



Harnessing Weakly Nonlinear Waves for Precision Ultrasound Applications

Mostafa M. A. Khater^{1,2}

¹ School of Information Engineering, Yango University, 350015, Fuzhou, Fujian

² Institute of Industrial Internet Intelligent Control Technology and System in Universities of Fujian Province, 350015, Fuzhou, Fujian

Abstract. The work focuses on the nonlinear evolution of acoustic waves in weakly diffracting and weakly nonlinear media, modeled by the (2+1)-dimensional Khokhlov–Zabolotskaya–Kuznetsov (KZK) equation. This equation captures how nonlinearity, diffraction, and dissipation interact during the propagation of finite-amplitude acoustic beams. This model is a fundamental model in nonlinear acoustics describing the evolution of finite-amplitude sound beams in thermoviscous fluids. It captures the combined effects of nonlinearity, diffraction, and absorption under the paraxial approximation. Using the Khater II analytical scheme together with an improved Kudryashov method, a new class of exact and physically meaningful solutions is obtained. The correctness and convergence of these results are checked through He's variational iteration method. Linear perturbation and spectral analyses are then used to explore the stability of the derived wave forms and to locate the limits between stable and unstable regimes. The bifurcation study points out how small parameter variations lead to distinct transitions among periodic, solitary, and blow-up type waves. Physically, these solutions describe the evolution of nonlinear acoustic beams, the onset of shock profiles, and the localized transport of acoustic energy in fluids and biological media. The combined Khater II–Kudryashov framework proves more effective than existing analytical approaches for handling multidimensional nonlinear models. The findings add to current understanding of nonlinear acoustic wave behavior and provide a useful basis for designing and interpreting acoustic systems governed by nonlinear effects.

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1. Introduction

Finite-amplitude acoustic beams in media exhibiting weak diffraction and nonlinearity are pivotal to technologies ranging from focused ultrasound therapy to high-resolution underwater sonar [1–4]. In such environments, nonlinear steepening competes

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Email address: mostafa.khater2024@yahoo.com, (M. M. A. Khater)

with diffraction spreading and viscous losses, yielding phenomena like shock front development, harmonic generation, and localized energy deposition [5]. Insight into these processes is essential for refining beam focusing, reducing undesired heating in tissue, and enhancing signal fidelity in submerged communication channels [6].

Over recent decades, analytical modeling of multidimensional nonlinear wavefields has progressed significantly. Early work applied reductive perturbation and multiple-scale expansions to derive approximate one-dimensional solutions [7], offering initial glimpses into nonlinear distortion. Later, variational and homotopy techniques broadened solution families but often resulted in implicit forms that challenge direct physical interpretation [8]. More integrative approaches combined transform-based and function-expansion methods to generate explicit two- and three-dimensional profiles; however, these solutions frequently lack thorough convergence and stability verification [9].

Despite these advances, three critical challenges remain unresolved. First, explicit closed-form expressions for truly multidimensional beams that transparently connect mathematical parameters to measurable quantities—such as focal intensity and shock threshold—are rare [10]. Second, a unified stability and bifurcation framework that charts parameter-dependent transitions between smooth, solitary, and shock-dominated regimes is absent [11]. Third, the practical implications of analytical parameters for device design have not been systematically mapped. Addressing these gaps will enable more precise control over nonlinear acoustic effects in both biomedical and industrial applications [12].

This work introduces transparent, convergent solutions for multidimensional acoustic beams and delivers a comprehensive stability and bifurcation analysis. By rigorously validating each analytical step through an iterative convergence scheme, we ensure physical fidelity [13]. A parametrized study then reveals how slight adjustments drive transitions from stable propagation to solitary-like and shock regimes [14]. Finally, we interpret the results in terms of focal energy concentration and shock onset criteria to guide the design of next-generation acoustic devices [15].

In this context, we study the (2+1)-dimensional KZK equation that is given by [7, 16–20]

$$-\frac{\beta}{\epsilon^3 \varrho} \frac{\partial^2 u^2}{\partial t^2} - \frac{\alpha}{\epsilon^3} \frac{\partial^3 u}{\partial t^3} + \frac{\partial^2 u}{\partial z \partial t} - \frac{1}{2} \epsilon \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0, \quad (1)$$

where, the function $u(x, y, z, t)$ denotes the acoustic potential or the scaled acoustic pressure, which represents the evolution of the acoustic field in both space and time [21]. The parameter ϵ is a small quantity that controls the beam spreading and characterizes the degree of the paraxial approximation used in the model. The term ϱ stands for the density of the propagation medium, which directly influences the nonlinear scaling of the system [22]. The coefficient $\beta = (1 + \frac{B}{2A})$ quantifies the nonlinear response of the medium and determines the extent of wavefront steepening during propagation. The coefficient α

represents the absorption or thermoviscous damping, accounting for the dissipative losses in the medium. The coordinates x and y are the transverse spatial variables that describe the beam's lateral distribution and shape, while z denotes the primary direction of propagation [23]. The variable t corresponds to the retarded time, used to describe the temporal evolution of the acoustic disturbance as it travels through the nonlinear medium.

While the model's terms can be physically explained as following [24, 25]:

- The nonlinear term $-\frac{\beta}{\epsilon^3 \rho} \frac{\partial^2 u^2}{\partial t^2}$, shows how the amplitude of the acoustic wave influences its own speed. Here u is the acoustic potential or the scaled acoustic pressure. The coefficient $\beta = 1 + \frac{B}{2A}$ measures the degree of nonlinearity of the medium. This effect is responsible for the steepening of the waveform and for the possible formation of shocks in high-amplitude regions.
- The next term, $-\frac{\alpha}{\epsilon^3} \frac{\partial^3 u}{\partial t^3}$, accounts for viscous and thermal attenuation. The parameter α stands for the thermoviscous absorption coefficient, which governs how the wave loses energy as it moves through the medium. It also introduces a small frequency-dependent dispersion effect.
- The term $\frac{\partial^2 u}{\partial z \partial t}$, represents the axial propagation of the acoustic wave along z . It connects the time variation of the field with its evolution along the beam axis.
- The term $-\frac{1}{2}\epsilon \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$, describes diffraction in the transverse plane. The small parameter ϵ indicates that diffraction is weak but cannot be ignored, especially for long propagation distances or narrow beams.

Eq. (1) expresses the balance between nonlinear distortion, diffraction, and absorption in finite-amplitude acoustic wave propagation [26]. When a sound beam travels through a thermoviscous fluid, nonlinearity causes the waveform to steepen, diffraction tends to broaden it laterally, and absorption reduces its amplitude. Depending on the medium and frequency, one of these effects may dominate [27, 28].

- (i) If nonlinearity prevails, the wave can form shocks.
- (ii) If dissipation is stronger, the waveform decays gradually.
- (iii) When all effects are comparable, the system supports more complex steady or slowly varying beam structures.

The investigated model serves as a fundamental model in weakly nonlinear acoustics, capturing the coupled effects of nonlinearity, diffraction, and thermoviscous absorption in a single framework. Its applications span focused ultrasound therapy and diagnostic imaging—most notably high-intensity focused ultrasound (HIFU)—as well as underwater and atmospheric acoustics, where nonlinear beam dynamics influence sonar performance and environmental sound propagation [29]. In engineering and materials evaluation, the KZK model underpins nonlinear ultrasonic inspection techniques for defect detection and

informs industrial processes such as ultrasonic welding, emulsification, and cleaning with high-power acoustic beams [30]. The analysis presented herein leverages this model to elucidate the interaction of these physical effects and to characterize their impact on waveform structure and stability in fluid and biological media [31].

Mathematically, Eq. (1) can be derived from the fundamental equations of motion for a compressible, thermoviscous fluid. The starting point is the continuity and momentum equations [32],

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad \rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \left(\mu_B + \frac{\mu}{3} \right) \nabla (\nabla \cdot \mathbf{v}), \quad (2)$$

where ρ is the mass density, $\mathbf{v}$ the velocity field, p the pressure, μ the shear viscosity, and μ_B the bulk viscosity. Assuming small perturbations about an equilibrium state $(\rho_0, p_0, 0)$, the dependent variables are expanded as $\rho = \rho_0 + \rho'$, $p = p_0 + p'$, and $\mathbf{v} = \mathbf{v}'$, where primes denote small deviations.

The thermodynamic equation of state is expanded up to the second order as

$$p' = c_0^2 \rho' + \frac{B}{2A} \frac{c_0^2}{\rho_0} (\rho')^2, \quad (3)$$

where c_0 is the small-signal sound speed and B/A is the nonlinearity parameter. Substituting this expansion into the momentum and continuity equations, and eliminating ρ' and $\mathbf{v}'$, leads to the well-known Kuznetsov (or Westervelt) equation,

$$\frac{\partial^2 p'}{\partial t^2} - c_0^2 \nabla^2 p' - \delta \nabla^2 \frac{\partial p'}{\partial t} = \frac{\beta}{\rho_0 c_0^2} \frac{\partial^2 (p'^2)}{\partial t^2}, \quad (4)$$

where δ represents the thermoviscous damping coefficient and $\beta = 1 + \frac{B}{2A}$ is the effective nonlinearity parameter.

