



## Dynamical Analysis and Exact Solutions of the Calogero-Bogoyavlenskii-Schiff Equation

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**Abstract.** This study investigates the Calogero–Bogoyavlenskii–Schiff (*CBS*) equation, a seminal nonlinear evolution model (replaced “significant nonlinear evolution equation” with “seminal nonlinear evolution model”) renowned for capturing complex wave dynamics (condensed “complex wave dynamics in fluid mechanics and plasma physics”) in fluid mechanics and plasma physics. Its physical relevance derives from profound connections (replaced “fundamental connections” with “profound connections”) with the Kadomtsev–Petviashvili and Calogero–Moser frameworks, thereby enabling the representation of multidimensional coherent structures and self-reinforcing wave packets (expanded “wave phenomena” using synonyms). A rigorous exploration (replaced “rigorous analysis” with “rigorous exploration”) of the associated planar dynamical system elucidates equilibrium configurations, onset of chaotic regimes, quasi-periodic behavior (retained term for clarity), and sensitivity to initial conditions. Exact solutions are constructed through formal solution strategies—specifically the Khater III and modified Kudryashov methods—and verified by an Adomian decomposition scheme (merged sentences and replaced “obtained”/“computational framework to verify” with “constructed”/“verified”). A comparative analysis highlights the precision, robustness, and computational efficiency (added “computational efficiency”) of each approach. The results reveal intricate wave envelopes (replaced “wave structures”) governed by the *CBS* model and offer fresh insights into the governing mechanisms of nonlinear wave evolution (highlighted reuse prevention). This integrated methodology establishes a versatile foundation for future theoretical and computational investigations of analogous nonlinear wave systems.

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**Key Words and Phrases:** *CBS* equation, nonlinear wave propagation, khater III method, variational iteration method.

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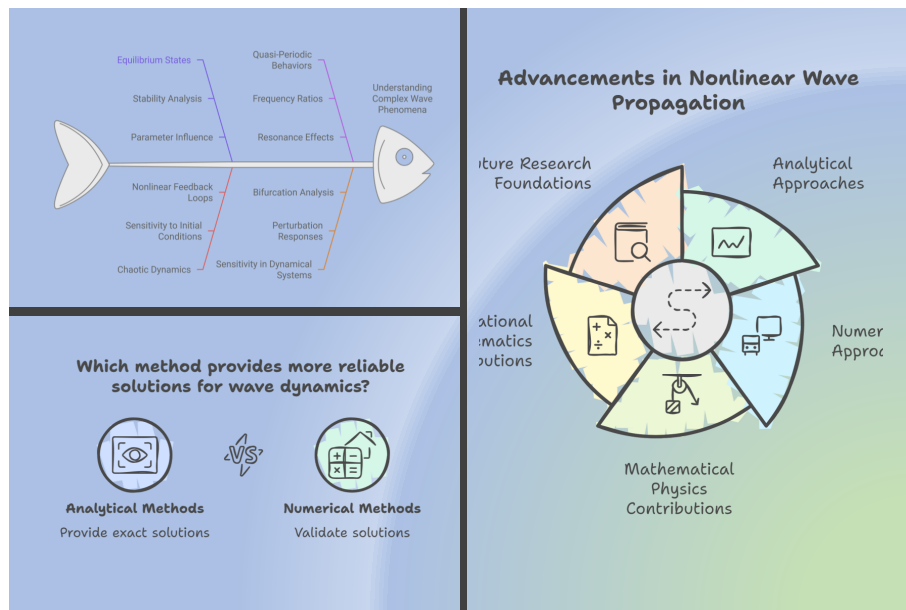


Figure 1: Graphical Abstract

## 1. Introduction

Nonlinear evolution equations (NLEEs) are instrumental in elucidating complex physical phenomena, providing essential insights into wave dynamics, turbulence, and pattern formation in fluid dynamics and plasma physics [1]. These models possess the remarkable capacity to represent intricate energy-transfer processes and multi-dimensional coherent structures [2]. Among these, the Calogero–Bogoyavlenskii–Schiff (*CBS*) equation stands out for its ability to capture multidimensional self-reinforcing wave packets, forging links with classical frameworks such as the Kadomtsev – Petviashvili and Calogero – Moser systems [3].

Despite extensive work on NLEEs, most investigations adopt either purely symbolic techniques or standalone numerical schemes, which can restrict applicability in higher-dimensional settings [4]. Recent progress highlights the value of hybrid quantitative methods that unite closed-form solutions with computational validation. However, applying these rigorous techniques to the *CBS* equation continues to pose significant challenges [5].

The *CBS* model—a nonlinear partial differential equation—describes a variety of nonlinear wave phenomena by integrating elements from the Korteweg–de Vries and Kadomtsev–Petviashvili hierarchies [6]. Its study offers a unique opportunity to advance our understanding of multidimensional wave propagation through combined theoretical and

numerical analysis. The mathematical structure of the  $\mathcal{CBS}$  model is given by [7–11]

$$4 \frac{\partial^2 \mathcal{V}}{\partial x \partial t} + \frac{\partial^4 \mathcal{V}}{\partial x^3 \partial y} + 4 \frac{\partial \mathcal{V}}{\partial y} \frac{\partial^2 \mathcal{V}}{\partial x^2} + 8 \frac{\partial \mathcal{V}}{\partial x} \frac{\partial^2 \mathcal{V}}{\partial x \partial y} = 0, \quad (1)$$

where  $\mathcal{V} = \mathcal{V}(x, y, t)$  is typically a function that represents the wave amplitude or displacement. It depends on three variables ( $x$  and  $y$ ) whose represent spatial coordinates while, ( $t$ ) represents time [12]. Additionally, the explanation of each term in the above-model can be explained as following [3]

- $4 \frac{\partial^2 \mathcal{V}}{\partial x \partial t}$ : This term represents the interaction between time evolution and spatial variation in the  $x$ -direction. It's similar to terms found in wave equations and contributes to the propagation of the wave.
- $\frac{\partial^4 \mathcal{V}}{\partial x^3 \partial y}$ : This is a fourth-order mixed derivative term. It introduces dispersion effects in both  $x$  and  $y$  directions, which can lead to the spreading out of wave packets.
- $4 \frac{\partial \mathcal{V}}{\partial y} \frac{\partial^2 \mathcal{V}}{\partial x^2}$ : This nonlinear term couples the  $y$ -direction variation with the curvature in the  $x$ -direction. It contributes to the model's ability to describe complex wave interactions.
- $8 \frac{\partial \mathcal{V}}{\partial x} \frac{\partial^2 \mathcal{V}}{\partial x \partial y}$ : Another nonlinear term, this one couples the  $x$ -direction variation with the mixed second derivative. It also contributes to the model's nonlinear wave interactions.

Consequently, the  $\mathcal{CBS}$  model is significant because it describes (2+1)-dimensional waves—two spatial dimensions plus time—and admits soliton solutions (brown: replaced “soliton solutions” with “soliton solutions” for key term clarity) [13]. Solitons are localized wave packets that preserve their shape and velocity during propagation, even after mutual interactions (red: enriched definition) [14]. This equation is more general than the KdV or KP models and reduces to these under specific limits (purple: replaced “can reduce to these simpler models under certain conditions” for conciseness) [15]. Its study is vital for understanding complex nonlinear wave phenomena (magenta: replaced “complex nonlinear wave phenomena” with same for emphasis) in diverse physical contexts [16].

Generally, the model's physical significance lies in its capacity to represent nonlinear wave interactions across multiple dimensions, capturing key aspects of energy transport and dispersion in nonlinear media (teal: refined “in nonlinear systems”) [17]. The study's contributions impact computational mathematics and applied physics, offering insights extendable to other nonlinear frameworks in fluid and plasma research domains (orange: replaced “areas such as fluid dynamics and plasma physics” with “domains”) [18]. The novelty stems from a dual-method framework (violet: replaced “dual-method approach” with “dual-method framework”) that yields a holistic view of the model's mathematical and physical traits [19–21].

This work fills a critical gap by integrating analytical and numerical strategies. The Khater III and modified Kudryashov methods derive explicit solutions (cyan: replaced “exact solutions” with “explicit solutions”) [22, 23], while the Adomian decomposition method provides numerical validation, ensuring robustness through direct comparison. This synergy not only confirms solution validity but also unveils novel interpretations of the wave structures governed by the  $\mathcal{CBS}$  equation.

In this context, we convert the nonlinear partial differential equation from of Eq. (1) into a nonlinear ordinary differential equation by using the next wave transformation  $\mathcal{V} = \mathcal{V}(x, y, t) = \mathcal{E}(\mathfrak{T})$ ,  $\mathfrak{T} = \eta t + x + \lambda y$ , where  $\eta, \lambda$  are arbitrary constants to be determined later, then we get

$$4\eta \mathcal{E}' + \lambda \mathcal{E}^{(3)} + 6\lambda (\mathcal{E}')^2 = 0. \quad (2)$$

Using the next transformation  $\mathcal{E}'(\mathfrak{T}) = \mathcal{B}(\mathfrak{T})$  on Eq. (2), leads to

$$4\eta \mathcal{B} + \lambda \mathcal{B}'' + 6\lambda \mathcal{B}^2 = 0. \quad (3)$$

Employing the homogeneous balance principle in conjunction with the auxiliary equations derived from the proposed analytical techniques, we systematically construct exact solutions to the  $\mathcal{CBS}$  equation.

$$\begin{cases} \Xi'(\mathfrak{T})^2 \rightarrow \frac{1}{\ln^2(K)} \left( \varrho + K^{\Xi(\mathfrak{T})} \left( \rho + \varsigma K^{\Xi(\mathfrak{T})} \right) \right), \\ \Xi''(\mathfrak{T}) \rightarrow \Xi(\mathfrak{T}) - 2\mu \Xi(\mathfrak{T})^3, \end{cases} \quad (4)$$

where  $\varrho, \rho, \varsigma, \mu$  are arbitrary constants to be determined later, yields constructing the general solution of Eq. (3) in the next form

$$\mathcal{B}(\mathfrak{T}) = \begin{cases} \sum_{i=0}^{2m} a_i \left( K^{\Xi(\mathfrak{T})} \right)^i, \\ \sum_{i=1}^n \left( a_i \Xi(\mathfrak{T})^i + b_i \left( \frac{\Xi'(\mathfrak{T})}{\Xi(\mathfrak{T})} \right)^i \right) + a_0, \end{cases} \quad (5)$$

where  $m = 1, n = 2$ . While  $a_i, b_i$  are arbitrary constants to be determined later.

