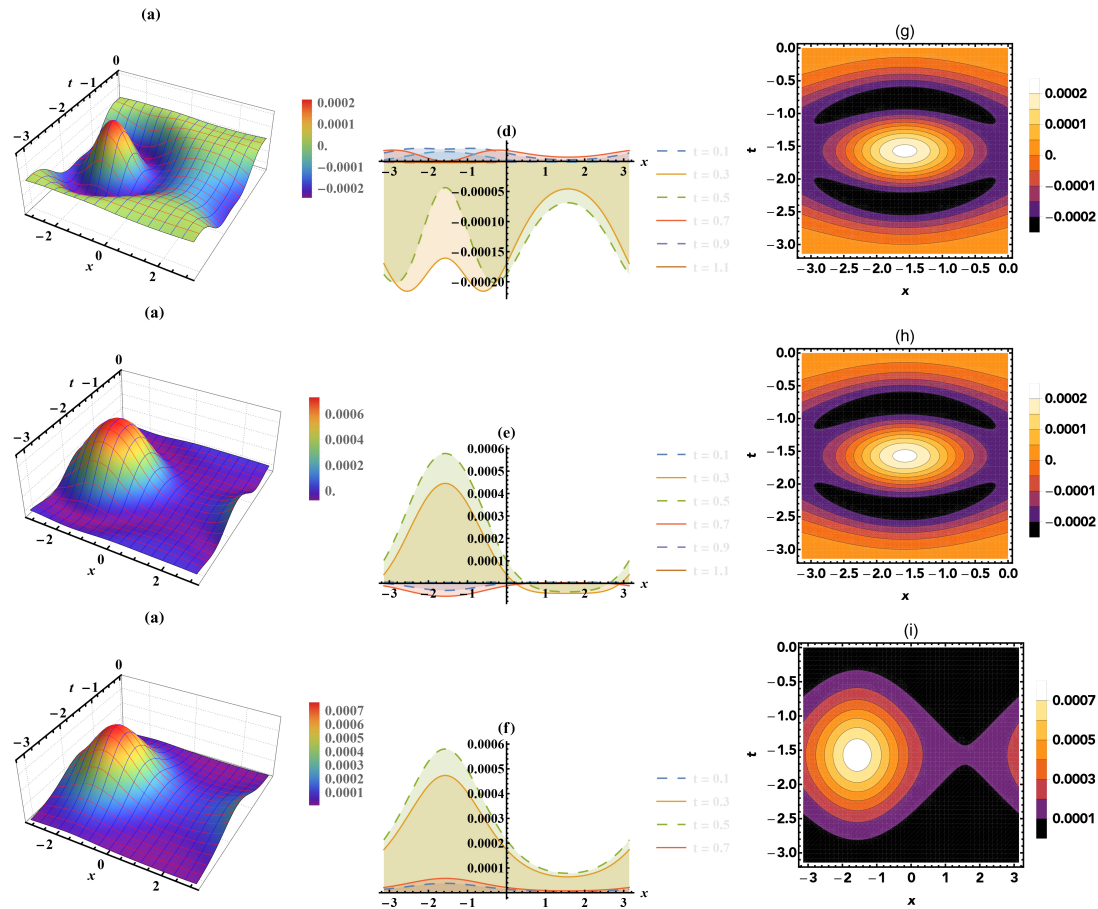


Figure 5: Graphical representation of the solution obtained via the analytical Khat II method. The subfigures (a, d, g) show the real component; (b, e, h) show the imaginary component; and (c, f, i) display the absolute value of the solution from Eq. (17). The evolution of the waveform is depicted, revealing key properties including amplitude, shape, and velocity. These figures visualize the spatial-temporal dynamics of the solution derived by the Khat II method. The real and imaginary components reflect the oscillatory behavior, while the absolute values provide insight into energy propagation. The solution showcases stability in amplitude over time, indicative of a soliton-like structure.

