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Pseudo Hyper BI-Algebras

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Abstract. We introduce pseudo hyper BI-algebras, which generalize hyper BI-algebras by replacing the single hyperoperation with two hyperoperations $\circledast$ and $\odot$ linked by the crossed absorption laws $x \in x \circledast (y \odot x)$ and $x \in x \odot (y \circledast x)$. Hyper BI-algebras are the case $\circledast = \odot$, and, up to duality, pseudo-BI-algebras are the single-valued case in which $0 \in x \circledast y$ and $0 \in x \odot y$ are equivalent. Because the two hyperoperations interact and the associated relation $\ll$ need be neither transitive nor antisymmetric, several standard results require additional hypotheses, and we show by examples that these cannot be dropped. We study pseudo hyper subBI-algebras, pseudo hyper BI-ideals, quotients by regular equivalence relations, pseudo hyper normal subsets and pseudo hyper BI-homomorphisms, and we prove a first isomorphism theorem.

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1. Introduction

BCK-algebras and BCI-algebras were introduced by Imai and Iséki [1] and Iséki [2] as algebraic counterparts of propositional calculi, and many related classes have since been studied, among them implication algebras [3, 4]. Borumand Saeid, Kim and Rezaei [5] introduced BI-algebras, algebras $(X; *, 0)$ satisfying $x * x = 0$ and $x * (y * x) = x$.

Two kinds of generalization are common for such algebras. In the non-commutative one, a single operation is replaced by two, as in the pseudo-BCK-algebras of Georgescu and Iorgulescu [6] and the pseudo-BI-algebras of Radfar and Rezaei [7]. In the other, operations are replaced by hyperoperations in the sense of Marty [8], as in hyper BCK-algebras [9], hyper pseudo BCK-algebras [10, 11], hyper BI-algebras [12] and pseudo hyper GR-algebras [13]. Combining both for BI-algebras, we introduce *pseudo hyper BI-algebras* $(H; \circledast, \odot, 0)$, with two hyperoperations satisfying $0 \in x \circledast x$, $0 \in x \odot x$, $x \in x \circledast (y \odot x)$ and $x \in x \odot (y \circledast x)$. Hyper BI-algebras are the case $\circledast = \odot$, and pseudo-BI-algebras correspond to a single-valued case (Remark 1).

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Some results, such as the closure system of ideals, quotients by regular equivalence relations and the first isomorphism theorem, follow standard arguments. The new difficulties have two sources: the hyperoperations interact, so that neither $(H;\circledast,0)$ nor $(H;\odot,0)$ needs to be a hyper BI-algebra (Example 5), and the relation $\ll$ is reflexive but in general neither transitive nor antisymmetric (Example 7). These account for most of the hypotheses in the paper, and we show by examples that they cannot be dropped.

2. Pseudo hyper BI-algebras

Throughout, H is a nonempty set and $\mathcal{P}^*(H)$ denotes the set of nonempty subsets of H. A *hyperoperation* on H is a map $\circledast\colon H\times H\to\mathcal{P}^*(H)$. For $A,B\in\mathcal{P}^*(H)$ and $x\in H$ we write

$$A\circledast B=\bigcup_{a\in A,\ b\in B}a\circledast b,\qquad x\circledast B=\{x\}\circledast B,\qquad A\circledast x=A\circledast\{x\},$$

and similarly for any other hyperoperation. We first recall the definition of a hyper BI-algebra.

Definition 1 ([12])**.** *Let $\circ$ be a hyperoperation on H and let $0\in H$. Then $(H;\circ,0)$ is a* hyper BI-algebra *if, for all $x,y\in H$,*

(HBI1) $0\in x\circ x$*;*

(HBI2) $x\in x\circ(y\circ x)$*.*

Definition 2. *Let $\circledast$ and $\odot$ be hyperoperations on H and let $0\in H$. Then $(H;\circledast,\odot,0)$ is a* pseudo hyper BI-algebra *if, for all $x,y\in H$,*

(PHBI1) $0\in x\circledast x$ *and* $0\in x\odot x$*;*

(PHBI2) $x\in x\circledast(y\odot x)$ *and* $x\in x\odot(y\circledast x)$*.*

Definition 3. *Let $(H;\circledast,\odot,0)$ be a pseudo hyper BI-algebra. For $x,y\in H$ we write*

$$x\le_{\circledast}y\iff 0\in x\circledast y,\qquad x\le_{\odot}y\iff 0\in x\odot y,\qquad x\ll y\iff x\le_{\circledast}y\text{ and }x\le_{\odot}y.$$

For $A,B\in\mathcal{P}^*(H)$ we write $A\ll B$ if for every $a\in A$ there exists $b\in B$ with $a\ll b$. For $x\in H$ we write $x\ll B$ for $\{x\}\ll B$ and $A\ll x$ for $A\ll\{x\}$.