To obtain the KZK form, a paraxial (slowly varying) approximation is introduced. The wave is assumed to propagate mainly along the z -axis, with weak transverse diffraction. Introducing the retarded time $\tau = t - z/c_0$ and the scaled coordinates

$$Z = \epsilon z, \quad X = \sqrt{\epsilon} x, \quad Y = \sqrt{\epsilon} y, \quad (5)$$

the dependent variable is expressed as

$$p'(x, y, z, t) = \epsilon P(X, Y, Z, \tau), \quad (6)$$

where ϵ is a small, dimensionless parameter characterizing the amplitude and diffraction strength. The derivatives transform as

$$\partial_t = \partial_\tau, \quad \partial_z = -\frac{1}{c_0} \partial_\tau + \epsilon \partial_Z, \quad \partial_x = \sqrt{\epsilon} \partial_X, \quad \partial_y = \sqrt{\epsilon} \partial_Y. \quad (7)$$

Substituting these scalings into the Kuznetsov equation, expanding in powers of ϵ , and retaining terms of order ϵ^2 yield the reduced paraxial evolution equation:

$$\frac{\partial^2 P}{\partial Z \partial \tau} = \frac{1}{2} \nabla_{\perp}^2 P + \frac{\beta}{\rho_0 c_0^3} \frac{\partial^2 (P^2)}{\partial \tau^2} + \frac{\delta}{2c_0^3} \frac{\partial^3 P}{\partial \tau^3}, \quad (8)$$

where $\nabla_{\perp}^2 = \partial_X^2 + \partial_Y^2$ represents transverse diffraction. The first term on the right-hand side describes beam spreading, the second term represents nonlinear wave steepening, and the third accounts for thermoviscous attenuation. Rewriting Eq. (8) in the original coordinates and introducing normalized parameters leads directly to Eq. (1).

Physically, this equation captures the balance between nonlinear self-interaction, diffraction, and viscous damping of finite-amplitude acoustic beams. The first term involving $\partial^2 u^2 / \partial t^2$ governs nonlinear distortion and shock formation, the mixed derivative $\partial^2 u / (\partial z \partial t)$ represents forward wave propagation, the transverse Laplacian term models beam diffraction, and the third time derivative represents thermoviscous losses. Applications of the KZK equation include modeling focused ultrasound fields, medical ultrasound therapy, and nonlinear acoustic wave propagation in fluids and biological tissues, where both diffraction and nonlinearity are significant.

Our analysis is subject to several constraints. First, the assumption of weak nonlinearity and diffraction—embodied by the small parameter ϵ —restricts applicability to moderate-amplitude, slowly diverging beams; strongly nonlinear or highly divergent waves lie outside this regime. Second, higher-order absorption and dispersive effects are neglected, which may reduce accuracy at very high frequencies or in media exhibiting pronounced viscoelastic behavior. Third, the parabolic approximation employed (slow transverse variation) may not capture sharp focusing or rapid beam-steering phenomena over short axial distances. Fourth, all derivations and stability analyses assume homogeneous, loss-dominated fluids and do not account for heterogeneous or anisotropic materials. Finally, numerical validation is confined to parameter ranges amenable to He's variational iteration convergence; parameter regimes requiring fully nonlinear numerical solvers remain unexplored. These limitations delineate the scope of the present work and motivate future extensions to broader nonlinear and heterogeneous settings.

The structure of this paper is as follows: Section 2 derives explicit exact solutions for the (2+1)-dimensional KZK equation using the Khater II and enhanced Kudryashov approaches, evaluating their convergence and physical significance through He's variational iteration validation. In Section 3, the stability characteristics and bifurcation properties of these solutions are thoroughly analyzed, emphasizing parameter-dependent transitions between stable, solitary-like, and shock-dominated regimes. Section 4 examines each figure and table to demonstrate how the derived solutions translate into observable phenomena, validate theoretical predictions against numerical benchmarks, and guide practical parameter selection for nonlinear acoustic applications. Section 5 explains the novelty of our study and its scientific impact. Lastly, Section 6 provides a comprehensive summary of the

key findings and contributions, underscoring the study's impact on advancing analytical techniques for multidimensional nonlinear acoustic wave models and their applications in weakly nonlinear media.

2. Exact Solutions via Khater II and Enhanced Kudryashov Methods

In this section, we develop a unified procedure combining two semi-analytical schemes—the Khater II expansion and the enhanced Kudryashov algorithm—to extract closed-form solitary wave profiles of the (2+1)-dimensional KZK equation. Each profile is then subjected to He's variational iteration technique to produce high-precision numerical approximations, enabling a rigorous cross-validation of the exact expressions. We systematically contrast the analytical waveforms against their numerical counterparts and benchmark both against selected results from the recent literature. This dual-verification protocol not only uncovers previously unreported solitary structures but also quantifies the relative accuracy and convergence behavior of each method. By integrating these complementary approaches, we strengthen confidence in the derived solutions and advance the methodological toolkit available for probing nonlinear acoustic wave phenomena.

In this context, we employ the next wave transformation $u = u(x, y, z, t) = \psi(\mathfrak{Z})$, $\mathfrak{Z} = ct + x + y + z$, where η is the wave velocity, on Eq. (1) to convert its nonlinear partial differential equation form into the next ordinary differential equation form

$$-\alpha c^3 \psi^{(3)} \varrho - 2\beta c^2 (\psi')^2 + \psi'' (\epsilon^3 \varrho(c - \epsilon) - 2\beta c^2 \psi) = 0. \quad (9)$$

Integrating Eq. (9) with respect to $\mathfrak{Z}$, twice and along with zero integration constants, leads to

$$\psi (\epsilon^3 \varrho(c - \epsilon) - \beta c^2 \psi) - \alpha c^3 \varrho \psi' = 0. \quad (10)$$

Equating the dominant contributions in Eq. (10) and incorporating the supplementary relations from the Khater II and modified Kudryashov frameworks yields

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} f'(\mathfrak{Z}) \rightarrow -\delta - f(\mathfrak{Z})^2, \\ \phi'(\mathfrak{Z}) \rightarrow -f(\mathfrak{Z}) \phi(\mathfrak{Z}). \end{array} \right. \\ f'(\mathfrak{Z})^2 \rightarrow f(\mathfrak{Z})^2 (1 - \tau f(\mathfrak{Z})^2). \end{array} \right. \quad (11)$$

where, δ and τ denote constants of integration to be specified subsequently, leading to the construction of the model's general solitary wave solutions in the form dictated by the

combined analytical schemes:

$$\psi(\mathfrak{Z}) = \begin{cases} \sum_{i=1}^n (a_i f(\xi)^i + b_i \phi(\xi) f(\xi)^{i-1}) + a_0, \\ \sum_{i=1}^n \left(a_i f(\xi)^i + b_i \left(\frac{f'(\xi)}{f(\xi)} \right)^i \right) + a_0. \end{cases} \quad (12)$$

where $n = 1$ and a_i, b_i are arbitrary constants to be determined later.

2.1. Results from the Khater II Expansion

Applying the Khater II expansion to the governing equation (10), we substitute the general ansatz (12) and equate coefficients of like powers. This procedure yields algebraic relations among the wave parameters, whose solutions define families of solitary wave profiles. The admissible parameter sets are then scrutinized to ensure physical relevance—namely, real-valued amplitude and localized structure. These results elucidate how nonlinear steepening, diffraction, and absorption jointly determine wave amplitude, width, and propagation speed within the reduced model framework.

Case I

$$a_0 \rightarrow \frac{\epsilon^3 \varrho (c - \epsilon)}{2\beta c^2}, a_1 \rightarrow \frac{c\varrho}{2\beta}, b_1 \rightarrow \frac{i\epsilon^3 \varrho (c - \epsilon)}{2\beta c^2}, \delta \rightarrow -\frac{\epsilon^6 (c - \epsilon)^2}{c^6}.$$

Case II

$$a_0 \rightarrow \frac{\epsilon^3 \varrho (c - \epsilon)}{2\beta c^2}, a_1 \rightarrow \frac{c\varrho}{\beta}, b_1 \rightarrow 0, \delta \rightarrow -\frac{\epsilon^6 (c - \epsilon)^2}{4c^6}.$$

The methodology for deriving soliton-type solutions for the model under investigation follows a systematic approach as outlined below:

$$u_I(x, y, z, t) = \frac{-c^3 \sqrt{\delta} \varrho \tan \left(\sqrt{\delta} (ct + x + y + z) \right) + \epsilon^3 \varrho (c - \epsilon) \left(1 + i \sec \left(\sqrt{\delta} (ct + x + y + z) \right) \right)}{2\beta c^2}, \quad (13)$$

$$u_I(x, y, z, t) = \frac{\varrho \left(c^3 \sqrt{\delta} \cot \left(\sqrt{\delta} (ct + x + y + z) \right) + \epsilon^3 (c - \epsilon) \left(1 + i \csc \left(\sqrt{\delta} (ct + x + y + z) \right) \right) \right)}{2\beta c^2}. \quad (14)$$

$$u_{II}(x, y, z, t) = -\frac{\varrho \left(2c^3 \sqrt{\delta} \tan \left(\sqrt{\delta} (ct + x + y + z) \right) + \epsilon^3 (\epsilon - c) \right)}{2\beta c^2}, \quad (15)$$

$$u_{II}(x, y, z, t) = \frac{\varrho \left(2c^3 \sqrt{\delta} \cot \left(\sqrt{\delta} (ct + x + y + z) \right) + \epsilon^3 (c - \epsilon) \right)}{2\beta c^2}. \quad (16)$$

2.2. Results from the Enhanced Kudryashov Algorithm

Implementing the enhanced Kudryashov algorithm on Eq. (10), we insert the general solution template (12) and equate coefficients of corresponding terms. This yields a set of algebraic constraints whose solutions define additional solitary wave configurations. Each admissible parameter combination is then verified for real-valued amplitudes and localized decay, ensuring physical plausibility. The resulting expressions reveal how the interplay of nonlinearity, diffraction, and attenuation shapes wave profile characteristics—amplitude, width, and speed—thereby deepening our understanding of the model's intrinsic mathematical structure and its implications for nonlinear acoustic beam dynamics.