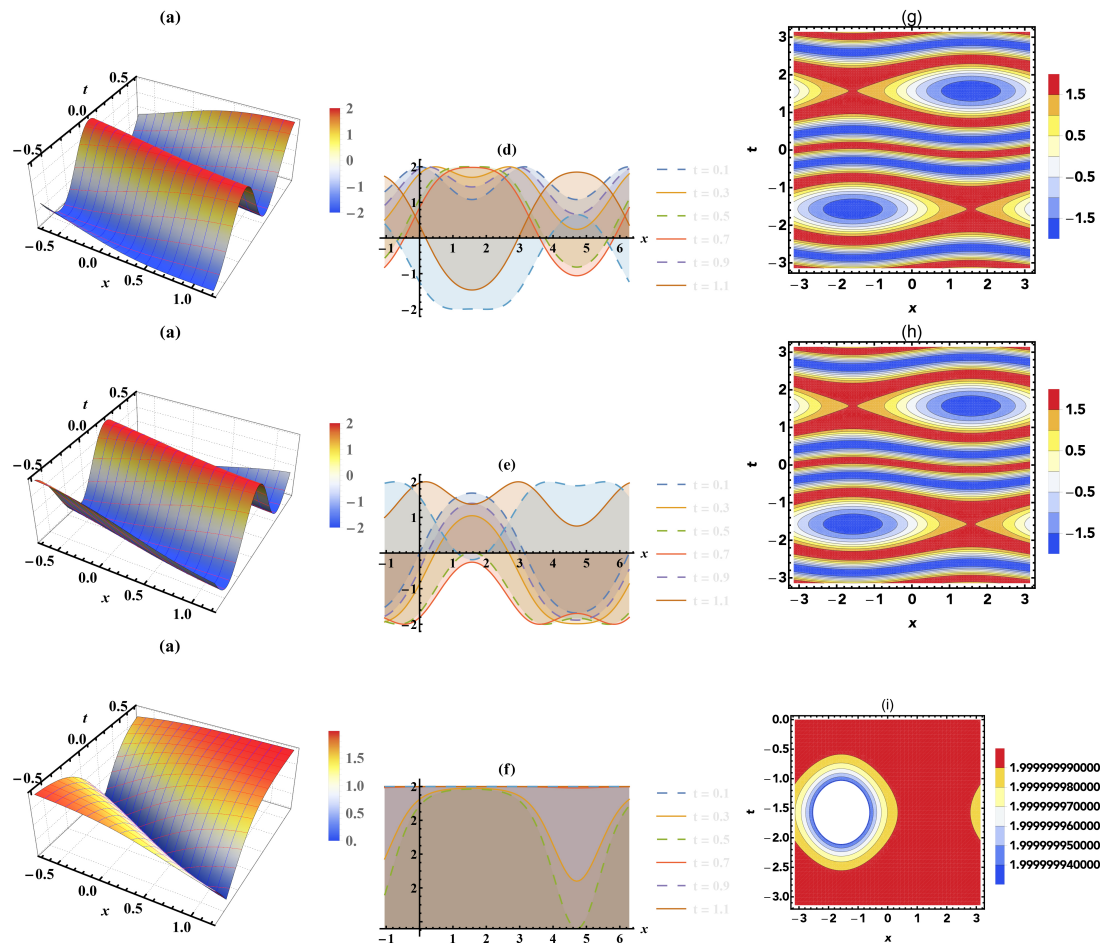


Figure 6: Solution representation via the Khat II method, showing the waveform's real (a, d, g), imaginary (b, e, h), and absolute (c, f, i) values from Eq. (21). These figures illustrate the waveform evolution for a different parameter set, providing insight into amplitude growth and waveform expansion. The periodic nature observed in imaginary components signifies energy transfer consistency, while the absolute value indicates solution stability and conservation properties.