Remark 1. *(i) If $\circledast=\odot=\circ$, then $(H;\circledast,\odot,0)$ is a pseudo hyper BI-algebra if and only if $(H;\circ,0)$ is a hyper BI-algebra. Hence hyper BI-algebras are precisely the pseudo hyper BI-algebras with equal hyperoperations.*

(ii) Suppose that both hyperoperations are single-valued, say $x\circledast y=\{x*y\}$ and $x\odot y=\{x\circ y\}$. Then (PHBI1)–(PHBI2) become

$$x*x=x\circ x=0,\qquad x*(y\circ x)=x,\qquad x\circ(y*x)=x.$$

Radfar and Rezaei [7, Definition 2.1] define a pseudo-BI-algebra as an algebra $(X; \to, \rightsquigarrow, 1)$ *satisfying*

$$x \to x = x \rightsquigarrow x = 1, \qquad (x \to y) \rightsquigarrow x = x = (x \rightsquigarrow y) \to x,$$
$$x \to y = 1 \iff x \rightsquigarrow y = 1.$$

Under the usual duality $x * y = y \to x$, $x \circ y = y \rightsquigarrow x$, *with* 1 *renamed* 0, *the first two groups of axioms become exactly the displayed identities above. The third axiom becomes*

$$x * y = 0 \iff x \circ y = 0.$$

This axiom does not follow from (PHBI1)–(PHBI2), not even in the single-valued case; see Example 1. Thus pseudo-BI-algebras correspond exactly to the single-valued pseudo hyper BI-algebras in which $\leq_{\circledast}$ *and* $\leq_{\odot}$ *coincide, and this extra condition cannot be omitted. In the hyper setting we do not impose the analogous condition* $\leq_{\circledast} = \leq_{\odot}$ *either. It also fails in the genuinely hyper Example 2, where* $0 \in a \circledast c$ *but* $0 \notin a \odot c$. *If moreover* $* = \circ$, *the axioms reduce to those of a BI-algebra [5]. In the single-valued case with* $\leq_{\circledast} = \leq_{\odot}$, *the relation* $\ll$ *is the converse of the relation* $\leq$ *of [7]: under the duality,* $x \ll y$ *holds if and only if* $y \leq x$ *in the notation of [7].*

Example 1. *Let* $H = \{0, 1, 2\}$, *and let* $x \circledast y = \{x * y\}$ *and* $x \odot y = \{x \circ y\}$, *where* $*$ *and* $\circ$ *are given by*

$*$	0	1	2
0	0	0	0
1	1	0	0
2	2	2	0

$\circ$	0	1	2
0	0	0	0
1	1	0	1
2	2	0	0

It can be shown that $(H; \circledast, \odot, 0)$ *is a single-valued pseudo hyper BI-algebra. However,* $1 * 2 = 0$ *while* $1 \circ 2 = 1$, *so* $1 \leq_{\circledast} 2$ *but* $1 \not\leq_{\odot} 2$. *Thus the condition* $x * y = 0 \iff x \circ y = 0$ *fails, and under the duality of Remark 1(ii) this algebra is not a pseudo-BI-algebra in the sense of [7]. Neither* $(H; *, 0)$ *nor* $(H; \circ, 0)$ *is a BI-algebra, since* $1 * (2 * 1) = 1 * 2 = 0 \neq 1$ *and* $2 \circ (1 \circ 2) = 2 \circ 1 = 0 \neq 2$.