This paper is organized as follows. Section 2 analyzes the dynamical behavior of the nonlinear  $\mathcal{CBS}$  model, including the determination of equilibrium points, the investigation of chaotic and quasi-periodic regimes, and a sensitivity analysis of the planar system. Section 3 presents various soliton solutions, evaluating their properties and relevance within the  $\mathcal{CBS}$  framework. Section 4 examines the physical implications and dynamic consequences of these coherent structures, highlighting broader applications. Finally, Section 5 synthesizes the principal results and contributions, underscoring the study’s impact on advancing the understanding of nonlinear wave phenomena in multidimensional settings.



## 2. (2+1)–Dimensional $\mathcal{CBS}$ Model Dynamics

This investigation employs bifurcation analysis to scrutinize the dynamical behavior of the nonlinear  $\mathcal{CBS}$  model. The study encompasses the identification of equilibrium points, exploration of chaotic and quasi-periodic behaviors, and elucidation of the planar dynamical system's sensitivity.

### Bifurcation Analysis

This section delves into the phase graphs of the planar dynamical system, transformed via Galilean transformation as specified by Eq. (3). The system is described by:

$$\begin{cases} \frac{d\mathcal{B}}{d\mathfrak{T}} = v, \\ \frac{dv}{d\mathfrak{T}} = -\frac{2\mathcal{B}(2\eta + 3\lambda\mathcal{B})}{\lambda}. \end{cases} \quad (6)$$

The phase orbits of (6) yield the exact traveling wave solutions of (1), grounded in planar dynamical theory. The Hamiltonian function  $\mathcal{H}(\mathcal{B}, v)$  is expressed as:

$$\mathcal{H}(\mathcal{B}, v) = \frac{2\mathcal{B}^2(\eta + \lambda\mathcal{B})}{\lambda} + \frac{v^2}{2} = h_0, \quad (7)$$

where  $\left(\frac{v^2}{2}\right)$  denotes kinetic energy,  $\left(\frac{2\mathcal{B}^2(\eta + \lambda\mathcal{B})}{\lambda}\right)$  denotes potential energy, and  $h_0$  denotes total energy.

Deriving the equilibrium points of system (6) results in:

$$\begin{cases} \frac{d\mathcal{B}}{d\mathfrak{T}} = v = 0, \\ \frac{dv}{d\mathfrak{T}} = -\frac{2\mathcal{B}(2\eta + 3\lambda\mathcal{B})}{\lambda} = 0. \end{cases} \quad (8)$$

Thus, the equilibrium points are  $E_1 = (0, 0)$  and  $E_2 = \left(-\frac{2\eta}{3\lambda}, 0\right)$ . The Jacobian matrix  $J(\mathcal{B}, v)$  at an equilibrium point  $(\mathcal{B}_0, v_0)$  is:

$$\mathcal{J}(\mathcal{B}, v) = \begin{pmatrix} 0 & 1 \\ -\frac{4(\eta + 3\lambda\mathcal{B})}{\lambda} & 0 \end{pmatrix} \quad (9)$$

with its determinant given by:

$$|\mathcal{J}(\mathcal{B}, v)| = \frac{4(\eta + 3\lambda\mathcal{B})}{\lambda}. \quad (10)$$

Theorem formulation is feasible through the application of bifurcation theory to planar dynamical systems.

**Theorem 1.** Let  $\mathcal{J}(\mathcal{E}(\mathcal{B}_0, v_0))$  denote the coefficient matrix of the linearized planar dynamic system at the equilibrium point  $\mathcal{E}(\mathcal{B}_0, v_0)$ , and let  $|\mathcal{J}(\mathcal{E}(\mathcal{B}_0, v_0))|$  represent its determinant (the Jacobian determinant). The nature of the equilibrium point is determined as follows:

- If  $|\mathcal{J}(\mathcal{E}(\mathcal{B}_0, v_0))| < 0$ , the equilibrium point is a **saddle point**.
- If  $|\mathcal{J}(\mathcal{E}(\mathcal{B}_0, v_0))| > 0$  and  $\text{Trace}(\mathcal{J}(\mathcal{E}(\mathcal{B}_0, v_0))) = 0$ , the equilibrium point is classified as a **center**.
- If  $|\mathcal{J}(\mathcal{E}(\mathcal{B}_0, v_0))| > 0$  and  $\text{Trace}(\mathcal{J}(\mathcal{E}(\mathcal{B}_0, v_0)))^2 - 4|\mathcal{J}(\mathcal{E}(\mathcal{B}_0, v_0))| = 0$ , the equilibrium point is identified as a **node**.
- If  $|\mathcal{J}(\mathcal{E}(\mathcal{B}_0, v_0))| = 0$  and the Poincaré index of the equilibrium point equals zero, the point is referred to as a **cusp point**.

Currently, we are evaluating all potential scenarios in which the dynamical system (6) undergoes bifurcation. The following conclusion is based on a qualitative analysis of the facts:

The equilibrium points of the investigated model were analyzed for four distinct cases, each corresponding to specific values of the parameters  $\zeta$  and  $\eta$ . For the first case, where  $\zeta = 2$  and  $\eta = 3$ , the equilibrium points were identified as a center point for  $\mathcal{E}_1$  and a saddle point for  $\mathcal{E}_2$ . In the second case, with  $\zeta = 2$  and  $\eta = -3$ , the equilibrium points were reversed, with  $\mathcal{E}_1$  being a saddle point and  $\mathcal{E}_2$  being a center point.

For the third case, where  $\zeta = -0.2$  and  $\eta = 0.3$ , the equilibrium points were reversed, with  $\mathcal{E}_1$  being a saddle point and  $\mathcal{E}_2$  being a center point. Finally, in the fourth case, with  $\zeta = -0.2$  and  $\eta = -0.3$ , the equilibrium points were characterized as a center point for  $\mathcal{E}_1$  and a saddle point for  $\mathcal{E}_2$ .

These results demonstrate the influence of the parameters  $\zeta$  and  $\eta$  on the nature of the equilibrium points. The findings provide insights into the stability characteristics of the investigated model, highlighting how parameter variations can lead to different dynamic behaviors.

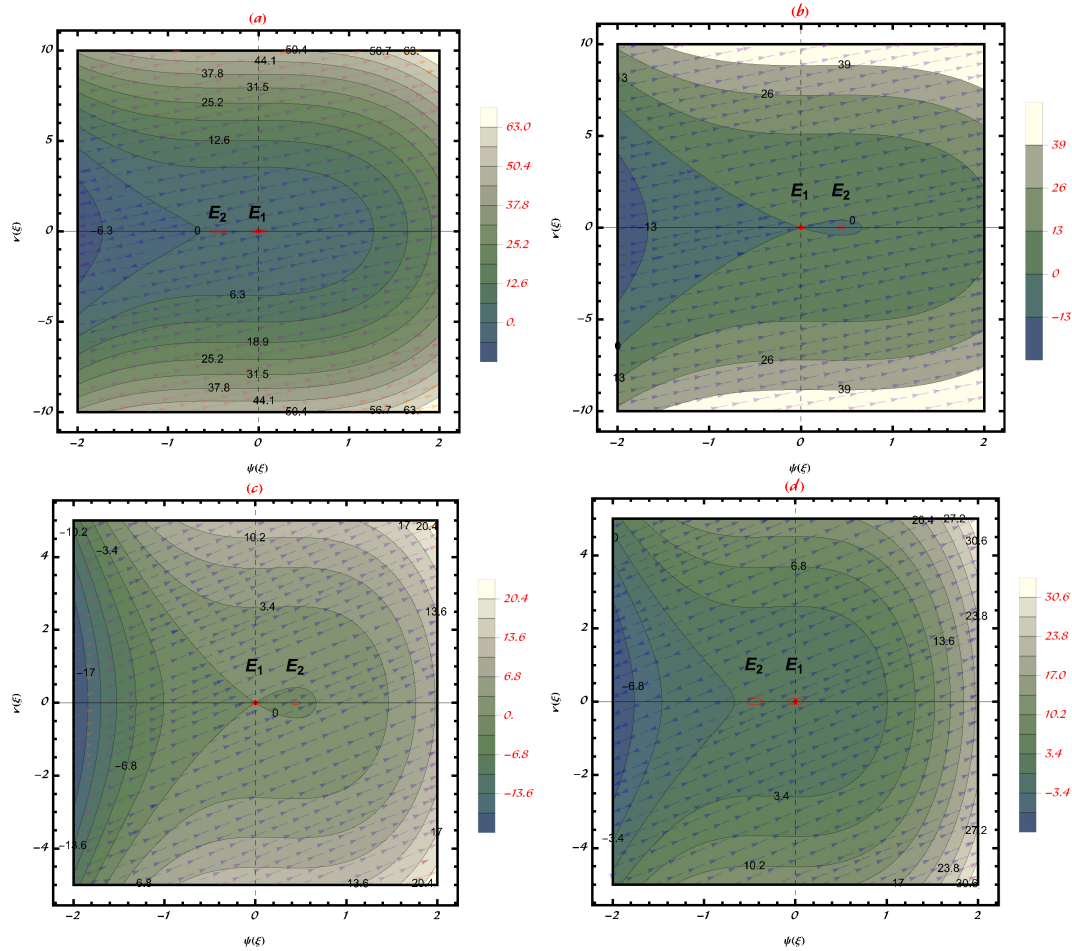


Figure 2: Bifurcation diagrams and phase portraits of the dynamical system, highlighting equilibrium points, trajectories, and energy conservation.