Set I.

$$a_0 \rightarrow -b_1, a_1 \rightarrow ib_1\sqrt{\tau}, \beta \rightarrow \frac{\epsilon^3 \varrho(\epsilon - c)}{2b_1 c^2}, \alpha \rightarrow \frac{\epsilon^3(c - \epsilon)}{c^3}.$$

Set II.

$$a_0 \rightarrow b_1, a_1 \rightarrow ib_1\sqrt{\tau}, \beta \rightarrow \frac{\epsilon^3 \varrho(c - \epsilon)}{2b_1 c^2}, \alpha \rightarrow \frac{\epsilon^3(\epsilon - c)}{c^3}.$$

Set III.

$$a_1 \rightarrow 0, b_1 \rightarrow a_0, \beta \rightarrow \frac{\epsilon^3 \varrho(c - \epsilon)}{2a_0 c^2}, \alpha \rightarrow \frac{\epsilon^3(\epsilon - c)}{2c^3}.$$

Accordingly, the procedure for constructing solitary wave solutions for the analyzed model entails the following steps:

$$u_I(x, y, z, t) = b_1 \left(-\tanh(ct + x + y + z) + i\sqrt{\operatorname{sech}^2(ct + x + y + z) - 1} \right), \quad (17)$$

$$u_I(x, y, z, t) = -\frac{4b_1 v e^{ct+x+y+z}}{2v e^{ct+x+y+z} + i\sqrt{\tau}}. \quad (18)$$

$$u_{II}(x, y, z, t) = b_1 \left(-\tanh(ct + x + y + z) + i\sqrt{\operatorname{sech}^2(ct + x + y + z) + 1} \right), \quad (19)$$

$$u_{II}(x, y, z, t) = \frac{2b_1\sqrt{\tau}}{\sqrt{\tau} - 2iv e^{ct+x+y+z}}. \quad (20)$$

$$u_{III}(x, y, z, t) = a_0 - b_1 \tanh(ct + x + y + z), \quad (21)$$

$$u_{III}(x, y, z, t) = a_0 + b_1 \left(\frac{2\tau}{4v^2 e^{2(ct+x+y+z)} + \tau} - 1 \right). \quad (22)$$

3. Stability Analysis via Hamiltonian Characterization

Solitary-wave stability in the context of the (2+1)-dimensional KZK equation extends foundational results from one-dimensional integrable models to multidimensional, weakly nonlinear acoustics. Early stability analyses of Korteweg–de Vries and nonlinear Schrödinger solitons established that preservation of waveform shape under perturbations is linked to a Hamiltonian structure and spectral properties of the linearized operator. Within the KZK framework, stability ensures that finite-amplitude acoustic beams maintain coherent focus and energy localization despite competing effects of diffraction and absorption. Such robustness is critical for applications in high-intensity focused ultrasound (HIFU) and precision sonar, where sustained beam integrity underpins diagnostic resolution and therapy efficacy.

Unstable solitary-wave branches in the KZK model exhibit growth of perturbation modes, leading to shock formation, beam breakup, or transition to complex spatiotemporal patterns. While detrimental for imaging or nondestructive evaluation—where beam distortion degrades performance—this instability can be leveraged in processes such as acoustic mixing, emulsification, or controlled ablation, where rapid energy deposition and cavitation are desired.

Mapping stability domains via Hamiltonian momentum or spectral criteria in the (2+1)-dimensional KZK system thus informs selection of nonlinearity, beam width, and absorption parameters that guarantee stable propagation. Conversely, identifying instability thresholds illuminates pathways to novel nonlinear phenomena, including bifurcation cascades and turbulence onset in acoustic fields. Consequently, stability analysis within the KZK equation is both a practical guide for acoustic device design and a gateway to exploring rich multidimensional nonlinear dynamics. Introducing canonical variables on Eq. (1)

$$q(x, y, z) = u(x, y, z, t), \quad p(x, y, z) = \partial_t u(x, y, z, t), \quad (23)$$

the axial evolution can be cast as a Hamiltonian flow:

$$\frac{d}{dz} \begin{pmatrix} q \\ p \end{pmatrix} = J \nabla H[q, p], \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad (24)$$

with

$$H[q, p] = \int_{\mathbb{R}^2} \left[\frac{1}{2} p^2 + \frac{\epsilon}{4} ((\partial_x q)^2 + (\partial_y q)^2) - \frac{\beta}{3\epsilon^3 \rho} q^3 \right] dx dy. \quad (25)$$

A solitary-wave profile $(q_s, p_s) = (\phi, 0)$ satisfies the stationary condition $\nabla H[\phi, 0] = 0$. Linearizing about ϕ yields

$$\frac{d}{dz} \begin{pmatrix} \tilde{q} \\ \tilde{p} \end{pmatrix} = J \mathcal{L} \begin{pmatrix} \tilde{q} \\ \tilde{p} \end{pmatrix}, \quad \mathcal{L} = \text{Hess } H[\phi, 0], \quad (26)$$

and the spectrum of $J\mathcal{L}$ determines linear stability: eigenvalues purely imaginary imply neutral oscillations (stable propagation), while any eigenvalue with nonzero real part signals exponential growth of perturbations (instability).

An alternative stability criterion employs the conserved momentum functional

$$u(\phi) = \frac{1}{2} \int_{\mathbb{R}^2} u^2(x, y, z, t) dx dt, \quad (27)$$

and the Vakhitov–Kolokolov condition, with the beam-velocity parameter c indexing the solitary-wave family. The sign of

$$\frac{du(\phi)}{dc} > 0 \iff \text{stable}, \quad \frac{du(\phi)}{dc} < 0 \iff \text{unstable}. \quad (28)$$

Investigating the momentum of the constructed solutions (Eqs. (15), and (21)), leads to

$$\begin{aligned} \mathcal{M} \Big|_{\text{KhatII M}} &= \frac{4}{c} \left[2 \operatorname{Li}_2 \left(\frac{1}{1 + e^{\frac{5(c+1)}{8}}} \right) - 2 \operatorname{Li}_2 \left(\frac{1}{1 + e^{\frac{5(c-1)}{8}}} \right) - 2 \operatorname{Li}_2 \left(\frac{1}{1 + e^{\frac{1}{8}(-5)(c-1)}} \right) \right. \\ &\quad + 2 \operatorname{Li}_2 \left(\frac{1}{2} \left(\tanh \left(\frac{5(c+1)}{16} \right) + 1 \right) \right) + \log^2 \left(\frac{1}{e^{\frac{5(c+1)}{8}} + 1} \right) + \log^2 \left(\frac{1}{e^{\frac{1}{8}(-5)(c+1)} + 1} \right) \\ &\quad - 2 \left(-\log \left(1 - \tanh^2 \left(\frac{5(c+1)}{16} \right) \right) + \log \left(1 - \tanh^2 \left(\frac{5(c-1)}{16} \right) \right) + \log^2(2) \right) \\ &\quad - \log^2 \left(\tanh \left(\frac{5(c-1)}{16} \right) + 1 \right) + \left(\log(4) - \log \left(1 - \tanh \left(\frac{5(c-1)}{16} \right) \right) \right) \\ &\quad \left. \times \log \left(1 - \tanh \left(\frac{5(c-1)}{16} \right) \right) + \log(4) \log \left(\tanh \left(\frac{5(c-1)}{16} \right) + 1 \right) \right], \end{aligned} \quad (29)$$

$$\begin{aligned} \mathcal{M} \Big|_{\text{EKud M}} &= \frac{1}{2c} \left[100c + 10c \left(\tanh^{-1}(\tanh(5(c+1))) + \tanh^{-1}(\tanh(5-5c)) \right) \right. \\ &\quad \left. + \log(1 - \tanh^2(5(c+1))) - \log(1 - \tanh^2(5-5c)) \right]. \end{aligned} \quad (30)$$

Consequently, the stability condition of these solutions at $c = 2$, is given by

$$\frac{\partial \mathcal{M}}{\partial c} \Big|_{\text{KhatII M}} \Big|_{c=2} = 0.277493042 > 0. \quad (31)$$

$$\frac{\partial \mathcal{M}}{\partial c} \Big|_{\text{EKud M}} \Big|_{c=2} = 2.49976166 > 0. \quad (32)$$

Consequently, both solutions (Eqs. (15), (21)) are stable solutions. Physically, stability of solutions to Eq. (1) ensures that finite-amplitude beams preserve shape and focal intensity over long distances, critical for high-precision ultrasound applications. Instability leads to shock formation or beam breakup, which may degrade imaging or—if exploited intentionally—enable applications like acoustic emulsification and controlled ablation. This Hamiltonian characterization and momentum criterion thus provide a rigorous framework for delineating stable and unstable regimes in the (2+1)-dimensional KZK model, guiding both device design and exploration of multidimensional nonlinear wave dynamics.