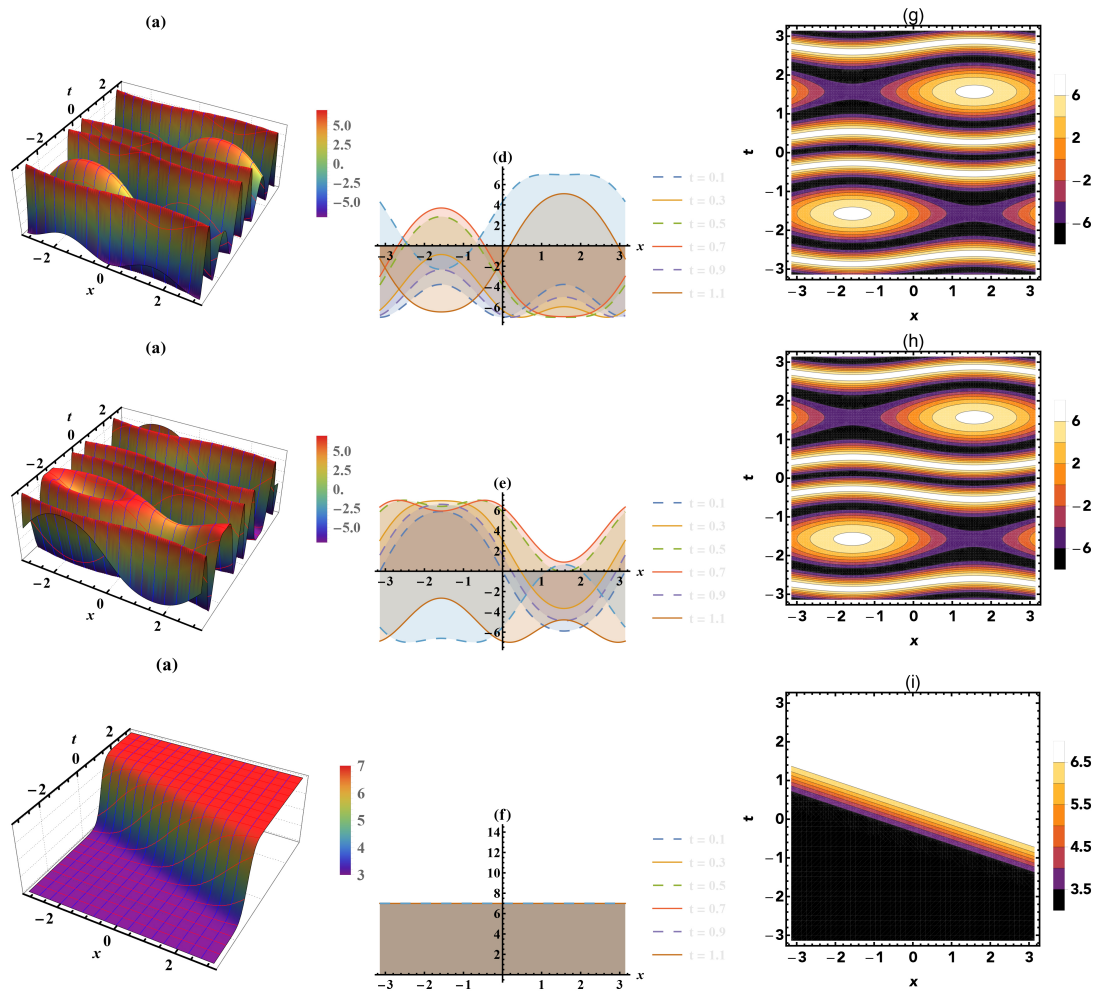


Figure 7: Graphical solution representation via the EKud method, displaying the real (a, d, g), imaginary (b, e, h), and absolute (c, f, i) values from Eq. (23). In this figure the EKud method's results reveal characteristics of the waveform, such as amplitude oscillation, phase shifts, and wave stability. The comparison between real and imaginary components allows for observing changes in wave phase, crucial for understanding energy distribution across space.