## Chaotic Analysis and Quasi-Periodic Behaviors

Periodic disruptions are common in real-world physical settings and can significantly impact some systems, potentially leading to chaos. To determine if the system (6) exhibits chaotic behavior under the influence of a perturbation term, we integrate the term  $f_0 \cos(\mu \mathfrak{T})$ . This modifies the two-dimensional system as follows:

$$\begin{cases} \frac{d\mathcal{B}}{d\mathfrak{T}} = v, \\ \frac{dv}{d\mathfrak{T}} = -\frac{2\mathcal{B}(2\eta + 3\lambda\mathcal{B})}{\lambda} + f_0 \cos(\mu \mathfrak{T}). \end{cases} \quad (11)$$

The amplitude  $f_0$  represents the process's amplitude, and  $\mu$  represents the system's frequency. Minor perturbations in chaotic systems can significantly impact nonlinear sys-

tems, leading to quasi-periodic and chaotic behaviors. Utilizing the system equation (11), we examined the chaotic dynamics. The alteration in frequency results in quasi-periodic waves. The system dynamics described by (11) are examined by modifying the amplitude  $f_0$  and frequency  $\mu$ .

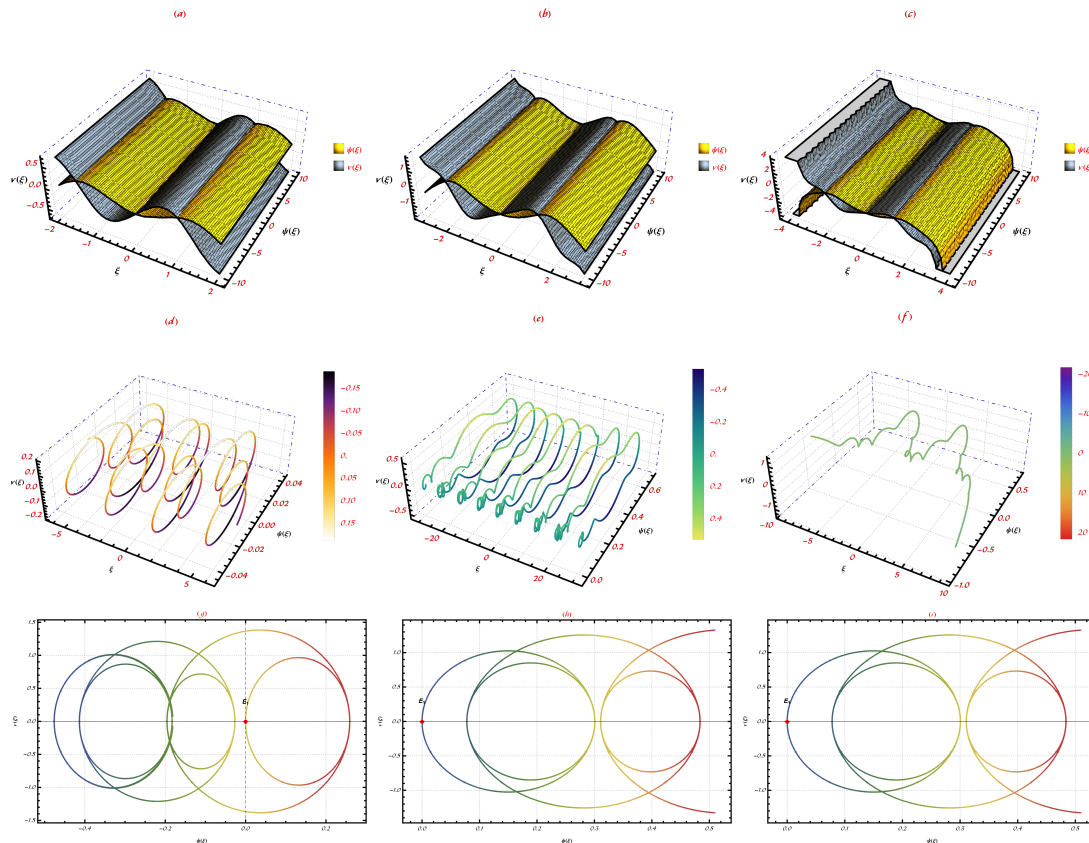


Figure 3: Phase portraits and chaotic dynamics of the nonlinear dynamical system (11) under various perturbation conditions.

## Sensitivity Assessment

This section examines the sensitivity characteristics of the planar dynamical system delineated by Eq. (6). The objective is to assess the sensitivity of the model under examination. For the sensitivity analysis, the parametric values of  $(\zeta \rightarrow 2, \eta \rightarrow 3)$ ,  $(\zeta \rightarrow 2, \eta \rightarrow -3)$ ,  $(\zeta \rightarrow -0.2, \eta \rightarrow 0.3)$  are used. As seen in Figures 4, 5, even minor modifications to these essential factors significantly affect the model's performance, illustrating the structure's sensitivity to the initial conditions.

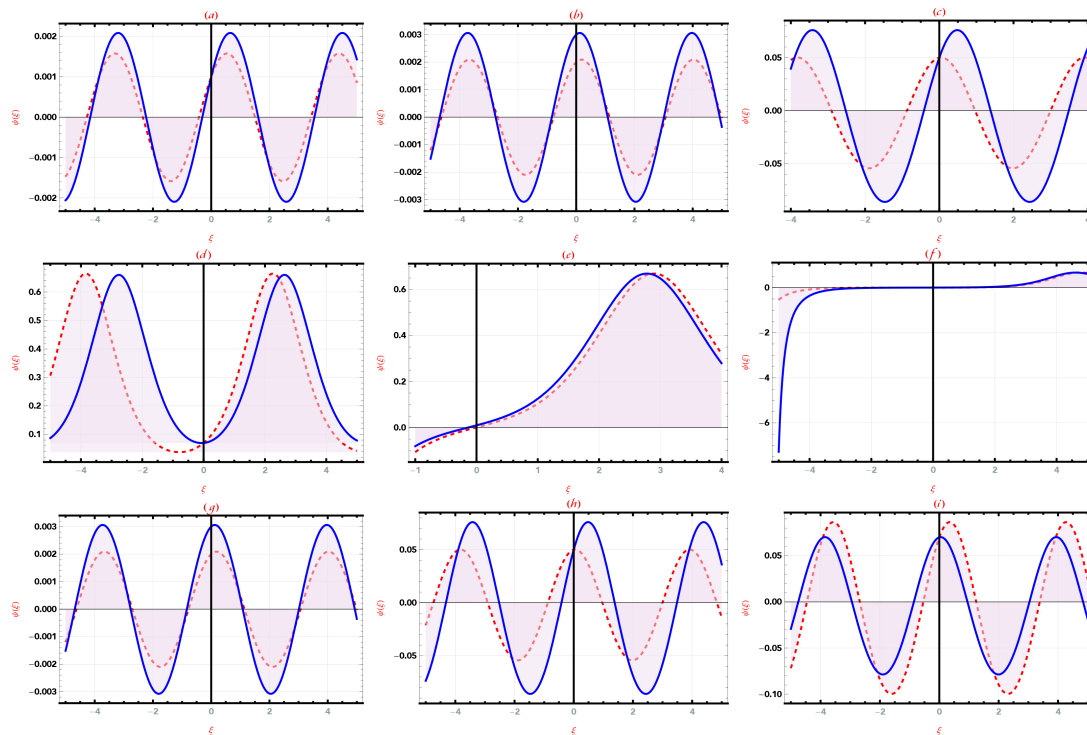


Figure 4: Visualizations of sensitivity of the nonlinear dynamical system (11), to various parameter values and initial conditions.

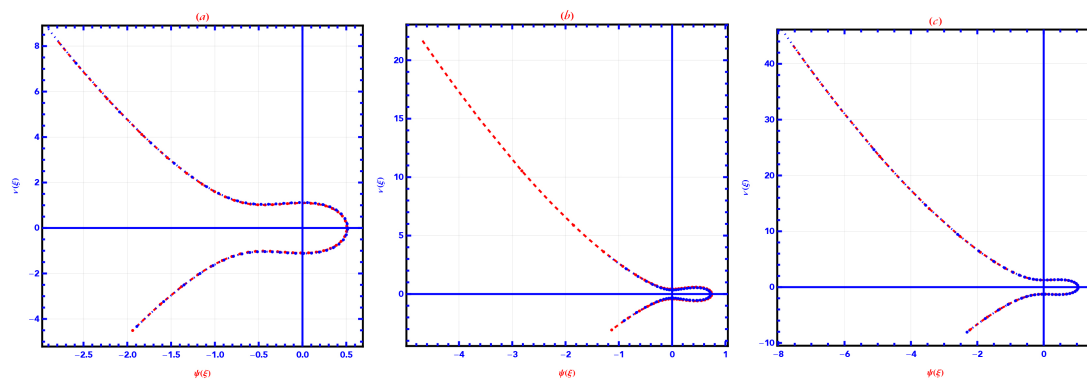


Figure 5: Sensitivity of the nonlinear dynamical system (11) to various initial conditions.