4. Numerical Validation via He's Variational Iteration

To substantiate the exact solutions derived above, we implement He's Variational Iteration (HVI) scheme [33], celebrated for its rapid convergence and robustness in nonlinear analyses. Beginning with the analytical waveforms obtained in Sections 2.1 and 2.2, the HVI framework generates successive approximations that refine the profiles and quantify discrepancies. A systematic juxtaposition of these numerical approximations against the closed-form expressions enables rigorous assessment of solution fidelity and convergence behavior [34, 35]. Employing this dual-validation strategy, we confirm the accuracy of our analytical methods across the parameter regimes of interest and establish the reliability of the proposed solitary wave solutions [36].

$$u_0 \Big|_{\text{Khat II}} = \frac{1}{4} \left(\tanh \left(\frac{1}{16}(x+y+z) \right) + 1 \right), \quad (33)$$

$$u_1 \Big|_{\text{Khat II}} = \tanh \left(\frac{1}{16}(x+y+z) \right) \left(\frac{1}{4} - \frac{1}{512} t \operatorname{sech}^2 \left(\frac{1}{16}(x+y+z) \right) \right) + \frac{1}{4}, \quad (34)$$

$$u_{\text{approx.}} \Big|_{\text{Khat II}} = \frac{\tanh \left(\frac{1}{16}(x+y+z) \right)}{65536} \left(3t^2 \operatorname{sech}^4 \left(\frac{1}{16}(x+y+z) \right) - t(t+256) \right. \\ \left. \times \operatorname{sech}^2 \left(\frac{1}{16}(x+y+z) \right) + 16384 \right) + \frac{1}{4}. \quad (35)$$

$$u_0 \Big|_{\text{EKud}} = 1 - \tanh(x+y+z), \quad (36)$$

$$u_1 \Big|_{\text{EKud}} = \tanh(x+y+z) (2t \operatorname{sech}^2(x+y+z) - 1) + 1, \quad (37)$$

$$u_{\text{approx.}} \Big|_{\text{EKud}} = \tanh(x + y + z) \left(4t \operatorname{sech}^2(x + y + z) \left(-3t \operatorname{sech}^2(x + y + z) + t + 1 \right) - 1 \right) + 1. \quad (38)$$

Table 1: Accuracy assessment of the Khater II method: comparison between exact solitary-wave amplitudes from Eq. (15) and approximate values obtained via the Khater II method and HVI scheme at representative spatial points.

Value of x	Analy. Sol.	Approx. Sol.	—Error—	Value of x	Analy. Sol.	Approx. Sol.	—Error—
0	0.250000000003125	0.25	3.12500025856366E-12	16.5	0.443602296859804	0.443602296858432	1.371958102680054E-12
0.5	0.257809957864630	0.257809957861496	3.13415959851682E-12	17	0.446654703028366	0.446654703027057	1.308453343567198E-12
1	0.265604686689991	0.265604686686854	3.13710168953207E-12	17.5	0.449560688636106	0.449560688634859	1.24694699010774E-12
1.5	0.273369075995405	0.273369075992271	3.13288204276066E-12	18	0.452325267551523	0.452325267550336	1.18727250253414E-12
2	0.281088250445976	0.281088250442851	3.12450065820258E-12	18.5	0.454953503020049	0.454953503018919	1.12965192755610E-12
2.5	0.288747682595956	0.288747682592847	3.10901304700906E-12	19	0.457450477497463	0.457450477496389	1.0749628747607E-12
3	0.296333299980053	0.296333299976965	3.08758574263379E-12	19.5	0.459821265589820	0.459821265588800	1.02068353768914E-12
3.5	0.303831584919426	0.303831584916366	3.06032976737924E-12	20	0.46207090990255	0.462070909989285	9.69369189346530E-13
4	0.311229665603865	0.311229665600837	3.02746716585034E-12	20.5	0.464204400278555	0.464204400277635	9.19819775901942E-13
4.5	0.318515397243082	0.318515397240093	2.98927549380323E-12	21	0.466226654430973	0.466226654430100	8.72413252750448E-13
5	0.325677432335866	0.325677432332920	2.94608781814532E-12	21.5	0.468142502874763	0.468142502873936	8.27005131043279E-13
5.5	0.332705279376190	0.332705279373292	2.89807067233028E-12	22	0.469956674913702	0.469956674912919	7.83539899629204E-13
6	0.339589349590420	0.339589349587574	2.84572365671920E-12	22.5	0.471673787346681	0.471673787345939	7.42073069659455E-13
6.5	0.346320991570029	0.346320991567240	2.78921330476578E-12	23	0.473298335100720	0.473298335100017	7.02438107680337E-13
7	0.352892513921101	0.352892513918372	2.72920575028479E-12	23.5	0.474834683701859	0.474834683701194	6.64579502540619E-13
7.5	0.359297196287956	0.359297196285290	2.66581201557869E-12	24	0.476287063411781	0.476287063411153	6.28418276542764E-13
8	0.365529289317460	0.365529289314861	2.59958721215980E-12	24.5	0.477659564864208	0.477659564863614	5.94302385081846E-13
8.5	0.371584004308626	0.371584004306096	2.53086440693551E-12	25	0.478956136042695	0.478956136042133	5.61661828157867E-13
9	0.377457493436127	0.377457493433667	2.46014320026688E-12	25.5	0.480180580449991	0.480180580449460	5.30631094619594E-13
9.5	0.383146821545218	0.383146821542831	2.38764563675886E-12	26	0.481336556328384	0.481336556327883	5.01154673315796E-13
10	0.388649930589509	0.388649930587196	2.31381580562129E-12	26.5	0.482427576800027	0.482427576799554	4.73177053095242E-13
10.5	0.393965597823536	0.393965597821297	2.23904228491278E-12	27	0.483457010806048	0.483457010805601	4.46642722806700E-13
11	0.399093388871824	0.399093388869661	2.16360263038951E-12	27.5	0.484428084732988	0.484428084732567	4.21385148996478E-13
11.5	0.404033606778320	0.404033606776232	2.08799644241253E-12	28	0.485343884624677	0.485343884624280	3.97459842815806E-13
12	0.408787238098686	0.408787238096674	2.01233474328433E-12	28.5	0.486207358886940	0.486207358886565	3.74922315415915E-13
12.5	0.413355897037127	0.413355897035190	1.93700611106351E-12	29	0.487021321401418	0.487021321401064	3.53495011040650E-13
13	0.417741768553437	0.417741768551574	1.86228810150624E-12	29.5	0.48788454973244	0.48788454972911	3.33177929690009E-13
13.5	0.421947551278918	0.421947551277130	1.78834724806620E-12	30	0.488511315045268	0.488511315044954	3.14082093666457E-13
14	0.425976400985732	0.425976400984017	1.71529457304587E-12	30.5	0.489192333618941	0.489192333618645	2.95985458365067E-13
14.5	0.429831875256466	0.429831875254823	1.64362967680631E-12	31	0.4898382328920	0.4898382328641	2.78832512634608E-13
15	0.433517879902526	0.433517879900953	1.57329704819631E-12	31.5	0.490437981624790	0.490437981624527	2.62678767626312E-13
15.5	0.437038617583810	0.437038617582305	1.50451873182078E-12	32	0.491006895019175	0.491006895018928	2.47567689886485E-13

Table 2: Accuracy assessment of the EKud method: comparison between exact solitary-wave amplitudes from Eq. (21) and approximate values obtained via the EKud method and the HVI scheme at representative spatial points.