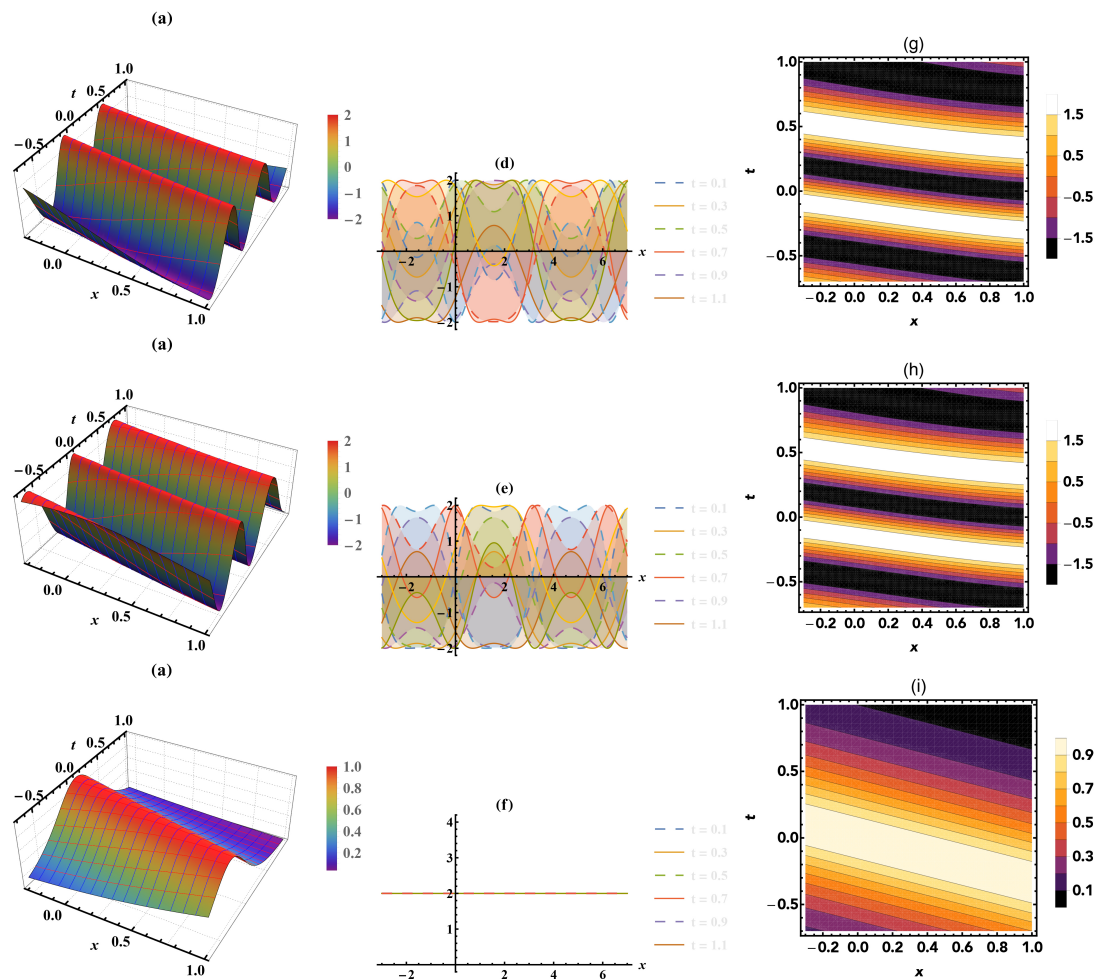


Figure 8: Solution representation for the investigated model obtained via the EKud method, with subfigures illustrating real, imaginary, and absolute values from Eq. (25). This figure shows the waveform behavior under different conditions, as predicted by the EKud method. This figure highlights phase stability and energy dispersion patterns, demonstrating the EKud method's utility in capturing the solution's intrinsic properties.