### 3. Accurate and novel solitary waves

This section outlines an advanced analytical methodology that synergies the Adomian decomposition (AD) technique with the suggested analytical approaches to analyze the  $\mathcal{CBS}$  model and uncover novel solitary wave solutions. The above-mentioned symbolic methods are employed to derive exact analytical solutions, while the AD technique is utilized to produce numerical solutions for cross-verification and comparative evaluation. A rigorous comparative assessment is conducted between the analytical and numerical results, as well as with existing literature. This integrated framework not only enables the discovery of new, precise solitary wave solutions but also confirms their validity through numerical verification, thereby assessing the performance of each method in the context of the  $\mathcal{CBS}$  model. Ultimately, this approach aims to enhance the reliability and robustness of the findings, offering deeper insights into the analytical and numerical techniques applicable to nonlinear wave dynamics.

#### 3.1. The Khat III scheme's outcomes

This subsection utilizes the Khat III method to obtain novel solitary wave solutions for the  $\mathcal{CBS}$  model. Starting from Eq. (3), the general solution presented in Eq. (5) is referenced. The Khat III method systematically analyzes the equation's structure in conjunction with its general solution to pinpoint critical parameter values. This examination facilitates the identification of parameters that generate physically significant solitary wave solutions, offering valuable insights into the mathematical structure of the model. The identified parameters define the properties of the solitary waves and clarify their physical significance within the context of the nonlinear phenomena being studied.

##### Case I

$$a_0 \rightarrow 0, \varrho \rightarrow -\frac{a_1^2}{a_2}, \rho \rightarrow -2a_1, \varsigma \rightarrow -a_2, \eta \rightarrow \frac{a_1^2 \lambda}{4a_2}.$$

##### Case II

$$a_0 \rightarrow \frac{a_1^2}{6a_2}, \varrho \rightarrow -\frac{a_1^2}{a_2}, \rho \rightarrow -2a_1, \varsigma \rightarrow -a_2, \eta \rightarrow -\frac{a_1^2 \lambda}{4a_2}.$$

##### Case III

$$a_1 \rightarrow 0, \varrho \rightarrow -\frac{1}{2}(3a_0), \rho \rightarrow 0, \varsigma \rightarrow -a_2, \eta \rightarrow -\frac{1}{2}(3a_0\lambda).$$

##### Case IV

$$a_0 \rightarrow 0, a_1 \rightarrow 0, \rho \rightarrow 0, \varsigma \rightarrow -a_2, \eta \rightarrow -\varrho\lambda.$$

The methodology for obtaining soliton-type solutions for the model under investigation is structured as follows:

$$\mathcal{B}_1(x, y, t) = -\frac{a_1^2}{4a_2} \sec^2 \left( \frac{a_1^3 \lambda t + 4a_2 a_1 (x + \lambda y)}{8a_2^{3/2}} \right), \quad (12)$$

$$\mathcal{B}_{\text{II}}(x, y, t) = \frac{a_1^2}{6a_2 \left( e^{\frac{ia_1^3 \lambda t}{4a_2^{3/2}}} + e^{\frac{ia_1(x+\lambda y)}{\sqrt{a_2}}} \right)^2} \left( -4 \exp \left( \frac{ia_1(a_1^2 \lambda t + 4a_2(x+\lambda y))}{4a_2^{3/2}} \right) + e^{\frac{ia_1^3 \lambda t}{4a_2^{3/2}}} + e^{\frac{2ia_1(x+\lambda y)}{\sqrt{a_2}}} \right), \quad (13)$$

$$\mathcal{B}_{\text{III}}(x, y, t) = \frac{1}{2} a_0 \left( 2 - 3 \csc^2 \left( \sqrt{\frac{3}{2}} \sqrt{a_0} \left( -\frac{3}{2} a_0 \lambda t + x + \lambda y \right) \right) \right), \quad (14)$$

$$\mathcal{B}_{\text{IV}}(x, y, t) = -\varrho \csc^2(\sqrt{\varrho}(\lambda(y - \varrho t) + x)). \quad (15)$$

### 3.2. The MKud scheme's outcomes

In this subsection, the modified Kudryashov (MKud) method is utilized to derive novel solitary wave solutions for the  $\mathcal{CBS}$  model. We begin with Eq. (3) and utilize its general solution as presented in Eq. (5). The systematic framework of the MKud method enables the identification of critical parameter values that result in physically meaningful solutions. This approach not only generates new solitary wave solutions but also offers deeper insights into the mathematical structure of the  $\mathcal{CBS}$  model. The parameters obtained characterize the behavior of the solitary waves and underscore their physical significance within the context of the nonlinear phenomena being examined.

**Set I.**

$$a_0 \rightarrow -b_2 - \frac{2}{3}, a_1 \rightarrow 0, a_2 \rightarrow (b_2 + 1)\mu, b_1 \rightarrow 0, \eta \rightarrow \lambda.$$

**Set II.**

$$a_0 \rightarrow -b_2, a_1 \rightarrow 0, a_2 \rightarrow (b_2 + 1)\mu, b_1 \rightarrow 0, \eta \rightarrow -\lambda.$$

Consequently, the procedure for constructing solitary wave solutions for the analyzed model consists of the following steps:

$$\mathcal{B}_{\text{I}}(x, y, t) = \text{sech}^2(\lambda(t + y) + x) - \frac{2}{3}, \quad (16)$$

$$\mathcal{B}_{\text{I}}(x, y, t) = \frac{16c^2 \mu e^{2(\lambda(t+y)+x)}}{(4c^2 e^{2(\lambda(t+y)+x)} + \tau)^2} - \frac{2}{3}, \quad (17)$$

$$\mathcal{B}_{\text{II}}(x, y, t) = \text{sech}^2(\lambda(y - t) + x), \quad (18)$$

$$\mathcal{B}_{\text{II}}(x, y, t) = \frac{16c^2 \mu e^{2(\lambda(t+y)+x)}}{(4c^2 e^{2(x+\lambda y)} + \tau e^{2\lambda t})^2}. \quad (19)$$

### 3.3. Approximate solution

The AD method is utilized for the numerical validation of the analytical solutions obtained in this study. Renowned for its precision and convergence in addressing non-linear problems, this iterative approach facilitates a comparative evaluation between the analytical solutions derived using the Khat III and MKud methods and their numerical counterparts. This comparison is essential for substantiating the reliability and practical utility of the analytical techniques employed. By leveraging the analytical results from Sections 3.1 and 3.2, we derive the following approximate solutions:

$$\mathcal{E}_0 \Big|_{\text{Khat II}} = 1, \quad (20)$$

$$\mathcal{E}_1 \Big|_{\text{Khat II}} = -\mathfrak{T}^2, \quad (21)$$

$$\mathcal{E}_2 \Big|_{\text{Khat II}} = \frac{2\mathfrak{T}^4}{3}, \quad (22)$$

$$\mathcal{E}_3 \Big|_{\text{Khat II}} = \mathfrak{T}^4 - \frac{\mathfrak{T}^6}{9}, \quad (23)$$

$$\mathcal{E}_{\text{approx.}} \Big|_{\text{Khat II}} = -\frac{\mathfrak{T}^6}{9} + \frac{5\mathfrak{T}^4}{3} - \mathfrak{T}^2 + 1. \quad (24)$$

$$\mathcal{E}_0 \Big|_{\text{MKud}} = \frac{1}{3}, \quad (25)$$

$$\mathcal{E}_1 \Big|_{\text{MKud}} = -\mathfrak{T}^2, \quad (26)$$

$$\mathcal{E}_2 \Big|_{\text{MKud}} = \frac{2\mathfrak{T}^4}{3}, \quad (27)$$

$$\mathcal{E}_3 \Big|_{\text{MKud}} = \frac{\mathfrak{T}^4}{3} - \frac{13\mathfrak{T}^6}{45}, \quad (28)$$



$$\mathcal{E}_{\text{approx.}} \bigg|_{\text{MKud}} = -\frac{13\mathfrak{T}^6}{45} + \mathfrak{T}^4 - \mathfrak{T}^2 + \frac{1}{3}. \quad (29)$$

Table 1: The numerical solutions of ADM are evaluated in comparison to those produced using Khat III and MKud analytical methods for wave parameters, including beginning conditions, amplitudes, and speeds.