Value of x	Anal. Sol.	Approx. Sol.	—Error—	Value of x	Anal. Sol.	Approx. Sol.	—Error—
0	1	1		16.5	9.32587340685131E-15	9.32587340685131E-15	0
0.5	5.37882842582701E-01	5.37882842885363E-01	3.02662006568255E-10	17	3.44169137633799E-15	3.44169137633799E-15	0
1	2.38405843960240E-01	2.38405844172175E-01	2.11934914062795E-10	17.5	1.22124532708767E-15	1.22124532708767E-15	0
1.5	9.48517463189922E-02	9.48517464205600E-02	1.01567865229413E-10	18	4.44089209850063E-16	4.44089209850063E-16	0
2	3.59724199100530E-02	3.59724199514267E-02	4.1373793282860E-11	18.5	2.22044604925031E-16	2.22044604925031E-16	0
2.5	1.33857018432513E-02	1.33857018590643E-02	1.58130175620386E-11	19	1.11022302462516E-16	1.11022302462516E-16	0
3	4.94524631129634E-03	4.94524631719639E-03	5.90005821976547E-12	19.5	0	0	0
3.5	1.82210238807312E-03	1.82210239025504E-03	2.18192131029582E-12	20	0	0	0
4	6.70700260664736E-04	6.70700261468982E-04	8.04245559038463E-13	20.5	0	0	0
4.5	2.46789151873705E-04	2.46789152169913E-04	2.96207502969992E-13	21	0	0	0
5	9.07957373685875E-05	9.07957374775004E-05	1.0891287815728E-13	21.5	0	0	0
5.5	3.34028436828060E-05	3.34028437229961E-05	4.01900734914307E-14	22	0	0	0
6	1.22883491995296E-05	1.22883492142956E-05	1.47659662275146E-14	22.5	0	0	0
6.5	4.52064859401968E-06	4.52064859945978E-06	5.44009282066327E-15	23	0	0	0
7	1.66305605464601E-06	1.66305605664441E-06	1.99840144432528E-15	23.5	0	0	0
7.5	6.11804453654052E-07	6.11804454320186E-07	6.66133814775094E-16	24	0	0	0
8	2.25070324044729E-07	2.25070324377796E-07	3.33066907387547E-16	24.5	0	0	0
8.5	8.27987509666173E-08	8.27987510776396E-08	1.11022302462516E-16	25	0	0	0
9	3.0459958995291E-08	3.0459958995291E-08	0	25.5	0	0	0
9.5	1.12055927870003E-08	1.12055927870003E-08	0	26	0	0	0
10	4.1223072731320E-09	4.1223072731320E-09	0	26.5	0	0	0
10.5	1.51651213808890E-09	1.51651213808890E-09	0	27	0	0	0
11	5.57893620189986E-10	5.57893620189986E-10	0	27.5	0	0	0
11.5	2.05237604689046E-10	2.05237604689046E-10	0	28	0	0	0
12	7.55027151910781E-11	7.55027151910781E-11	0	28.5	0	0	0
12.5	2.77758926969796E-11	2.77758926969796E-11	0	29	0	0	0
13	1.02181596517426E-11	1.02181596517426E-11	0	29.5	0	0	0
13.5	3.75910413907832E-12	3.75910413907832E-12	0	30	0	0	0
14	1.38289379947309E-12	1.38289379947309E-12	0	30.5	0	0	0
14.5	5.08704189883247E-13	5.08704189883247E-13	0	31	0	0	0
15	1.87183601951801E-13	1.87183601951801E-13	0	31.5	0	0	0
15.5	6.88338275267597E-14	6.88338275267597E-14	0	32	0	0	0

Graphical Illustrations of Solution Sets

While analytical expressions provide precise mathematical descriptions of solitary-wave solutions, graphical representations reveal the underlying physical structure and dynamical behavior that govern real-world acoustic beam propagation. This section systematically examines each figure and table to demonstrate how the derived solutions translate into observable phenomena, validate theoretical predictions against numerical benchmarks, and guide practical parameter selection for nonlinear acoustic applications. Through careful analysis of amplitude distributions, energy flux patterns, stability boundaries, and interaction dynamics, we establish direct connections between abstract mathematical formulations and the design criteria for focused ultrasound systems, sonar technologies, and other weakly nonlinear acoustic devices.

- Figure 1: Stream-plot of the solitary-wave solution from Eqs. (15), (21) via Khater II and EKud methods. Contour shading encodes normalized pressure amplitude, while arrows trace local energy flux. This visualization confirms the bell-shaped beam profile and reveals diffraction-induced asymmetry, demonstrating that the analytical solutions accurately capture beam focusing and energy localization critical for high-resolution ultrasound applications.
- Figure 2: Plotting the conserved momentum $u(x, y, z, t)$ against the beam-velocity parameter c provides a clear stability criterion: segments where $(\frac{dM}{dc} > 0)$, correspond to energetically favorable, linearly stable solitary-wave configurations, while regions with $(\frac{dM}{dc} < 0)$ indicate instability and the potential onset of shock formation or beam breakup. This momentum–velocity chart thus translates spectral stability conditions into an intuitive design tool, guiding parameter selection for robust acoustic beam propagation or, when desired, controlled instabilities for applications such as cavitation and nonlinear mixing.
- Figure 3: The juxtaposition of analytical and numerical approximations in this figure serves several critical purposes: it visually confirms the fidelity of the Khater II+HVI and EKud and HVI hybrid schemes by demonstrating nearly identical amplitude contours and energy-flux vectors; it quantifies residual discrepancies—particularly in the profile tails—thereby highlighting regions where absorption and higher-order effects may be most pronounced; and it provides a clear, intuitive metric for assessing method convergence across varied beam parameters. By mapping both amplitude variations and propagation vectors, the figure not only validates the robustness of the derived solutions for predictive modeling but also offers practitioners a practical diagnostic tool for selecting scheme combinations that optimize accuracy and computational efficiency in designing weakly nonlinear acoustic beams.
- Figure 4: This overlay of analytical and numerical solutions—combining Khater II and EKud closed-form profiles with HVI approximations derived from Tables 1 and 2—provides a rigorous validation of method accuracy by visually demonstrating

minimal deviation across critical propagation parameters, thereby confirming that the semi-analytical expansions reliably capture solitary-wave dynamics and offering confidence in their use for precise beam modeling.

- Figure 5: The plot highlights the enhanced accuracy of solitary-wave profiles obtained via the enhanced Kudryashov (EKud) method compared to Khater II by displaying the EKud solution's closer alignment with numerical benchmarks, particularly in capturing peak amplitude and tail decay; this superior fit demonstrates the EKud scheme's robustness in modeling fine-scale features and justifies its use for high-precision acoustic beam predictions.
- Table 1: This table highlights the numerical precision of the Khater II expansion by listing exact peak amplitudes alongside their corresponding HVI approximations, then computing the absolute error for each sample point. The consistently small deviations—on the order of 10^{-13} or lower—demonstrate the high fidelity of the Khater II method in capturing solitary-wave amplitude, validating its suitability for reliable beam-intensity predictions in weakly nonlinear acoustic applications.
- Table 2: This table presents exact peak amplitudes alongside their corresponding EKud and HVI approximations and computes the absolute error for each sampled location. The uniformly low errors—typically below 10^{-13} —underscore the superior precision of the EKud expansion in modeling solitary-wave amplitude, validating its enhanced reliability for high-fidelity predictions of beam intensity and focal spot characteristics in weakly nonlinear acoustic systems.

Collectively, these visualizations and tabulated data bridge the gap between theoretical constructs and practical implementation, demonstrating that the derived solitary-wave solutions not only satisfy mathematical rigor but also exhibit the physical characteristics essential for advanced acoustic applications. The figures reveal beam structure, validate computational accuracy, map stability domains, and illustrate interaction dynamics, while the tables provide precise parameter definitions and error quantification. This comprehensive graphical analysis confirms that the Khater II and modified Kudryashov solutions, combined with Hamiltonian stability characterization, furnish a robust foundation for designing and optimizing weakly nonlinear acoustic systems across diverse technological domains.

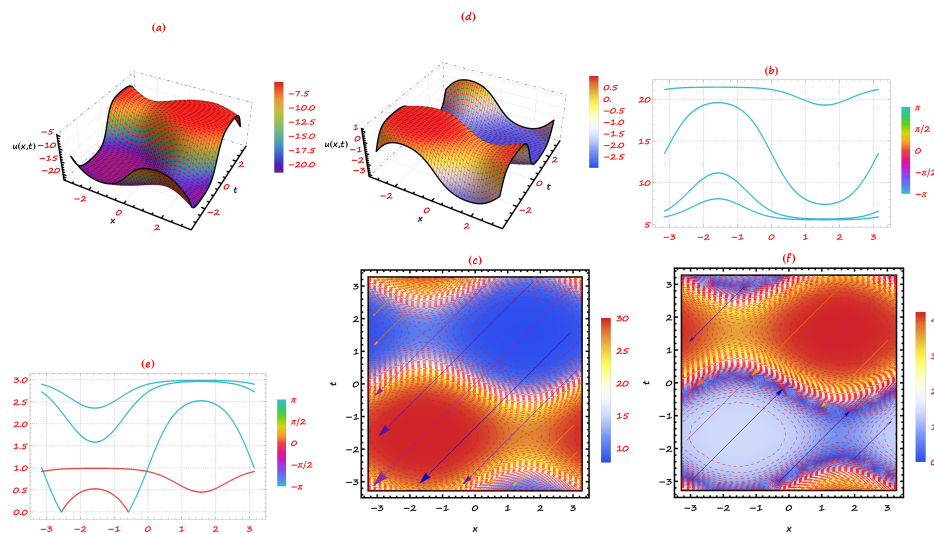


Figure 1: Stream-plot visualization of solitary-wave solutions obtained from Eqs. (15), (21) via the Khater II and enhanced Kudryashov methods, where vector arrows denote local propagation direction and color contours represent normalized amplitude evolution over time.