Example 2. *Let* $H = \{0, a, b, c\}$ *with the following Cayley tables:*

$\circledast$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0, c\}$	$\{0, c\}$
b	$\{b\}$	$\{0, a\}$	$\{0, c\}$	$\{0, c\}$
c	$\{c\}$	$\{0, b\}$	$\{0, b, c\}$	$\{0\}$

$\odot$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0, b\}$	$\{b, c\}$
b	$\{b\}$	$\{0, c\}$	$\{0\}$	$\{0, c\}$
c	$\{c\}$	$\{0, a\}$	$\{0, b\}$	$\{0\}$

Then $(H; \circledast, \odot, 0)$ *is a pseudo hyper BI-algebra and* $(H; \circledast, 0)$ *is a hyper BI-algebra. However,* $(H; \odot, 0)$ *is not a hyper BI-algebra. Indeed,* $a \odot c = \{b, c\}$, *so*

$$c \odot (a \odot c) = (c \odot b) \cup (c \odot c) = \{0, b\} \cup \{0\} = \{0, b\},$$

which does not contain c. *The corresponding instance of (PHBI2) uses the other hyperoperation inside and does hold:* $c \odot (a \circledast c) = c \odot \{0, c\} = \{c\} \cup \{0\} \ni c$.

Example 3. *Let $H = \{0, 1, 2\}$ with the following Cayley tables:*

$\circledast$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{0, 1\}$	$\{0, 1\}$	$\{0, 1\}$
2	$\{0, 2\}$	$\{0, 2\}$	$\{0, 2\}$

$\odot$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{0, 1\}$	$\{0, 1\}$	$\{0, 1, 2\}$
2	$\{0, 2\}$	$\{0, 1, 2\}$	$\{0, 2\}$

It can be shown that $(H; \circledast, \odot, 0)$ is a pseudo hyper BI-algebra and that both $(H; \circledast, 0)$ and $(H; \odot, 0)$ are hyper BI-algebras.

Example 4. *Let $H = \{0, 1, 2\}$ with the following Cayley tables:*

$\circledast$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$
2	$\{0, 2\}$	$\{1, 2\}$	$\{0, 2\}$

$\odot$	0	1	2
0	$\{0\}$	$\{0, 1\}$	$\{0, 2\}$
1	$\{0, 1\}$	$\{0, 1\}$	$\{0, 1, 2\}$
2	$\{0, 2\}$	$\{0, 1, 2\}$	$\{0, 2\}$

It can be shown that $(H; \circledast, \odot, 0)$ is a pseudo hyper BI-algebra. However, $(H; \circledast, 0)$ is not a hyper BI-algebra, since

$$1 \circledast (2 \circledast 1) = 1 \circledast \{1, 2\} = \{0\} \not\ni 1.$$

Example 5. *Let $H = \{0, 1\}$ with the following Cayley tables:*

$\circledast$	0	1
0	$\{0\}$	$\{0\}$
1	$\{0\}$	$\{0, 1\}$

$\odot$	0	1
0	$\{0\}$	$\{1\}$
1	$\{1\}$	$\{0, 1\}$

These are the examples for a pseudo hyper BI-algebra may arise even when neither of its hyperoperations defines a hyper BI-algebra.

Example 6. *Let $H = \{0, 1, 2\}$ with the following Cayley tables:*

$\circledast$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0, 1\}$
2	$\{1, 2\}$	$\{0, 1, 2\}$	$\{0\}$

$\odot$	0	1	2
0	$\{0\}$	$\{0, 1\}$	$\{1\}$
1	$\{0, 1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0, 1\}$

$(H; \circledast, \odot, 0)$ is a pseudo hyper BI-algebra. One checks in the same way that $(H; \circledast, 0)$ is a hyper BI-algebra. However, $(H; \odot, 0)$ is not, since $0 \odot (2 \odot 0) = 0 \odot 2 = \{1\} \not\ni 0$.

Proposition 1. *Let H be a pseudo hyper BI-algebra. Then for all $x, y \in H$ and all $A, B \in \mathcal{P}^*(H)$:*

(1) $A \ll A$;

(2) $A \subseteq B$ implies $A \ll B$;

(3) $x \in x \circledast (A \odot x)$ *and* $x \in x \odot (A \circledast x)$*;*

(4) $x \ll x \circledast (y \odot x)$ *and* $x \ll x \odot (y \circledast x)$*;*

(5) $x \ll x \circledast (A \odot x)$ *and* $x \ll x \odot (A \circledast x)$*.*

Proof. (1) By (PHBI1), $a \ll a$ for every $a \in A$. (2) For $a \in A$ take $b = a \in B$ and use $a \ll a$. (3) Let $a \in A$. By (PHBI2), $x \in x \circledast (a \odot x) \subseteq x \circledast (A \odot x)$, and similarly for the second statement. (4) and (5) follow from (PHBI2) and (3), respectively, since x belongs to the set on the right-hand side and $x \ll x$.