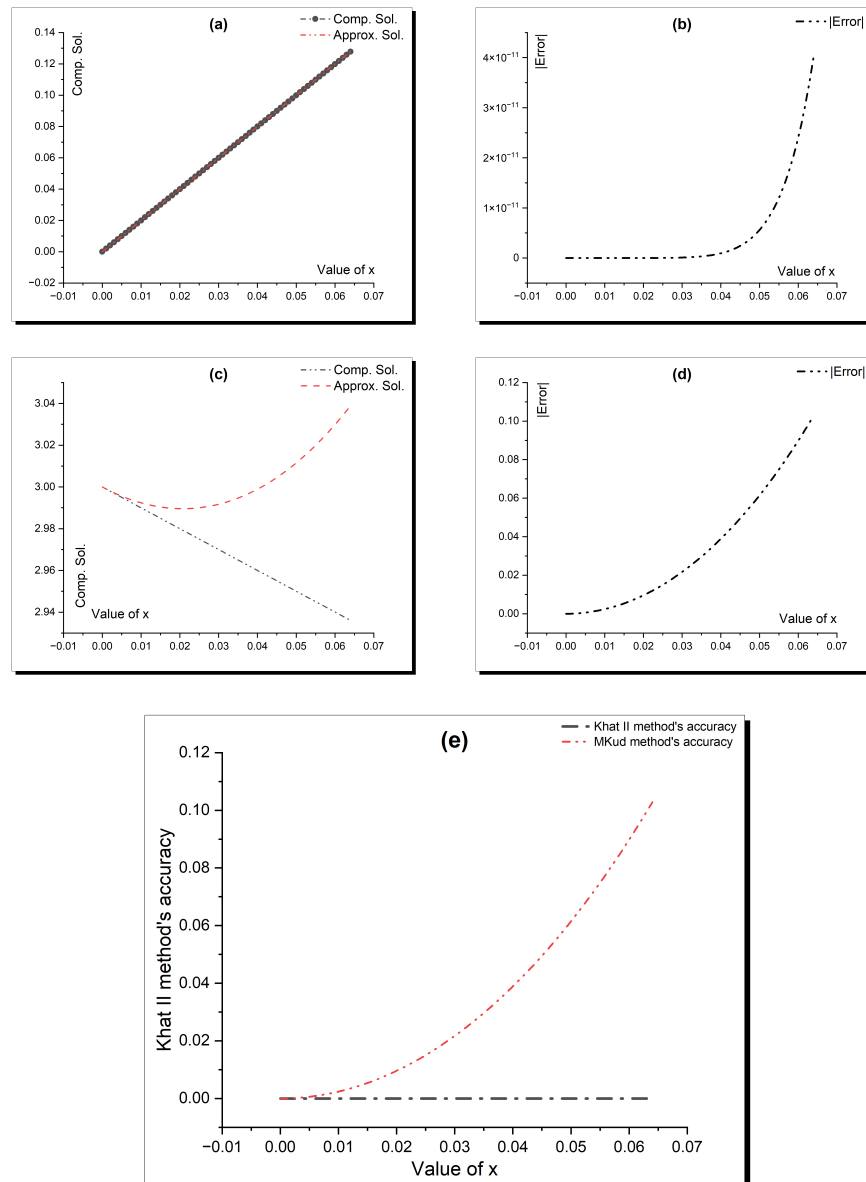


Figure 9: Comparative Analysis of Analytical and Numerical Solutions Using Khat II and EKud Methods: Subfigure (a) and subfigure (b) show the alignment between the analytical solution from the Khat II method (Eq. (17)) and the numerically constructed solution via the Adomian decomposition scheme, demonstrating high consistency and accuracy. Subfigure (c) and subfigure (d) illustrate the comparison between the EKud method's analytical solution (Eq. (23)) and its numerical counterpart, highlighting areas of agreement and variance. Subfigure (e) contrasts the Khat II and EKud methods in terms of accuracy and precision, underscoring the Khat II method's superior performance in approximating the model's behavior, particularly in stability and reliability over time.

5. Results and Discussion

This section synthesizes the analytical, numerical, and dynamical findings obtained for the $(2 + 1)$ -dimensional fractional FKMN equation. The analysis proceeds in three steps: (i) dynamical characterization of the model via bifurcation and sensitivity studies, (ii) construction of solitary-wave solutions using the KII and EKud methods, and (iii) validation of analytical results through comparison with the ADM. Together, these results provide insight into the stability, coherence, and parameter dependence of fractional nonlinear waveforms.

Dynamical characterization

Bifurcation diagrams and phase-space trajectories (Figs. 1, 2) demonstrate transitions from stable equilibria to quasiperiodic and chaotic regimes under parameter variation. These transitions highlight the role of dispersion and nonlinearity coefficients in shaping solution stability. Sensitivity analyses (Figs. 3, 4) further confirm that small changes in parameters or initial conditions can qualitatively alter long-term dynamics, underscoring the complexity of the system. Such features are consistent with the expected behavior of nonlinear dispersive systems governed by fractional operators.