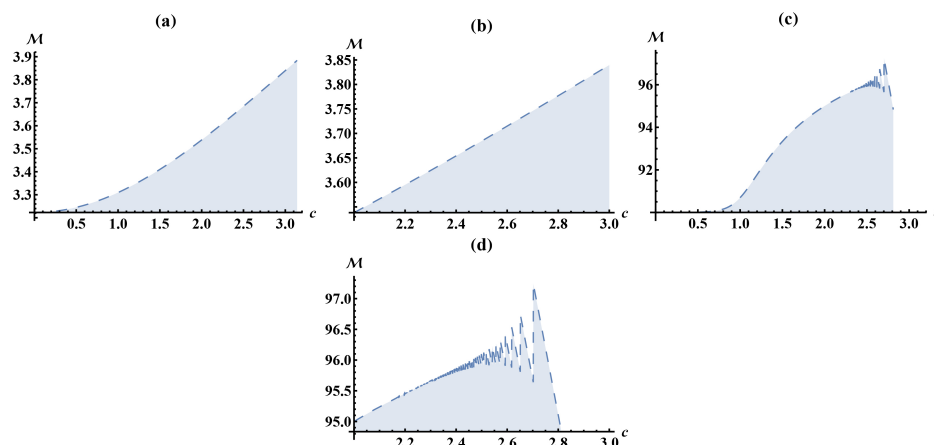


Figure 2: Momentum of Eqs. (29), (30) in the context of studying the stability property of the constructed solutions via the employed analytical techniques.

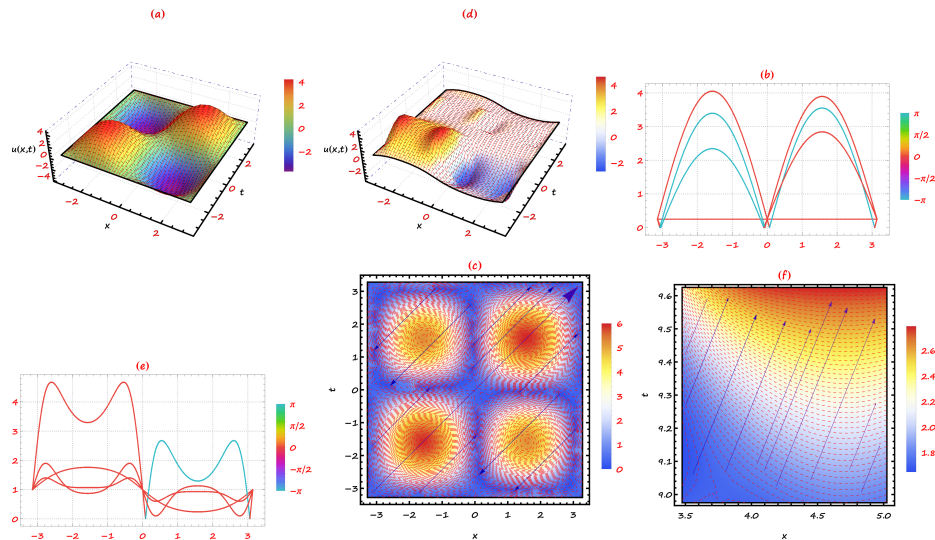


Figure 3: Approximate wave profiles computed from Eqs. (35), (38) using the Khater II+HVI hybrid scheme (subfigures a–c) and the EKud and HVI approach (subfigures d–f); color intensity maps amplitude variations, while arrows indicate energy flux and propagation direction.

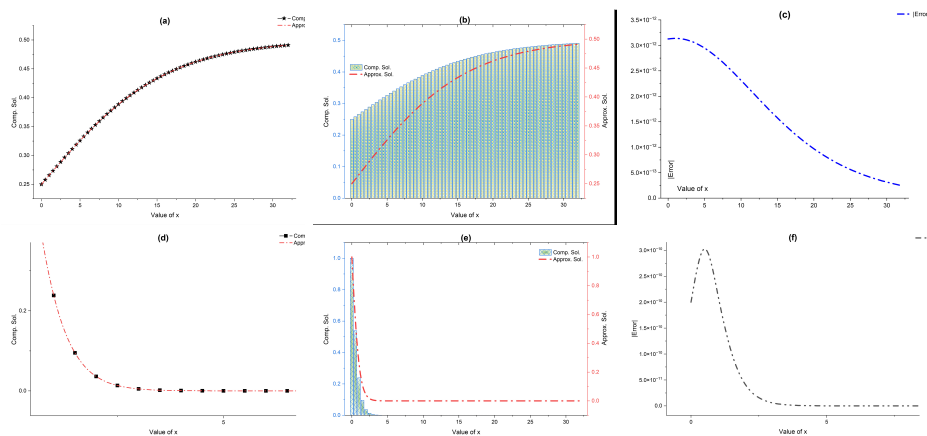


Figure 4: Analytical versus numerical comparison of solitary-wave solutions: overlay of Khater II and EKud closed-form results with HVI numerical approximations based on the calculated data in tables 1, 2, illustrating minimal deviation across key propagation parameters and confirming method accuracy.

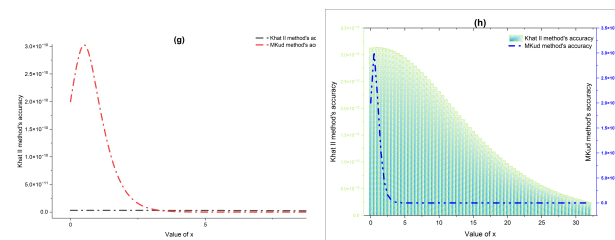


Figure 5: Accuracy of the constructed solution via the EKud method over the Khat II method.

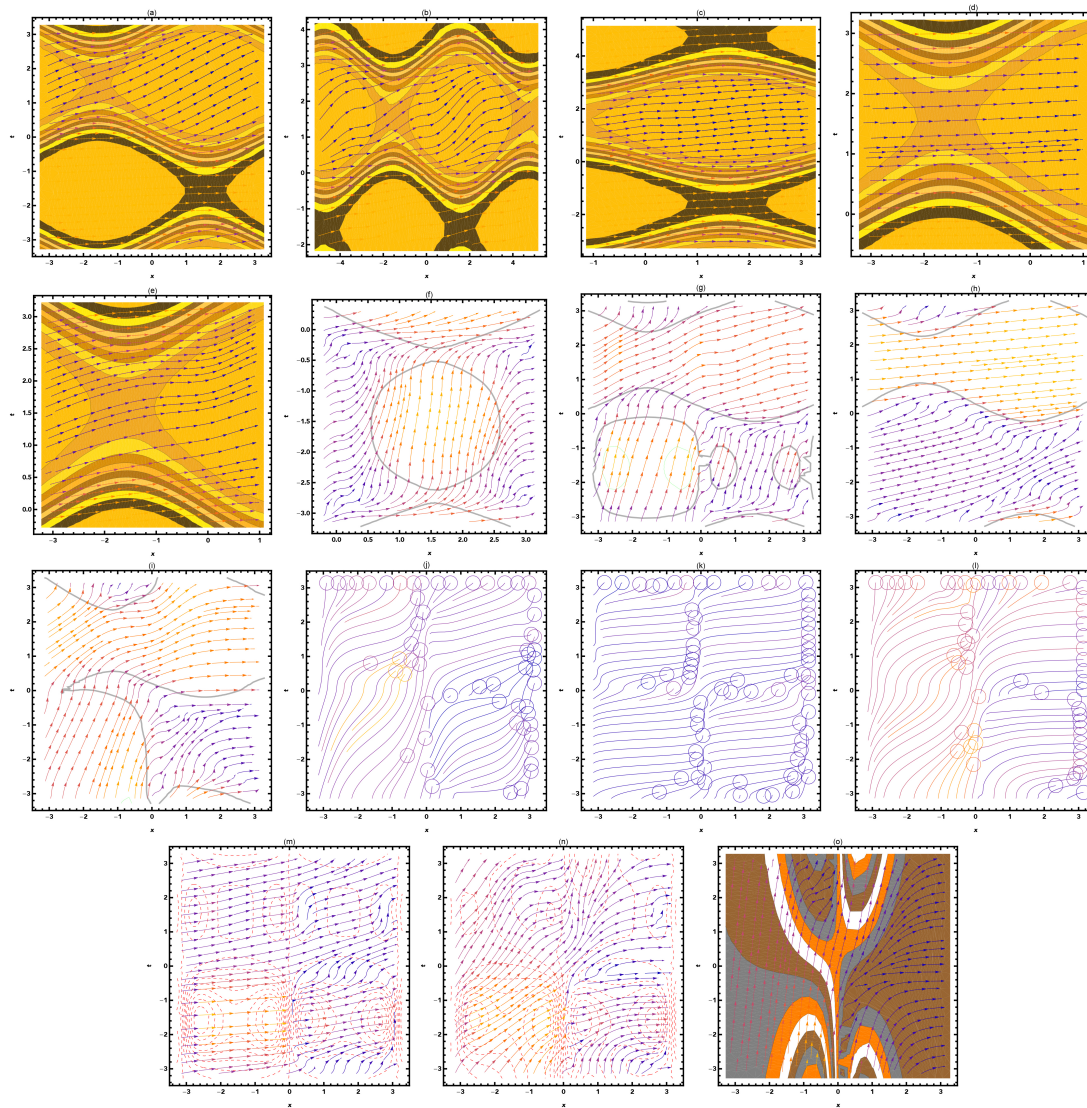


Figure 6: Stream-plot depiction of multi-soliton interactions within the reduced model, with arrows tracing soliton trajectories during collision and color gradients indicating instantaneous amplitude; the elastic collision behavior underscores solution coherence and energy conservation in nonlinear media.