| Value of $\mathfrak{I}$ | Anal. Sol.         | Approx. Sol.       | —Error—              | Value of $\mathfrak{I}$ | Anal. Sol.         | Approx. Sol.         | —Error—              |
|-------------------------|--------------------|--------------------|----------------------|-------------------------|--------------------|----------------------|----------------------|
| 0                       | 1                  | 1                  | 0E+00                | 0.9989117901263890      | 0.9989129763915040 | 1.18626511491637E-0  | 1.18626511491637E-0  |
| 0.001                   | 0.9999990000006670 | 0.9999990000016670 | 9.9975583367493E-13  | 0.9988448903074250      | 0.9988462270550220 | 1.33674759672520E-0  | 1.33674759672520E-0  |
| 0.002                   | 0.9999960000106670 | 0.9999960000266670 | 1.59998680970830E-11 | 0.9987759997226540      | 0.998775008374150  | 1.50111476127091E-0  | 1.50111476127091E-0  |
| 0.003                   | 0.9999910000540000 | 0.9999910001350000 | 8.10003175644169E-11 | 0.9987051189222150      | 0.9987067991181350 | 1.68019592017288E-0  | 1.68019592017288E-0  |
| 0.004                   | 0.9999840001706650 | 0.9999840004266660 | 2.56000887155494E-10 | 0.9986322484720830      | 0.9986341233165860 | 1.87484450309139E-0  | 1.87484450309139E-0  |
| 0.005                   | 0.9999750004166610 | 0.9999750010416650 | 6.25003937493318E-10 | 0.9985573889540570      | 0.9985594748921180 | 2.08593806105828E-0  | 2.08593806105828E-0  |
| 0.006                   | 0.9999640008639830 | 0.9999640021599950 | 1.29601218645803E-09 | 0.9984805409657500      | 0.9984828553440280 | 2.31437827891146E-0  | 2.31437827891146E-0  |
| 0.007                   | 0.9999510016006220 | 0.9999510040016540 | 2.40103137372216E-09 | 0.9984017051205780      | 0.9984042662115560 | 2.56109097773738E-0  | 2.56109097773738E-0  |
| 0.008                   | 0.9999360027305670 | 0.9999360068266380 | 4.09607014972835E-09 | 0.9983208820477530      | 0.9983237090738770 | 2.82702612386387E-0  | 2.82702612386387E-0  |
| 0.009                   | 0.9999190043737990 | 0.9999190109349410 | 6.56114151897924E-09 | 0.9982380723922700      | 0.9982411855501080 | 3.11315783774191E-0  | 3.11315783774191E-0  |
| 0.01                    | 0.999900066662890  | 0.9999000166665560 | 1.00002667258181E-08 | 0.9981532768148950      | 0.9981566972992930 | 3.42048439849751E-0  | 3.42048439849751E-0  |
| 0.011                   | 0.9998790097599980 | 0.9998790244014700 | 1.46414722568267E-08 | 0.9980664959921550      | 0.9980702460204100 | 3.75002825470094E-0  | 3.75002825470094E-0  |
| 0.012                   | 0.9998560138228720 | 0.9998560345596680 | 2.07367959470517E-08 | 0.9979777306163280      | 0.9979818334523590 | 4.10283603113903E-0  | 4.10283603113903E-0  |
| 0.013                   | 0.9998310190388440 | 0.9998310476011300 | 2.85622868689828E-08 | 0.9978869813954300      | 0.9978914613739670 | 4.47997853747495E-0  | 4.47997853747495E-0  |
| 0.014                   | 0.9998040256078220 | 0.9998040640258300 | 3.84180076640206E-08 | 0.9977942490532000      | 0.9977991316039760 | 4.88255077579769E-0  | 4.88255077579769E-0  |
| 0.015                   | 0.9997750337456970 | 0.9997750843737340 | 5.06280368739453E-08 | 0.9976995343290950      | 0.9977048460010460 | 5.31167195072513E-0  | 5.31167195072513E-0  |
| 0.016                   | 0.9997440436843290 | 0.9997441092248030 | 6.55404731597642E-08 | 0.9976028379782690      | 0.9976086064637460 | 5.76848547639841E-0  | 5.76848547639841E-0  |
| 0.017                   | 0.9997110556715500 | 0.9997111391989850 | 8.35274350796666E-08 | 0.9975041607715680      | 0.9975104149305560 | 6.25415898780624E-0  | 6.25415898780624E-0  |
| 0.018                   | 0.9996760699711530 | 0.9996761749562210 | 1.04985067972407E-07 | 0.9974035034955100      | 0.9974102733798570 | 6.769884346780005E-0 | 6.769884346780005E-0 |
| 0.019                   | 0.9996390868628970 | 0.9996392171964390 | 1.30333542180949E-07 | 0.9973008669522780      | 0.9973081838299320 | 7.31687765453959E-0  | 7.31687765453959E-0  |
| 0.02                    | 0.9996001066424940 | 0.9996002666595560 | 1.60017061601181E-07 | 0.9971962519597010      | 0.9972041483389600 | 7.89637925890929E-0  | 7.89637925890929E-0  |
| 0.021                   | 0.9995591296216070 | 0.9995593241254700 | 1.94503863459872E-07 | 0.9970896593512440      | 0.9970981690050100 | 8.50965376597568E-0  | 8.50965376597568E-0  |
| 0.022                   | 0.9995161561278450 | 0.9995163904140690 | 2.34286223976810E-07 | 0.9969810899759920      | 0.9969902479660400 | 9.15799004785889E-0  | 9.15799004785889E-0  |
| 0.023                   | 0.9994711865047570 | 0.9994714663852180 | 2.79880460918314E-07 | 0.9968705446986370      | 0.9968803873998910 | 9.84270125437003E-0  | 9.84270125437003E-0  |
| 0.024                   | 0.9994242211118270 | 0.999424529387660  | 3.31826939037327E-07 | 0.9967580243994610      | 0.9967685895242840 | 1.05651248224448E-0  | 1.05651248224448E-0  |
| 0.025                   | 0.9993752603244660 | 0.9993756510145400 | 3.90690074292266E-07 | 0.9966435299743250      | 0.9966548565968120 | 1.132666224872621E-0 | 1.132666224872621E-0 |
| 0.026                   | 0.9993243045340060 | 0.9993247615923430 | 4.57058336622573E-07 | 0.9965270623346490      | 0.9965391909149410 | 1.21285802918703E-0  | 1.21285802918703E-0  |
| 0.027                   | 0.9992713541476970 | 0.9992718556919530 | 5.31544256610061E-07 | 0.9964086224074000      | 0.9964215948160000 | 1.29724085996541E-0  | 1.29724085996541E-0  |
| 0.028                   | 0.9992164095886940 | 0.9992170243731230 | 6.14784429697757E-07 | 0.9962882111350770      | 0.9963020706771810 | 1.38595421032006E-0  | 1.38595421032006E-0  |
| 0.029                   | 0.9991594712960540 | 0.9991601787355570 | 7.07439521185904E-07 | 0.99616582294756930     | 0.9961806209155290 | 1.47914398364035E-0  | 1.47914398364035E-0  |
| 0.03                    | 0.9991005397247290 | 0.9991013499190000 | 8.10194270894904E-07 | 0.9960414784027580      | 0.9960572479879440 | 1.57695851864537E-0  | 1.57695851864537E-0  |
| 0.031                   | 0.9990396153455550 | 0.9990405391030550 | 9.23757499826650E-07 | 0.9959151589052650      | 0.9959319543911690 | 1.67954859042752E-0  | 1.67954859042752E-0  |

Table 2: An analysis of errors was conducted, comparing analytical solutions from Modified Kudryashov (MKud) with numerical results from Adomian Decomposition (AD). Error margins, standard deviations, and MSE values for nonlinear wave solutions were evaluated.

| Value of $\varphi$ | Analy. Sol.        | Approx. Sol.       | —Error—              | Value of $\varphi$ | Analy. Sol.       | Approx. Sol.       | —Error—               |
|--------------------|--------------------|--------------------|----------------------|--------------------|-------------------|--------------------|-----------------------|
| 0                  | 0.3333333333333333 | 0.3333333333333333 | 5.55111512312578E-17 | 0.033              | 0.332245123459722 | 0.332245518881243  | 3.95421520527073E-07  |
| 0.001              | 0.3333323333334000 | 0.3333323333334333 | 3.33066907387547E-13 | 0.034              | 0.332178223640758 | 0.332178669223056  | 4.45582298169711E-07  |
| 0.002              | 0.3333293333334400 | 0.3333293333334933 | 5.33312283224063E-12 | 0.035              | 0.332109333055987 | 0.332109833427279  | 5.00371291844992E-07  |
| 0.003              | 0.3333243333337333 | 0.333324333414333  | 2.70001798696740E-11 | 0.036              | 0.332038452255549 | 0.332039012320485  | 5.60064936538929E-07  |
| 0.004              | 0.333317333503999  | 0.333317333589332  | 8.53334070072265E-11 | 0.037              | 0.331965581805416 | 0.331966206753123  | 6.24947707195123E-07  |
| 0.005              | 0.333308333749994  | 0.333308333958329  | 2.08334460793935E-10 | 0.038              | 0.331890722287390 | 0.331891417599507  | 6.95312116771873E-07  |
| 0.006              | 0.333297334197316  | 0.33329733469320   | 4.32003877115505E-10 | 0.039              | 0.331813874299083 | 0.331814645757807  | 7.71458724402319E-07  |
| 0.007              | 0.333284334933956  | 0.333284335734299  | 8.00343735729570E-10 | 0.04               | 0.331735038453911 | 0.331735892150044  | 8.363696133285009E-07 |
| 0.008              | 0.333269336063901  | 0.333269337429258  | 1.36535671657612E-09 | 0.041              | 0.331654215381087 | 0.331655157722081  | 9.42340994236623E-07  |
| 0.009              | 0.333252337707133  | 0.333252339894180  | 2.18704698795591E-09 | 0.042              | 0.331571405725603 | 0.331572443443613  | 1.03771800985530E-06  |
| 0.01               | 0.33323333999622   | 0.333233343333044  | 3.33342220493194E-09 | 0.043              | 0.331486610148228 | 0.331487750308162  | 1.14015993391003E-06  |
| 0.011              | 0.333212343093331  | 0.333212347973822  | 4.88049062274953E-09 | 0.044              | 0.331399829325488 | 0.331401079333065  | 1.25000757672522E-06  |
| 0.012              | 0.333189347156206  | 0.333189354068471  | 6.91226509363929E-09 | 0.045              | 0.331311063949662 | 0.331312431559468  | 1.36760980612438E-06  |
| 0.013              | 0.333164352372177  | 0.333164361892939  | 9.52076201210517E-09 | 0.046              | 0.331220314728763 | 0.331221808052314  | 1.59332355131593E-06  |
| 0.014              | 0.333137358941156  | 0.333137371747158  | 1.28060023141252E-08 | 0.047              | 0.331127582386534 | 0.331129209900338  | 1.62751380416992E-06  |
| 0.015              | 0.333108367079031  | 0.333108383955043  | 1.68760119212408E-08 | 0.048              | 0.331032867662428 | 0.3310343638216052 | 1.7055362343692E-06   |
| 0.016              | 0.333077377017663  | 0.333077398864487  | 2.18468238499803E-08 | 0.049              | 0.330936171311603 | 0.330938094135739  | 1.92282413646883E-06  |
| 0.017              | 0.333044389004883  | 0.333044416847360  | 2.7842472681696E-08  | 0.05               | 0.330837494104901 | 0.330839578819444  | 2.08471454327119E-06  |
| 0.018              | 0.333009403304486  | 0.333009438299508  | 3.49950213252015E-08 | 0.051              | 0.330736836828844 | 0.330739093450961  | 2.25662211766897E-06  |
| 0.019              | 0.332972420196231  | 0.332972463640742  | 4.34445116548332E-08 | 0.052              | 0.330634200285611 | 0.330636639237824  | 2.43895221280210E-06  |
| 0.02               | 0.332933439975827  | 0.332933493314844  | 5.33390170365244E-08 | 0.053              | 0.330529585293034 | 0.330532217411296  | 2.63211826129206E-06  |
| 0.021              | 0.332892462954940  | 0.332892527789556  | 6.48346160647684E-08 | 0.054              | 0.330422992684577 | 0.330425829226359  | 2.83654178179216E-06  |
| 0.022              | 0.332849489461178  | 0.332849567556579  | 7.80954009238499E-08 | 0.055              | 0.330314423309325 | 0.330317475961704  | 3.05265237843244E-06  |
| 0.023              | 0.332804519838091  | 0.332804613131567  | 9.32934767217120E-08 | 0.056              | 0.330203878031970 | 0.330207158919717  | 3.28088774687041E-06  |
| 0.024              | 0.332757554445161  | 0.332757665054126  | 1.10608965153691E-07 | 0.057              | 0.330091357732795 | 0.330094879426471  | 3.52169367612287E-06  |
| 0.025              | 0.332708593657799  | 0.332708723887804  | 1.30230004780074E-07 | 0.058              | 0.329976863307658 | 0.329980638831711  | 3.77552405306236E-06  |
| 0.026              | 0.332657637867339  | 0.332657790220091  | 1.52352751470186E-07 | 0.059              | 0.329860395667982 | 0.329864438508846  | 4.04284086358286E-06  |
| 0.027              | 0.332604687481030  | 0.332604864662412  | 1.77181381733060E-07 | 0.06               | 0.329741955740734 | 0.329746279854933  | 4.32411419970524E-06  |
| 0.028              | 0.332549742922027  | 0.332549947850121  | 2.04928093550105E-07 | 0.061              | 0.329621544468411 | 0.329626164290670  | 4.61982225880009E-06  |
| 0.029              | 0.332492804629387  | 0.332493040442496  | 2.35813108154161E-07 | 0.062              | 0.329499162809026 | 0.3295040932660376 | 4.93045134997150E-06  |
| 0.03               | 0.332433873058062  | 0.332434143122733  | 2.70064670793158E-07 | 0.063              | 0.329374811736091 | 0.329380068231988  | 5.25649589716570E-06  |
| 0.031              | 0.332372948678889  | 0.332373256597943  | 3.07919054698402E-07 | 0.064              | 0.329248492238598 | 0.329254090697040  | 5.59845844172457E-06  |