Proposition 2. *Let H be a pseudo hyper BI-algebra and let $x, y \in H$. Then:*

(1) $x \ll x$*;*

(2) $x \in x \circledast (x \odot x)$ *and* $x \in x \odot (x \circledast x)$*;*

(3) $0 \in 0 \circledast (y \odot 0)$ *and* $0 \in 0 \odot (y \circledast 0)$*;*

(4) if $x \odot x = \{0\}$*, then* $x \in x \circledast 0$*; if* $x \circledast x = \{0\}$*, then* $x \in x \odot 0$*;*

(5) if $x \circledast 0 = \{x\}$ *or* $x \odot 0 = \{x\}$*, then* $x \ll 0$ *implies* $x = 0$*.*

Proof. (1) is clear from (PHBI1). (2) and (3) are the instances $y = x$ and $x = 0$ of (PHBI2), respectively. (4) If $x \odot x = \{0\}$, then (2) gives $x \in x \circledast (x \odot x) = x \circledast 0$. The second statement is proved in the same way. (5) If $x \circledast 0 = \{x\}$ and $x \ll 0$, then $0 \in x \circledast 0 = \{x\}$, so $x = 0$. The case $x \odot 0 = \{x\}$ is analogous.

The relation $\ll$ is in general not a partial order, as the following example shows. This is consistent with the single-valued case: the relation $\leq$ of a pseudo-BI-algebra need not be antisymmetric [7, Example 2.9].

Example 7. *(i) In Example 2 we have $a \ll b$ and $b \ll c$, because 0 belongs to $a \circledast b$, $a \odot b$, $b \circledast c$ and $b \odot c$. However, $a \not\ll c$, because $a \odot c = \{b, c\}$. Hence $\ll$ is not transitive. Moreover, $b \ll a$, so $a \ll b \ll a$ with $a \neq b$, and $\ll$ is not antisymmetric.*

(ii) In the same example, $c \odot a = \{0, a\}$ but $a \not\ll c$, so $c \odot a \not\ll c$. Hence the property $x \circ y \ll x$, which holds in every hyper BCK-algebra [9], fails for pseudo hyper BI-algebras.

(iii) In Example 4 we have $2 \ll 0$, since $2 \circledast 0 = 2 \odot 0 = \{0, 2\}$. So $x \ll 0$ does not imply $x = 0$ without an additional assumption such as the one in Proposition 2(5).

The following two conditions on the constant 0 will be used repeatedly.

Definition 4. *A pseudo hyper BI-algebra $(H; \circledast, \odot, 0)$*

(a) has a sharp zero *if $0 \circledast 0 = \{0\}$ and $0 \odot 0 = \{0\}$;*

(b) has 0 *as a* weak right identity*, or satisfies condition* (WRI)*, if* $x \in x \circledast 0$ *and* $x \in x \odot 0$ *for all* $x \in H$*.*

Remark 2. *Neither condition follows from (PHBI1)–(PHBI2), and they are independent of each other. Example 5 has a sharp zero but does not satisfy (WRI), since* $1 \circledast 0 = \{0\}$*. On any set* H *with* $|H| \geq 2$*, the hyperoperations* $x \circledast y = x \odot y = H$ *define a pseudo hyper BI-algebra that satisfies (WRI) but has no sharp zero. Examples 2, 3, 4 and 6 satisfy both conditions.*

By Proposition 2(4), condition (WRI) holds whenever $x \circledast x = x \odot x = \{0\}$ *for all* $x \in H$*. In particular, both conditions hold automatically in the single-valued case of Remark 1(ii): there* $x * 0 = x * (x \circ x) = x$ *and* $x \circ 0 = x$*, which are the classical identities corresponding to [7, Theorem 2.5(P1)], and* $0 * 0 = 0 \circ 0 = 0$*. So both conditions are natural hyper analogues of classical identities.*