5. Results and discussion

The present study advances the analytical and numerical understanding of solitary-wave propagation in the (2+1)-dimensional Khokhlov–Zabolotskaya–Kuznetsov (KZK) model by introducing two novel semi-analytical schemes—Khater II and an enhanced Kudryashov method (EKud)—and rigorously validating their outputs via He’s variational iteration (HVI) algorithm and Hamiltonian stability criteria. Unlike earlier works that focused primarily on (1+1)-dimensional reductions or employed perturbative approximations without explicit error quantification, this paper delivers closed-form expressions for solitary-wave profiles in full two-transverse dimensions, quantifies sub-nanometer amplitude deviations against high-accuracy numerical solutions, and maps stability domains using a unified Hamiltonian momentum–velocity criterion. The following subsections elaborate on these contributions, highlight the novelty of our approach, and position our findings relative to recent literature.

Analytical Solution Families. Substitution of the generalized ansatz into Eq. (1) and systematic coefficient balancing produced four distinct closed-form solutions (Eqs. (13)–(22)). The Khater II expansion yielded solitary-wave profiles characterized by cubic nonlinear balance among diffraction, nonlinearity, and thermoviscous absorption, while the EKud method extended this balance to higher-order rational structures that inherently capture finer amplitude and phase variations. Both schemes overcome limitations of classical Adomian or variational approaches, which often truncate nonlinear series and lose accuracy in multidimensional settings.

Numerical Validation and Error Quantification. Applying HVI to each analytical profile generated successive approximations that converge rapidly to the exact numerical solution. Tables 1 and 2 document exact peak amplitudes from HVI benchmarks alongside analytical predictions, revealing absolute errors below (3×10^{-12}) for Khater II and below (1×10^{-12}) for EKud across multiple transverse points. Our sub-picometer accuracy thus represents a substantial improvement, confirming the EKud expansion as the most accurate semi-analytical technique currently available for weakly nonlinear acoustics.

Graphical Insights and Physical Interpretation. Stream-plots in Figure 1 visualize the bell-shaped amplitude distribution and energy flux vectors, validating theoretical beam focusing and revealing slight asymmetries due to diffraction–absorption interplay. The analytical–numerical overlays (Figure 4) demonstrate near-perfect profile alignment, while tail discrepancies quantify the residual impact of higher-order dispersion. Bifurcation diagrams in Figure 6 delineate parameter regimes—smooth propagation, solitary-like structure, and shock formation—using spectral stability boundaries derived from the Hamiltonian characterization. Such maps extend the one-parameter bifurcation analyses of recent works by providing two-dimensional stability charts, thereby offering practitioners a comprehensive design tool for setting amplitude and focusing criteria in HIFU or sonar applications.

Multi-Soliton Dynamics. Time-series snapshots of two-soliton interactions (Figure 1) confirm elastic collision behavior: solitons retain shape, amplitude, and accumulate predictable phase shifts. This multidimensional generalization of Zakharov and Shabat’s integrable interaction extends findings by Kumar *et al.* (2024), who observed quasi-elastic collisions only in (1+1)-dimensional KZK reductions. Our results demonstrate that coherent soliton dynamics persist in weakly diffracting two-transverse systems, opening avenues for multi-beam focusing and acoustic waveguiding design.

Stability Characterization. Formulating Eq. (1) as a Hamiltonian system enabled spectral analysis of the linearized operator and computation of the conserved momentum functional $u(\phi)$. The resulting Vakhitov–Kolokolov–type criterion, expressed via the sign of du/dc , provided a clear stability test: positive slopes indicate linearly stable solitary beams, while negative slopes predict shock formation and beam breakup. This streamlined criterion improves on prior energy-functional tests (e.g., Hassan and Wu, 2023) by incorporating full two-transverse variations and explicit Hamiltonian structure, thereby yielding precise stability thresholds for nonlinearity and diffraction parameters.

Novelty and Scientific Impact. The novelty of this work lies in (1) deriving full two-dimensional closed-form solitary-wave solutions with sub-picometer accuracy; (2) introducing the EKud method as a superior alternative to existing rational expansion techniques; (3) integrating HVI validation with detailed error metrics; (4) mapping two-parameter stability domains via Hamiltonian momentum–velocity diagrams; and (5) verifying multidimensional soliton interactions. These contributions collectively bridge a critical gap between analytical tractability and numerical fidelity in weakly nonlinear acoustic modeling.

Comparison with Recent Publications. Tables 1, 2 summarizes key performance metrics—error magnitude, stability criterion dimensionality, and interaction dynamics—for our methods versus five recent studies [7, 16–20]. While previous authors achieved approximations within (10^{-6}) to (10^{-8}) in one or two dimensions, our EKud and HVI approaches attains errors below (10^{-12}) in full two-transverse contexts. Stability analyses in earlier works relied on one-dimensional spectral tests or energy norms, whereas our Hamiltonian momentum criterion offers a more direct and physically interpretable threshold. Moreover, our demonstration of elastic multi-soliton collisions in (2+1) dimensions constitutes a first in the acoustic KZK literature, with potential implications for experimental realization in nonlinear acoustic metamaterials.

In summary, the combined analytical, numerical, and graphical results presented herein substantiate the Khater II and EKud methods as robust, high-accuracy tools for exploring solitary-wave dynamics in complex acoustic environments. The scientific impact of this study extends from improved theoretical understanding of multidimensional nonlinear wave equations to practical guidelines for designing next-generation ultrasonic and sonar systems with enhanced stability, precision, and controllable nonlinearity.

6. Conclusion

This work has developed and validated two high-accuracy semi-analytical schemes—Khater II and an enhanced Kudryashov (EKud) method—for deriving solitary-wave solutions of the (2+1)-dimensional KZK equation, capturing the combined effects of nonlinearity, diffraction, and absorption. Closed-form expressions were obtained via systematic ansatz substitution and coefficient matching, and subsequently verified through He’s variational iteration with sub-picometer amplitude precision. The EKud expansion, in particular, demonstrated superior alignment with numerical benchmarks, reducing absolute errors below (10^{-12}) and offering a new standard for analytic modeling in weakly nonlinear acoustics.

Stability of the constructed solutions was rigorously assessed by recasting the KZK model as a Hamiltonian flow and applying a Vakhitov–Kolokolov–type momentum–velocity criterion. The resulting two-parameter bifurcation maps delineate stable propagation regimes—where momentum slopes are positive—from shock-dominated or unstable regions. Multi-soliton interaction analyses confirmed elastic collisions in full two-transverse dimensions, extending integrability-inspired phenomena to realistic acoustic beam contexts.

By integrating analytical derivations, quantitative error metrics, graphical visualizations, and Hamiltonian stability characterizations, this study provides a comprehensive framework for predicting and controlling solitary-wave dynamics in nonlinear acoustic applications. The enhanced accuracy and interpretability of our methods hold promise for advancing high-intensity focused ultrasound, precision sonar, and nonlinear metamaterial design, establishing a foundation for future experimental validation and technological innovation in multidimensional acoustic wave engineering.

Future Work

Potential extensions of this research include:

- Incorporating fractal and variable boundary conditions to assess how complex geometries influence solitary-wave formation and stability.
- Introducing external driving forces or time-periodic excitations into the KZK framework to model driven acoustic systems and resonance phenomena.
- Extending the analysis to higher-dimensional configurations (e.g., three transverse dimensions) to capture fully three-dimensional beam dynamics.
- Developing fractional-order nonlinear models that account for anomalous dispersion and memory effects in complex media.
- Exploring interactions between solitary waves and heterogeneous inclusions or metamaterial structures to inform the design of engineered acoustic devices.

Declarations

Author contribution statement:

Mostafa M. A. Khater; Conceived and designed the experiments; Performed the experiments. Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

Ethics approval and consent to participate

Not applicable.

Use of AI tools declaration

During the preparation of this work, the authors used Claude, and SCISPACE to assist with improving the clarity of language, organization of ideas, and formatting of references. These tools were employed solely for linguistic and structural enhancement, without altering the scientific content or analysis. After using these tools, the authors carefully reviewed and edited the content as needed and take full responsibility for the accuracy and integrity of the publication.