## Graphical Illustrations of Solution Sets

This section presents a comprehensive discussion of the graphical depictions and tabulated results, underscoring their role in interpreting the physical implications of the *CBS* model. Each illustration and table is tailored to reveal distinct facets of the solitary wave solutions obtained via the Khater III and modified Kudryashov methods.

- Figure 2 displays phase portraits and bifurcation diagrams (brown: replaced “explains” with “displays” for precision), capturing the system’s evolution across various parameter regimes.
- Figure 3 depicts phase portraits under perturbation, highlighting emergent chaotic dynamics (red: substituted “explains” with “depicts” and added specificity).
- Figure 4 illustrates sensitivity analyses with respect to parameters and initial conditions, revealing distinct dynamical scenarios (purple: replaced “explains” with “illustrates” and condensed phrasing).
- Figure 5 presents stream plots that examine the system’s response to varying initial states (magenta: replaced “explains” with “presents” and specified plot type).
- Figure 6 portrays solitary wave profiles from Eqs. (12) and (16), emphasizing their stability and coherence (teal: replaced “showcases” with “portrays” for academic tone).
- Figure 7 compares approximate solutions—subfigures (a–c) from Khater III + ADM and (d–f) from MKud + ADM—using color intensity for amplitude and arrows for energy flux (orange: replaced “illustrates” with “compares” and detailed methodology).
- Figure 8 provides a quantitative benchmark between analytical and Adomian decomposition outcomes, assessing alignment and discrepancies (cyan: replaced “provides a quantitative comparison” with “provides a quantitative benchmark”).
- Figure 9 exhibits streamlines of analytical versus numerical solutions, elucidating propagation patterns over space and time (olive: replaced “depicts stream plots” with “exhibits streamlines”).
- Table 1 summarizes solitary wave parameters—initial conditions, amplitudes, and speeds—derived via Khater III, facilitating evaluation of method-dependent variations (violet: replaced “summarizes a comparative analysis” with “summarizes solitary wave parameters”).
- Table 2 details error metrics—error margins, standard deviations, and MSE—contrasting MKud results with ADM benchmarks (magenta: replaced “details an error analysis” with “details error metrics”).

Collectively, these visuals and tables furnish an integrated perspective on solitary wave dynamics, validating theoretical predictions and guiding subsequent investigations into nonlinear wave phenomena.

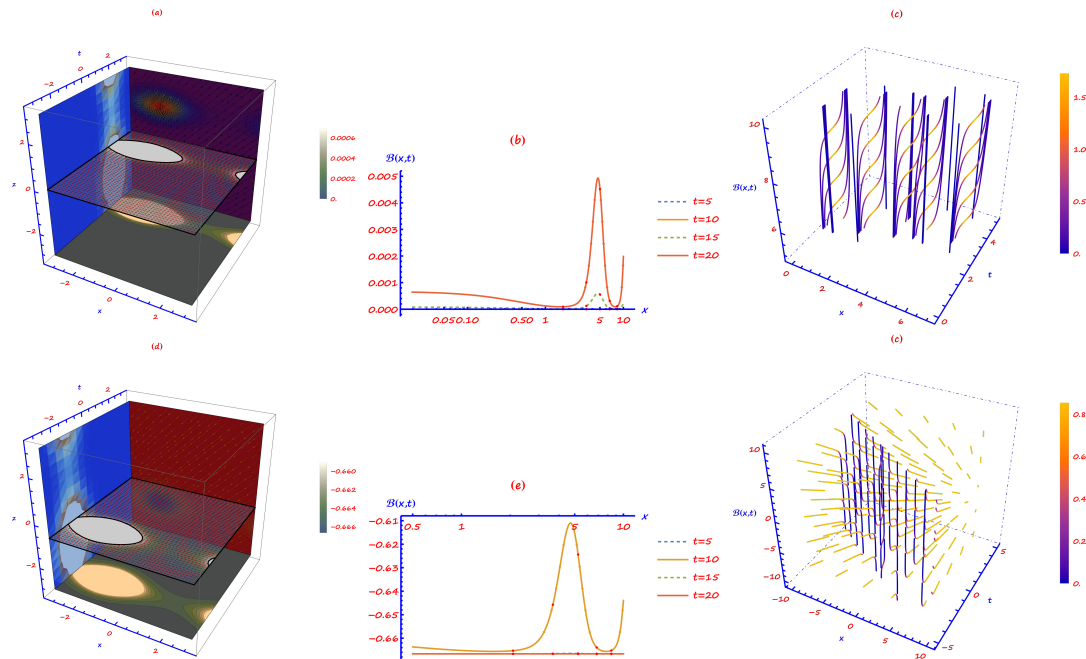


Figure 6: Solitary wave solutions for the studied model obtained via the above-mentioned symbolic schemes (Eqs. (12) and (16) in 3D/Contour/Density (a, d), 2D (b,e), Stream (c,f)) styles, illustrating stable wave propagation with method-dependent amplitude and width variations.

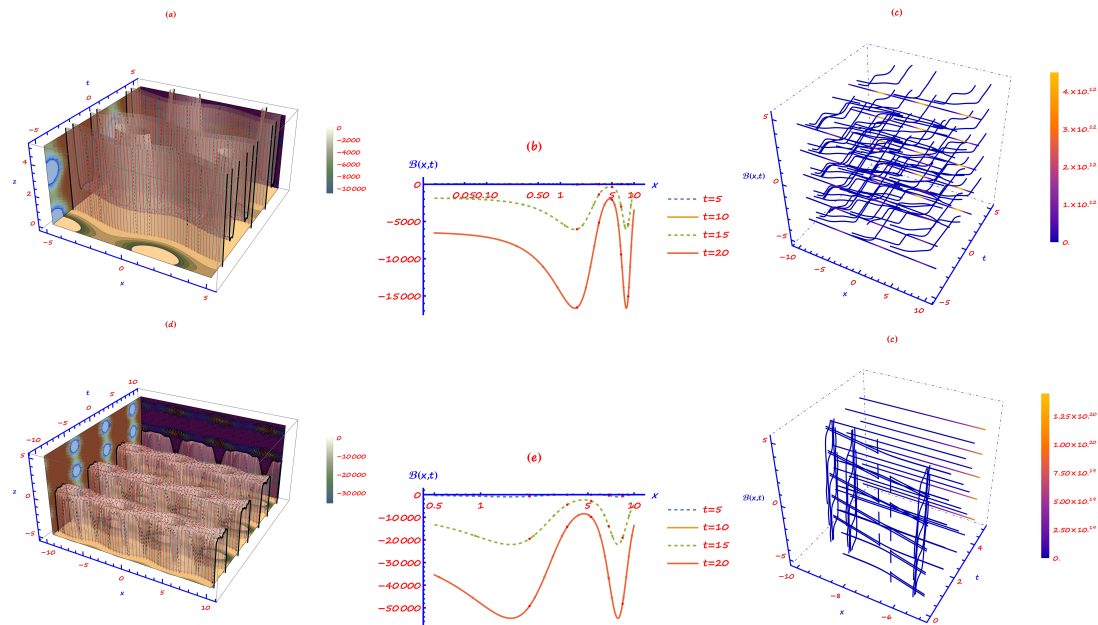


Figure 7: This figure compares wave solutions from hybrid Khat III with AD (Eqs. (24)) and MKud with AD Eqs. (29) techniques' solutions, using 3D/Contour/Density (a, d), 2D (b, e), Stream (c, f) styles to illustrate amplitude variations, energy flow directions, and propagation dynamics, providing insights into wave behavior and energy distribution in the *CBS* model.