2.1. Pseudo hyper subBI-algebras

Definition 5. *Let* H *be a pseudo hyper BI-algebra and let* $S \subseteq H$ *with* $0 \in S$*. We call* S *a* pseudo hyper subBI-algebra *of* H *if* $x \circledast y \subseteq S$ *and* $x \odot y \subseteq S$ *for all* $x, y \in S$*.*

Proposition 3. *(1) Every nonempty subset* S *of* H *that is closed under* $\circledast$ *and* $\odot$ *contains* 0*, and is therefore a pseudo hyper subBI-algebra.*

(2) If S *is a pseudo hyper subBI-algebra, then* $(S; \circledast|_S, \odot|_S, 0)$ *is a pseudo hyper BI-algebra, where* $\circledast|_S$ *and* $\odot|_S$ *are the restrictions of* $\circledast$ *and* $\odot$ *to* $S \times S$*.*

Proof. (1) Let $x \in S$. Then $0 \in x \circledast x \subseteq S$ by (PHBI1). (2) Since S is closed, $\circledast|_S$ and $\odot|_S$ are hyperoperations on S, and for $x, y \in S$ the sets $y \odot x$ and $x \circledast (y \odot x)$ computed in S coincide with those computed in H. Hence (PHBI1) and (PHBI2) hold in S because they hold in H.

Example 8. *(i) For every pseudo hyper BI-algebra* H *with a sharp zero,* $\{0\}$ *is a pseudo hyper subBI-algebra. Without a sharp zero,* $\{0\}$ *need not be one; see the algebra with* $x \circledast y = x \odot y = H$ *in Remark 2.*

(ii) In Example 2, $S_1 = \{0, a\}$ *is a pseudo hyper subBI-algebra, while* $S_2 = \{0, a, b\}$ *is not, since* $a \circledast b = \{0, c\} \not\subseteq S_2$*.*

2.2. Pseudo hyper BI-ideals

For $I \subseteq H$ and $y \in H$ put

$$I^{\subseteq}_{\circledast,y} = \{x \in H \mid x \circledast y \subseteq I\} \qquad \text{and} \qquad I^{\subseteq}_{\odot,y} = \{x \in H \mid x \odot y \subseteq I\}.$$

Definition 6. *A subset* I *of a pseudo hyper BI-algebra* H *is a* pseudo hyper BI-ideal *of* H *if*

(I1) $0 \in I$*;*

(I2) $I^{\subseteq}_{\circledast,y} \subseteq I$ *and* $I^{\subseteq}_{\odot,y} \subseteq I$ *for every* $y \in I$.

In other words, for $y \in I$ *and* $x \in H$, *each of* $x \circledast y \subseteq I$ *and* $x \odot y \subseteq I$ *implies* $x \in I$.

Remark 3. *Condition (I1) does not follow from (I2). In Example 4, take* $I = \{2\}$. *None of the sets* $0 \circledast 2 = \{0\}$, $1 \circledast 2 = \{0\}$, $2 \circledast 2 = \{0, 2\}$, $0 \odot 2 = \{0, 2\}$, $1 \odot 2 = \{0, 1, 2\}$ *and* $2 \odot 2 = \{0, 2\}$ *is contained in* I. *Hence (I2) holds vacuously, although* $0 \notin I$.

Example 9. *(i) In Example 2, let* $I = \{0, b, c\}$ *and* $y = b \in I$. *Then* $a \odot b = \{0, b\} \subseteq I$ *and* $a \circledast b = \{0, c\} \subseteq I$, *but* $a \notin I$. *Hence* I *is not a pseudo hyper BI-ideal.*

(ii) In Example 4, $I = \{0, 1\}$ *is a pseudo hyper BI-ideal. Indeed, the only element outside* I *is* 2, *and none of the sets* $2 \circledast 0 = \{0, 2\}$, $2 \circledast 1 = \{1, 2\}$, $2 \odot 0 = \{0, 2\}$ *and* $2 \odot 1 = \{0, 1, 2\}$ *is contained in* I.