Closed-form solutions

Application of the KII and EKud schemes yields explicit solitary-wave solutions. The KII Tech.-derived solutions (Figs. 5, 6) exhibit three consistent features: oscillatory real components, periodic imaginary components, and nearly constant modulus profiles. These collectively indicate soliton-like propagation with sustained coherence. Solutions obtained by the EKud method (Figs. 7, 8) show similar amplitude preservation and stable phase evolution, with parameter-dependent modulation that remains bounded over time. These results confirm that both analytical methods are effective for constructing families of fractional solitary waves.

Numerical validation

Tables 1 and 2 compare analytical results with ADM approximations. In both cases, absolute errors remain uniformly small across sampled points, confirming the reliability of ADM in reproducing closed-form solutions. The agreement between ADM reconstructions and analytical forms demonstrates that the solutions are not only mathematically consistent but also numerically robust under fractional operators.

Interpretation

The persistence of amplitude and phase coherence across a broad parameter range suggests that fractional temporal memory stabilizes nonlinear dispersive waveforms more effectively than integer-order formulations. The enlarged stability window observed in this

study supports the hypothesis that memory effects, encoded by fractional derivatives, can suppress instabilities and extend soliton-like behavior. These findings complement earlier work on fractional wave equations [8, 9], while providing new explicit solution families that align with semi-analytical approximations.

Limitations

Two limitations warrant consideration. First, truncation of fractional kernels may introduce approximation bias near bifurcation thresholds. Second, while ADM provides accurate reconstructions, its accuracy depends on truncation order and step size; small deviations in amplitude under parameter variation likely stem from these numerical constraints rather than model inconsistency. A systematic quantification of these effects is left for future work.

Implications

The results establish a hybrid analytical–numerical framework for studying fractional nonlinear equations. The derived solitary–wave families demonstrate stability and coherence across parameter regimes relevant to optical fibers, magnetized plasmas, and shallow–water flows. These findings indicate that the FKMN model is well suited for capturing nonlocal and memory–driven dynamics in dispersive media, and that fractional operators provide a powerful extension of classical nonlinear evolution equations for both theoretical investigation and applied modeling.

6. Conclusion

This study has investigated the $(2 + 1)$ -dimensional fractional FKMN equation formulated with truncated Mittag–Leffler and β -fractional operators. By applying the KII and EKud methods, we derived families of solitary–wave solutions that exhibit oscillatory coherence, amplitude preservation, and soliton-like stability across a range of parameter regimes. Independent validation through the ADM confirmed close agreement between analytical solutions and semi-analytical reconstructions, with consistently small errors. These results demonstrate that fractional temporal memory can stabilize nonlinear dispersive waveforms and broaden the conditions under which coherent propagation persists.

Despite these advances, several limitations remain. The truncation of fractional operators may introduce approximation errors, particularly near regime boundaries, and the Adomian decomposition method, while accurate, is sensitive to truncation order and discretization. These factors may explain minor discrepancies in amplitude profiles under certain parameter variations.

Future research should extend the present analysis in several directions: (i) exploring alternative fractional kernels and higher-order approximations to assess their effect on sta-

bility margins; (ii) conducting systematic parameter studies to map bifurcation structures and delineate precise stability boundaries; and (iii) comparing the fractional model against integer-order baselines to isolate the specific influence of memory effects. Extensions incorporating spatial heterogeneity, cross-diffusion, or stochastic perturbations would further clarify the role of fractional dynamics in realistic environments.

Overall, this work establishes the FKMN equation as a rigorous and flexible framework for modeling nonlinear dispersive systems with memory effects. The analytical and semi-analytical solutions presented here contribute to the broader theory of fractional evolution equations and provide a foundation for future applications in optical, fluid, and plasma systems where robust, soliton-like waveforms are of central importance.

Appendix

Here, we provide the definition of the truncated Mittag-Leffler function and outline the fundamental properties of the truncated β -fractional derivative, following the framework established in [25].

Definition 1. *The truncated Mittag-Leffler function with a single parameter is defined as follows [26]:*

$$\mathcal{Q}_{\mathfrak{Z}}^i(\kappa) = \sum_{n=0}^i \frac{\kappa^n}{\Gamma(\mathfrak{Z}n + 1)}, \quad (27)$$

where $\mathfrak{Z} > 0$ and $\kappa \in \mathbb{C}$.