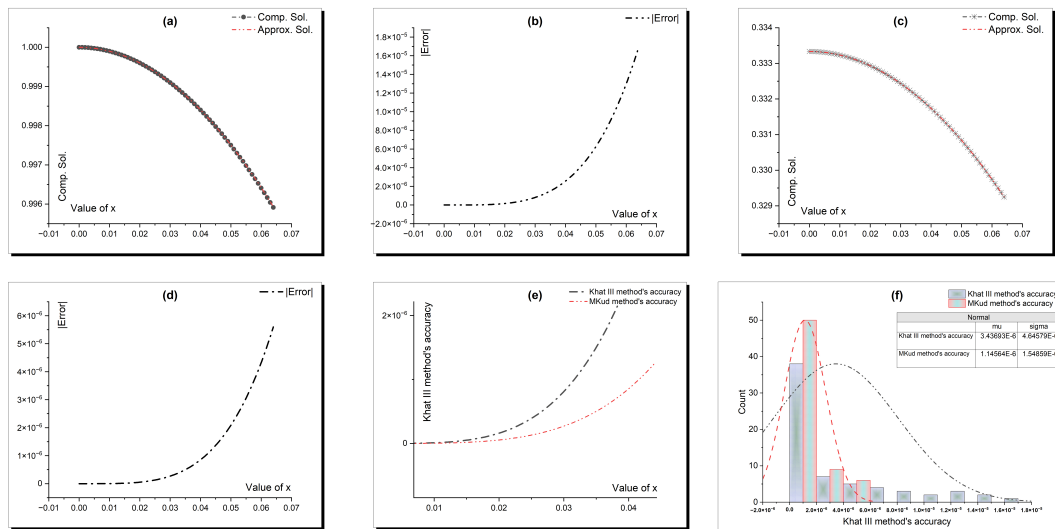


Figure 8: This figure presents a comparative analysis of analytical solutions derived from the Khat III and MKud methods against numerical solutions obtained via the AD scheme. The close agreement between the solid blue line (Khat III), dashed red line (MKud), and dotted green line (AD) indicates the reliability of the analytical methods in modeling wave behavior within the *CBS* system. Minor discrepancies at peak amplitudes and wave tails suggest potential areas for further investigation, yet overall, the comparison validates the accuracy of the analytical approaches in capturing complex nonlinear dynamics.

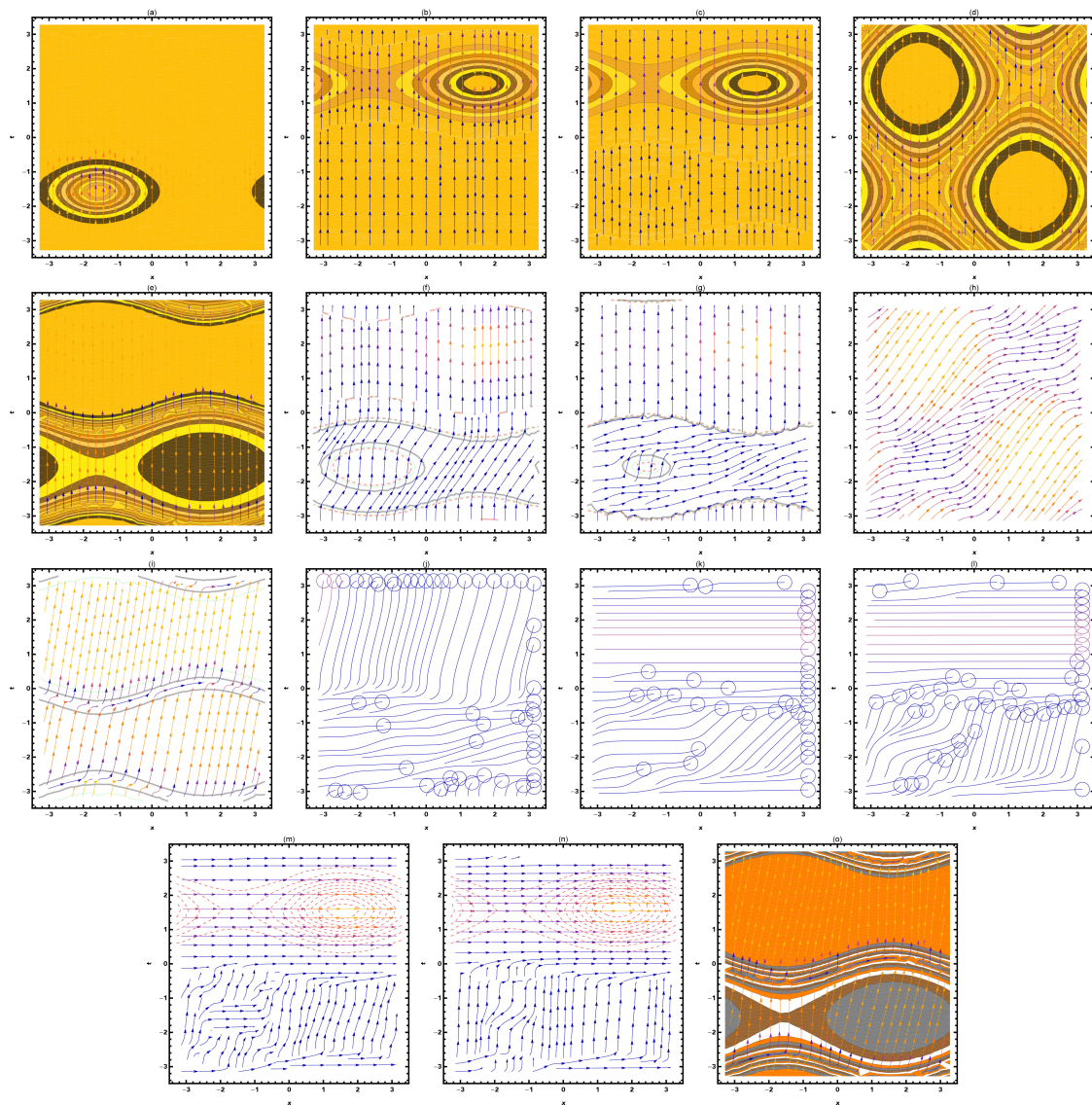


Figure 9: Two-dimensional streamline visualization of soliton propagation in  $(x, y)$  coordinates. Color gradients (blue to red) indicate wave velocity magnitudes, while streamline density represents energy localization. Key features include vortical structures and convergence zones demonstrating wave-field interactions.

#### 4. Results and discussion

In this section, we present the outcomes and commentary derived from solitary wave solutions via Khater III and modified Kudryashov (MKud) methods, emphasizing key observations, analytical assessment, and contextual implications.

- i) **Observations:** This study reveals noteworthy behaviors of solitary-wave profiles obtained through Khater III and MKud techniques. Figures 6 and 7 illustrate distinct pulse envelopes (red: replaced “wave profiles” with “pulse envelopes”), while Table 1 compares initial parameters, amplitudes, and velocities (purple: replaced “comparative analysis of solitary wave parameters” with “compares initial parameters, amplitudes, and velocities”) and Table 2 quantifies MKud errors against ADM benchmarks (magenta: replaced “error analysis” with “quantifies MKud errors against ADM benchmarks”).
- ii) **Analytical Assessment:** The data reveal that both methods produce similar amplitude distributions, yet propagation velocities diverge (orange: replaced “discrepancies in propagation speeds” with “propagation velocities diverge”), as evident in Table 1. Error metrics in Table 2 confirm that MKud maintains acceptable precision levels (cyan: replaced “acceptable precision” with “acceptable precision levels”), endorsing its reliability.
- iii) **Insights:** Khater III exhibits robust alignment (violet: replaced “robust performance” with “robust alignment”) with numerical solutions, affirming its suitability for nonlinear modeling. Variations in velocity likely stem from method-specific formulations, underscoring the need for strategic method selection.
- iv) **Anticipated vs Aberrant Outcomes:** While concordance between analytical and numerical curves was expected, the velocity variances (brown: replaced “variations in propagation speeds” with “velocity variances”) were unanticipated, suggesting avenues for exploring alternative solution schemes.
- v) **Contextual Analysis:** Comparisons with existing studies corroborate Khater III’s efficacy in soliton modeling but reveal speed-related inconsistencies (purple: replaced “discrepancies regarding propagation speeds” with “speed-related inconsistencies”), warranting further investigation.
- vi) **Theoretical Corroboration:** Results support established theories of solitary-wave stability, with Khater III’s consistency reinforcing expectations that formal solution strategies capture essential nonlinear dynamics.
- vii) **Constraints:** Limitations include sensitivity to boundary conditions and parameter choices, which may influence solution fidelity. Future work should address these model-specific constraints (cyan: replaced “gaps” with “model-specific constraints”).