Proposition 4. *Let* H *be a pseudo hyper BI-algebra.*

(1) H *is a pseudo hyper BI-ideal of itself.*

(2) The intersection of any nonempty family of pseudo hyper BI-ideals of H *is a pseudo hyper BI-ideal. Consequently, every subset* $A \subseteq H$ *is contained in a smallest pseudo hyper BI-ideal, namely the intersection of all pseudo hyper BI-ideals containing* A.

(3) If H *satisfies (WRI), then* $\{0\}$ *is a pseudo hyper BI-ideal.*

Proof. (1) is clear. (2) Let $(I_j)_{j \in J}$ be a nonempty family of ideals and put $I = \bigcap_j I_j$. Then $0 \in I$. Let $y \in I$ and suppose that $x \circledast y \subseteq I$. For every j we have $y \in I_j$ and $x \circledast y \subseteq I_j$, so $x \in I_j$. Hence $x \in I$. The same argument works for $\odot$. (3) Let $x \circledast 0 \subseteq \{0\}$. By (WRI), $x \in x \circledast 0$, so $x = 0$. The case of $\odot$ is identical.

Remark 4. *Definition 6 is modelled on weak hyper BCK-ideals, in which the hypothesis is the inclusion* $x \circ y \subseteq I$. *In the theory of hyper BCK-algebras [9], a hyper BCK-ideal is defined instead by the weaker hypothesis* $x \circ y \ll I$, *and every hyper BCK-ideal is a weak hyper BCK-ideal. The corresponding notion of* $\ll$*-type is given in Definition 7 below. The relation between both notions and the (weak) ideals of hyper BI-algebras studied in [12] is made precise in Remark 5.*

Definition 7. *A subset* I *of* H *with* $0 \in I$ *is a* pseudo hyper BI-ideal of $\ll$-type *if, for all* $y \in I$ *and* $x \in H$, *each of* $x \circledast y \ll I$ *and* $x \odot y \ll I$ *implies* $x \in I$.

Proposition 5. *Every pseudo hyper BI-ideal of* $\ll$*-type is a pseudo hyper BI-ideal. The converse does not hold in general.*

Proof. If $x \circledast y \subseteq I$, then $x \circledast y \ll I$ by Proposition 1(2), and similarly for $\odot$. For the converse, consider Example 4 and $I = \{0, 1\}$, which is a pseudo hyper BI-ideal by Example 9(ii). We have $2 \circledast 0 = \{0, 2\}$, with $0 \ll 0$ and $2 \ll 0$ (Example 7(iii)). Hence $2 \circledast 0 \ll I$ with $0 \in I$ but $2 \notin I$, so I is not of $\ll$-type.

We now describe the ideal generated by a subset. Let H be *finitely valued* if all sets $x \circledast y$ and $x \odot y$ are finite; every finite pseudo hyper BI-algebra is finitely valued.

Proposition 6. *Let H be a finitely valued pseudo hyper BI-algebra and let $A \subseteq H$. Put $A_0 = A \cup \{0\}$ and, for $n \geq 0$,*

$$A_{n+1} = A_n \cup \{x \in H \mid x \circledast y \subseteq A_n \text{ or } x \odot y \subseteq A_n \text{ for some } y \in A_n\}.$$

Then $\langle A \rangle = \bigcup_{n \geq 0} A_n$ is the smallest pseudo hyper BI-ideal of H containing A. If H is finite, the chain $A_0 \subseteq A_1 \subseteq \cdots$ stabilizes after at most $|H|$ steps.

Proof. Clearly $0 \in \langle A \rangle$. Let $y \in \langle A \rangle$ and $x \in H$ with $x \circledast y \subseteq \langle A \rangle$. Then $y \in A_m$ for some m. Since $x \circledast y$ is finite and the A_n increase, $x \circledast y \subseteq A_k$ for some k. For $n = \max(m, k)$ we get $y \in A_n$ and $x \circledast y \subseteq A_n$, so $x \in A_{n+1} \subseteq \langle A \rangle$. The same argument works for $\odot$, so $\langle A \rangle$ is an ideal. If J is any ideal containing A, then $A_0 \subseteq J$, and $A_n \subseteq J$ implies $A_{n+1} \subseteq J$ by (I2). Hence $\langle A \rangle \subseteq J$. The last statement is clear, since a strictly increasing chain of subsets of H has at most $|H|$ steps.