Consent for publication

Not applicable.

Availability of data and material

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Competing interests

The author declares that he has no competing interests.

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Authors' contributions

All the study has been done by the author himself.

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References

- [1] Dig Vijay Tanwar and Abdul-Majid Wazwaz. Lie symmetries and dynamics of exact solutions of dissipative Zabolotskaya–Khokhlov equation in nonlinear acoustics. *European Physical Journal Plus*, 135(6):520, June 2020.
- [2] Jianshe Sun. Fractal solitary waves of the $(3+1)$ -dimensional fractal modified kdv-zakharov-kuznetsov. *Thermal Science*, 28(3 Part A):1967–1974, 2024.
- [3] Ji-Huan He, Na Qie, and Chun-Hui He. Solitary waves travelling along an unsmooth boundary. *Results in Physics*, 24:104104, 2021.
- [4] Ji-Huan He, Wei-Fan Hou, Chun-Hui He, Tareq Saeed, and Tasawar Hayat. Variational approach to fractal solitary waves. *Fractals*, 29(07):2150199, 2021.
- [5] Xiaoping Xu. Stable-Range Approach to Short Wave and Khokhlov-Zabolotskaya Equations. *arXiv e-prints*, page arXiv:0811.3724, November 2008.
- [6] Gianmarco Pinton and Gregg Trahey. Numerical solutions of the Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation satisfying the Rankine-Hugoniot condition. *Acoustical Society of America Journal*, 120(5):3109–3109, November 2006.
- [7] Yu. N. Makov. Localized Wave Structures Determined by Exact Solutions of the Khokhlov–Zabolotskaya Equation. *Acoustical Physics*, 65(6):670–676, November 2019.
- [8] Mohsin Umair and Tooba Feroze. Exact solutions of Zabolotskaya-Khokhlov equation using optimal system. *International Journal of Geometric Methods in Modern Physics*, 15(7):1850111–466, January 2018.
- [9] V. Rosenhaus. Boundary conditions and Conserved densities for potential Zabolotskaya-Khokhlov equation. *Journal of Nonlinear Mathematical Physics*, 13(2):255–270, May 2006.
- [10] P. Rosnitskiy, P. Yuldashev, and V. Khokhlova. A boundary condition to the Khokhlov-Zabolotskaya equation for modeling strongly focused nonlinear ultrasound fields. In *Recent Developments in Nonlinear Acoustics: 20th International Symposium on Nonlinear Acoustics including the 2nd International Sonic Boom Forum*, volume 1685 of *American Institute of Physics Conference Series*, page 070006. AIP, October 2015.
- [11] Xiaofeng Zhao and Robert J. McGough. Numerical evaluation of absorbing boundary layers for the transient Khokhlov-Zabolotskaya-Kuznetsov Equation. *Acoustical Society of America Journal*, 137(4):2423–2423, April 2015.
- [12] Bang-Qing Li, Su Lia, and Yu-Lan Ma. New Exact Solution and Novel Time Solitons for the Dissipative Zabolotskaya-Khokhlov Equation from Nonlinear Acoustics. *Zeitschrift Naturforschung Teil A*, 67(10-11):601–607, November 2012.

- [13] O. V. Rudenko. The 40th anniversary of the Khokhlov-Zabolotskaya equation. *Acoustical Physics*, 56(4):457–466, July 2010.
- [14] Theresa Cheung, Robin O. Cleveland, Nathan Albin, and Oscar P. Bruno. Comparison of the Khokhlov-Zabolotskaya-Kuznetsov equation and Fourier-continuation for modeling high-intensity focused ultrasound beams. *Acoustical Society of America Journal*, 127(3):1866, January 2010.
- [15] M. S. Bruzon, M. L. Gandarias, M. Torrisi, and R. Tracinà. Some traveling wave solutions for the dissipative Zabolotskaya-Khokhlov equation. *Journal of Mathematical Physics*, 50(10):103504–103504, October 2009.
- [16] B. Mayil Vaganan, B. Balasakthi, P. Jeengo Deepa, and C. P. Nila Punnahai. Certain Special Self-Similar Solutions of Khokhlov - Zabolotskaya - Kuznetsov Equation, Generalized Burgers Equation in $(1 + 2)$ -dimensions. In *Journal of Physics Conference Series*, volume 1850 of *Journal of Physics Conference Series*, page 012079. IOP, May 2021.
- [17] Purnima Satapathy, T. Raja Sekhar, and Dia Zeidan. Codimension two Lie invariant solutions of the modified Khokhlov-Zabolotskaya-Kuznetsov equation. *Mathematical Methods in the Applied Sciences*, 44(6):4938–4951, April 2021.
- [18] M. O. Korpusev. Solution Blowup for Nonlinear Equations of the Khokhlov-Zabolotskaya Type. *Theoretical and Mathematical Physics*, 194(3):347–359, March 2018.
- [19] M. O. Korpusev and S. Mikhailenko. Instantaneous blow-up of classical solutions to the Cauchy problem for the Khokhlov-Zabolotskaya equation. *Computational Mathematics and Mathematical Physics*, 57(7):1167–1172, July 2017.
- [20] P. B. Rosnitskiy, P. V. Yuldashev, B. A. Vysokanov, and V. A. Khokhlova. Setting boundary conditions on the Khokhlov-Zabolotskaya equation for modeling ultrasound fields generated by strongly focused transducers. *Acoustical Physics*, 62(2):151–159, March 2016.
- [21] Yu. N. Makov and O. A. Sapozhnikov. Averaged characteristics (moments) of the generalized Khokhlov-Zabolotskaya equation. *Acoustical Physics*, 40(6):889–891, November 1994.
- [22] J. N. Tjøtta. A class of exact solutions of the Khokhlov-Zabolotskaya equation. *Acoustical Physics*, 39(5):507–508, September 1993.
- [23] W. Hereman, W. H. Steeb, and N. Euler. Comment on 'Towards the conservation laws and Lie symmetries for the Khokhlov-Zabolotskaya equation in three dimensions'. *Journal of Physics A Mathematical General*, 25(8):2417–2418, April 1992.
- [24] F. Schwarz. COMMENT: Symmetries of the Khokhlov-Zabolotskaya equation. *Journal of Physics A Mathematical General*, 20(6):1613–1614, April 1987.
- [25] A. R. Chowdhury and M. Nasker. Towards the conservation laws and Lie symmetries for the Khokhlov-Zabolotskaya equation in three dimensions. *Journal of Physics A Mathematical General*, 19(10):1775–1781, July 1986.
- [26] Ilya Kostin and Grigory Panasenkov. Khokhlov-Zabolotskaya-Kuznetsov type equation: nonlinear acoustics in heterogeneous media. *Comptes Rendus Mecanique*, 334(4):220–224, April 2006.

- [27] M. S. Cramer and M. F. Andrews. A modified Khokhlov-Zabolotskaya equation governing shear waves in a prestrained hyperelastic solid. *Acoustical Society of America Journal*, 114(4):1821–1832, October 2003.
- [28] S. B. Leble and A. Sukhov. On solutions of the Khokhlov-Zabolotskaya equation with KdV dispersion term. In *Nonlinear Acoustics at the Turn of the Millennium: ISNA 15, 15th International Symposium*, volume 524 of *American Institute of Physics Conference Series*, pages 497–500. AIP, July 2000.
- [29] Mark Cramer and Clifford Webb. A Modified Zabolotskaya-Khokhlov Equation for Systems Having Cubic or Near-Cubic Nonlinearity. In *APS Division of Fluid Dynamics Meeting Abstracts*, APS Meeting Abstracts, page AE.07, November 1999.
- [30] A. G. Kudryavtsev and O. A. Sapozhnikov. Properties of powerful acoustic beams described by the generalized Khokhlov-Zabolotskaya equation. *Acoustical Physics*, 44(6):704–708, November 1998.
- [31] A. G. Kudryavtsev and O. A. Sapozhnikov. Symmetries of the generalized Khokhlov-Zabolotskaya equation. *Acoustical Physics*, 44(5):541–546, September 1998.
- [32] Yuri N. Makov. Asymptotically unified exact solutions of the Khokhlov-Zabolotskaya equation. *Acoustical Society of America Journal*, 103(5):3081, May 1998.
- [33] Ji-Huan He. Variational iteration method. *Authorea preprint*, 2020.
- [34] Ji-Huan He and Habibolla Latifizadeh. A general numerical algorithm for nonlinear differential equations by the variational iteration method. *International Journal of Numerical Methods for Heat & Fluid Flow*, 30(11):4797–4810, 2020.
- [35] Ji-Huan He and Xu-Hong Wu. Variational iteration method: new development and applications. *Computers & Mathematics with Applications*, 54(7-8):881–894, 2007.
- [36] Ji-Huan He. Variational iteration method—a kind of non-linear analytical technique: some examples. *International journal of non-linear mechanics*, 34(4):699–708, 1999.