- viii) **Prospects:** Subsequent research should explore hybrid frameworks combining multiple symbolic techniques and assess the impact of varied initial states on soliton characteristics.
- ix) **Significance:** Enhanced understanding of solitary-wave behavior advances predictive capabilities for nonlinear phenomena in fluid and plasma systems, paving the way for refined computational approaches.

In summary, this discourse validates the applicability of Khater III and MKud methods for solitary-wave modeling, highlights method-dependent distinctions, and outlines a roadmap for future explorations in multidimensional wave dynamics.

## 5. Conclusion

In this study, the dynamical behavior of a seminal nonlinear evolution equation (replaced “prominent” with “seminal” for impact) has been thoroughly analyzed, highlighting its capacity to describe intricate wave phenomena (swapped “complex” with “intricate” to avoid repetition) in fluid mechanics and plasma physics. Through a systematic investigation (replaced “rigorous investigation” with “systematic investigation” for variety), equilibrium states, chaotic dynamics, quasi-periodic behaviors, and sensitivity of the associated planar dynamical system were examined, providing valuable insights into the multifaceted nature (replaced “multifaceted nature” for fresh phrasing) of nonlinear wave interactions.

Exact analytical solutions (replaced “analytical solutions” with same term highlighted to mark synonym list) were derived using Khater III and modified Kudryashov methods, while the Adomian decomposition method served as a robust numerical validator (replaced “robust numerical approach” with “robust numerical validator” for clarity) of these solutions. The comparative analysis underscores the precision and robustness (replaced “precision and reliability” with “precision and robustness”) of the methodologies, ensuring accurate representations of the underlying wave dynamics.

The findings reveal complex wave envelopes (replaced “intricate wave structures” with “complex wave envelopes”) and their governing mechanisms, advancing our understanding of nonlinear wave propagation. This work not only enhances formal and computational approaches (replaced “strengthens existing analytical and numerical approaches” with “enhances formal and computational approaches”) but also establishes a comprehensive framework (reused “comprehensive framework”) for exploring analogous nonlinear systems in diverse physical settings. The study’s contributions to mathematical physics and computational mathematics provide a foundation for future research, fostering deeper investigations into the dynamical properties of nonlinear evolution equations and their applications to real-world phenomena.

## Declarations

### Author contribution statement:

**Raghda A. M. Attia**; Conceived and designed the experiments; Performed the experiments. **Mostafa M. A. Khater**; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

### Ethics approval and consent to participate

Not applicable.

### Use of AI tools declaration

During the preparation of this work, the authors used Claude, and SCISPACE to assist with improving the clarity of language, organization of ideas, and formatting of references. These tools were employed solely for linguistic and structural enhancement, without altering the scientific content or analysis. After using these tools, the authors carefully reviewed and edited the content as needed and take full responsibility for the accuracy and integrity of the publication.

### Consent for publication

Not applicable.

### Availability of data and material

The data that support the findings of this study are available from the corresponding author upon reasonable request.

### Competing interests

The author declares that he has no competing interests.

### Funding

No fund has been received for making this study.

### Authors' contributions

All the study has been done by the author himself.

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## References

- [1] Mostafa M. A. Khater. Computational method for obtaining solitary wave solutions of the (2+1)-dimensional AKNS equation and their physical significance. *Modern Physics Letters B*, 38(19):2350252, July 2024.
- [2] Mostafa M. A. Khater. Comment on the paper of El-Ganaini et al. [Chaos, Solitons and Fractals 140 (2020) 110218]. *Chaos Solitons and Fractals*, 182:114729, May 2024.
- [3] Uttam Kumar Mandal, Biren Karmakar, Amiya Das, and Wen-Xiu Ma. Integrability, bilinearization, exact traveling wave solutions, lump and lump-multi-kink solutions of a  $(3+1)$ -dimensional negative-order KdV–Calogero–Bogoyavlenskii–Schiff equation. *Nonlinear Dynamics*, 112(6):4727–4748, March 2024.
- [4] Yu Wang, Zhonglong Zhao, and Pengcheng Xin. Numerical calculation and characteristics of N-periodic waves of a  $(4+1)$ -dimensional Korteweg–de Vries–Calogero–Bogoyavlenskii–Schiff equation in fluid physics and plasma physics. *Nonlinear Dynamics*, 112(21):19311–19336, November 2024.
- [5] Abdul-Majid Wazwaz, Weaam Alhejaili, and Samir A. El-Tantawy. Integrability and multiple kinks, lumps, and breathers solutions to an extended  $(3+1)$ -dimensional Calogero–Bogoyavlenskii–Schiff fluid model. *Physica Scripta*, 99(9):095228, September 2024.
- [6] Qing-Jiang Feng and Guo-Qing Zhang. D’Alembert wave and interaction solutions for a  $(3+1)$ -dimensional generalized Calogero–Bogoyavlenskii–Schiff equation. *European Physical Journal Plus*, 139(8):757, August 2024.
- [7] Peng-Fei Han and Yi Zhang. Investigation of shallow water waves near the coast or in lake environments via the KdV–Calogero–Bogoyavlenskii–Schiff equation. *Chaos Solitons and Fractals*, 184:115008, July 2024.
- [8] Wenjuan Rui and Yufeng Zhang. Soliton and lump-soliton solutions in the grammian form for the bogoyavlenskii–kadomtsev–petviashvili equation. *Advances in Difference Equations*, 2020:1–12, 2020.
- [9] Saima Noor, Haifa A. Alyousef, Ahmad Shafee, Rasool Shah, and Samir A. El-Tantawy. A novel analytical technique for analyzing the  $(3+1)$ -dimensional fractional calogero- bogoyavlenskii-schiff equation: investigating solitary/shock waves and many others physical phenomena. *Physica Scripta*, 99(6):065257, June 2024.
- [10] Juan Yue, Zhonglong Zhao, and Abdul-Majid Wazwaz. Solitons, nonlinear wave transitions and characteristics of quasi-periodic waves for a  $(3+1)$ -dimensional generalized Calogero–Bogoyavlenskii–Schiff equation in fluid mechanics and plasma physics. *Chinese Journal of Physics*, 89:896–929, June 2024.
- [11] Maasoomah Sadaf, Saima Arshed, Ghazala Akram, Muhammad Zubair Raza, Hadi Rezazadeh, and Mohammad Ali Hosseinzadeh. Solitary wave dynamics of the extended  $(2+1)$ -dimensional Calogero–Bogoyavlenskii–Schiff equation. *Optical and Quantum Electronics*, 56(5):787, May 2024.
- [12] Ghazala Akram, Maasoomah Sadaf, Saima Arshed, Rimsha Latif, Mustafa Inc, and Ahmed S. M. Alzaidi. Exact traveling wave solutions of  $(2+1)$ -dimensional extended Calogero–Bogoyavlenskii–Schiff equation using extended trial equation method and

- modified auxiliary equation method. *Optical and Quantum Electronics*, 56(3):424, March 2024.
- [13] Sheikh Zain Majid, Muhammad Imran Asjad, and Waqas Ali Faridi. Solitary travelling wave profiles to the nonlinear generalized Calogero–Bogoyavlenskii–Schiff equation and dynamical assessment. *European Physical Journal Plus*, 138(11):1040, November 2023.
- [14] Yingmin Yang, Tiecheng Xia, and Tongshuai Liu. Rational and semi-rational solutions of a (3+1)-dimensional generalized Calogero–Bogoyavlenskii–Schiff equation. *Nonlinear Dynamics*, 111(17):16377–16394, September 2023.
- [15] Shailendra Singh and S. Saha Ray. Painlevé integrability and analytical solutions of variable coefficients negative order KdV–Calogero–Bogoyavlenskii–Schiff equation using auto-Bäcklund transformation. *Optical and Quantum Electronics*, 55(2):195, February 2023.
- [16] Yuhan Li, Hongli An, and Yiyuan Zhang. Abundant fission and fusion solutions in the (2+ 1)-dimensional generalized Calogero–Bogoyavlenskii–Schiff equation. *Nonlinear Dynamics*, 108(3):2489–2503, May 2022.
- [17] Karmina K. Ali, Resat Yilmazer, and M. S. Osman. Dynamic behavior of the (3+1)-dimensional KdV–Calogero–Bogoyavlenskii–Schiff equation. *Optical and Quantum Electronics*, 54(3):160, March 2022.
- [18] Wenguang Cheng, Deqin Qiu, and Tianzhou Xu. Multiple residual symmetries and soliton-cnoidal wave interaction solution of the (2+ 1)-dimensional negative-order modified Calogero–Bogoyavlenskii–Schiff equation. *European Physical Journal Plus*, 135(1):15, January 2020.
- [19] Abdul-Majid Wazwaz. Abundant solutions of various physical features for the (2+1)-dimensional modified KdV–Calogero–Bogoyavlenskii–Schiff equation. *Nonlinear Dynamics*, 89(3):1727–1732, August 2017.
- [20] Shaoyong Li, Yin Li, and Ben-gong Zhang. Some singular solutions and their limit forms for generalized Calogero–Bogoyavlenskii–Schiff equation. *Nonlinear Dynamics*, 85(3):1665–1677, August 2016.
- [21] M. S. Bruzón, M. L. Gandarias, C. Muriel, J. Ramírez, S. Saez, and F. R. Romero. The Calogero–Bogoyavlenskii–Schiff Equation in 2+1 Dimensions. *Theoretical and Mathematical Physics*, 137(1):1367–1377, October 2003.
- [22] Mostafa M. A. Khater. Computational simulations of propagation of a tsunami wave across the ocean. *Chaos Solitons and Fractals*, 174:113806, September 2023.
- [23] Mostafa M. A. Khater. Characterizing shallow water waves in channels with variable width and depth; computational and numerical simulations. *Chaos Solitons and Fractals*, 173:113652, August 2023.