Definition 2. *Let $\mathcal{U} : [0, \infty) \rightarrow \mathbb{R}$. The truncated β -fractional derivative of the function $\mathcal{U}$ of order ε is defined by the following expression [27]:*

$${}_i\mathcal{D}_{\beta}^{\varepsilon, \mathfrak{Z}}(\mathcal{U}(\xi)) = \lim_{\varepsilon \rightarrow 0} \frac{\mathcal{U}(\xi \mathcal{E}_{\mathfrak{Z}}^i(\varepsilon \xi^{1-\varepsilon})) - \mathcal{U}(\xi)}{\varepsilon}, \quad \forall \xi > 0, \quad (28)$$

where $0 < \varepsilon < 1$ and $\mathfrak{Z} > 0$. If the truncated β -fractional derivative of the function $\mathcal{U}$ of order ε exists, then the function $\mathcal{U}$ is said to be ε -differentiable.

Theorem 1. *Let $0 < \varepsilon \leq 1$, $\mathfrak{Z} > 0$, and $\delta_1, \delta_2 \in \mathbb{R}$, with $\mathcal{U}$ and $\mathcal{Z}$ being ε -differentiable*

functions at a point $\xi > 0$. Then, the following properties hold [28]:

$$\left\{ \begin{array}{l} {}_i\mathcal{D}_{\beta}^{\varepsilon,3}(\delta_1 \mathcal{U}(\xi) + \delta_2 \mathcal{Z}(\xi)) = \delta_1 {}_i\mathcal{D}_{\beta}^{\varepsilon,3}(\mathcal{U}(\xi)) + \delta_2 {}_i\mathcal{D}_{\beta}^{\varepsilon,3}(\mathcal{Z}(\xi)), \\ {}_i\mathcal{D}_{\beta}^{\varepsilon,3}(\mathcal{U}(\xi) \cdot \mathcal{Z}(\xi)) = \mathcal{U}(\xi) {}_i\mathcal{D}_{\beta}^{\varepsilon,3}(\mathcal{Z}(\xi)) + \mathcal{Z}(\xi) {}_i\mathcal{D}_{\beta}^{\varepsilon,3}(\mathcal{U}(\xi)), \\ {}_i\mathcal{D}_{\beta}^{\varepsilon,3}\left(\frac{\mathcal{U}(\xi)}{\mathcal{Z}(\xi)}\right) = \frac{\mathcal{Z}(\xi) {}_i\mathcal{D}_{\beta}^{\varepsilon,3}(\mathcal{U}(\xi)) - \mathcal{U}(\xi) {}_i\mathcal{D}_{\beta}^{\varepsilon,3}(\mathcal{Z}(\xi))}{(\mathcal{Z}(\xi))^2}, \\ {}_i\mathcal{D}_{\beta}^{\varepsilon,3}(\delta) = 0 \quad \text{for a constant } \delta = \mathcal{U}(\xi). \end{array} \right. \quad (29)$$

For a differentiable function $\mathcal{U}$, the truncated β -fractional derivative is given by

$${}_i\mathcal{D}_{\beta}^{\varepsilon,3}(\mathcal{U}(\xi)) = \frac{\xi^{1-\varepsilon}}{\Gamma(3+1)} \frac{d\mathcal{U}(\xi)}{d\xi}. \quad (30)$$

Declarations

Author contribution statement:

Mostafa M. A. Khater; Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

Ethics approval and consent to participate

Not applicable.

Use of AI tools declaration

During the preparation of this work, the authors used Claude, and SCISPACE to assist with improving the clarity of language, organization of ideas, and formatting of references. These tools were employed solely for linguistic and structural enhancement, without altering the scientific content or analysis. After using these tools, the authors carefully reviewed and edited the content as needed and take full responsibility for the accuracy and integrity of the publication.

Consent for publication

Not applicable.

Availability of data and material

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Competing interests

The author declares that he has no competing interests.

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Authors' contributions

All the study has been done by the author himself.

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