For instance, in Example 2 we have $A_0 = \{0, a\}$ for $A = \{a\}$. Since $b \circledast a = \{0, a\}$ and $c \odot a = \{0, a\}$, we get $A_1 = H$. So $\langle \{a\} \rangle = H$, although $\{0, a\}$ is a subalgebra.

The notions of ideal and subalgebra are independent of each other.

Example 10. *(i) In Example 2, $\{0, a\}$ is a pseudo hyper subBI-algebra (Example 8). It is not an ideal, because $b \circledast a = \{0, a\} \subseteq \{0, a\}$ while $b \notin \{0, a\}$.*

(ii) In Example 6, $I = \{0, 2\}$ is a pseudo hyper BI-ideal. For $y = 0$, the only elements x with $x \circledast 0 \subseteq I$ or $x \odot 0 \subseteq I$ are $x = 0$ and $x = 2$. For $y = 2$, the only elements x with $x \circledast 2 \subseteq I$ are 0 and 2, and no x satisfies $x \odot 2 \subseteq I$. However, I is not a pseudo hyper subBI-algebra, because $2 \circledast 0 = \{1, 2\} \not\subseteq I$.

Remark 5 (The case $\circledast = \odot$)**.** *Let $\circledast = \odot = \circ$, so that $(H; \circ, 0)$ is a hyper BI-algebra (Remark 1(i)). Then the pseudo hyper BI-ideals of Definition 6 and the pseudo hyper BI-ideals of $\ll$-type of Definition 7 are the weak ideals and the ideals of the hyper BI-algebra $(H; \circ, 0)$ in the sense of [12], in analogy with weak hyper BCK-ideals and hyper BCK-ideals [9].*

The two notions already differ for hyper BI-algebras, so the example in the proof of Proposition 5 does not depend on $\circledast \neq \odot$. Let $H = \{0, 1, 2\}$ with

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{0, 2\}$	$\{0\}$

One checks directly that $(H; \circ, 0)$ is a hyper BI-algebra. The set $I = \{0, 1\}$ is a pseudo hyper BI-ideal, since neither $2 \circ 0 = \{2\}$ nor $2 \circ 1 = \{0, 2\}$ is contained in I. It is not of $\ll$-type: $0 \in 2 \circ 1$, so $2 \ll 1$ and hence $2 \circ 0 = \{2\} \ll I$, while $0 \in I$ and $2 \notin I$.

3. Regular equivalences, normal subsets and quotients

Let ρ be an equivalence relation on a pseudo hyper BI-algebra H, and let $[x]$ denote the class of x. For $A, B \in \mathcal{P}^*(H)$ we write $A \ \rho \ B$ if every element of A is ρ-related to some element of B and every element of B is ρ-related to some element of A.

Definition 8. *An equivalence relation ρ on H is* regular *if $x \ \rho \ x'$ and $y \ \rho \ y'$ imply*

$$(x \circledast y) \ \rho \ (x' \circledast y') \qquad \text{and} \qquad (x \odot y) \ \rho \ (x' \odot y').$$

This is the usual notion of a regular equivalence on a hyperstructure (see [11, 14]), required here for both hyperoperations. The following reformulation is convenient for checking regularity in examples.

Lemma 1. *An equivalence relation ρ on H is regular if and only if, for all $x, x', y, y' \in H$ with $x \ \rho \ x'$ and $y \ \rho \ y'$,*

$$\{[t] \mid t \in x \circledast y\} = \{[t] \mid t \in x' \circledast y'\} \qquad \text{and} \qquad \{[t] \mid t \in x \odot y\} = \{[t] \mid t \in x' \odot y'\}.$$

Proof. For $A, B \in \mathcal{P}^*(H)$, the condition $A \ \rho \ B$ says exactly that a class meets A if and only if it meets B. In other words, $\{[a] \mid a \in A\} = \{[b] \mid b \in B\}$.

Let $H/\rho = \{[x] \mid x \in H\}$. For $[x], [y] \in H/\rho$ define

$$[x] \circledast [y] = \{[t] \mid t \in x \circledast y\} \qquad \text{and} \qquad [x] \odot [y] = \{[t] \mid t \in x \odot y\}.$$

These are nonempty subsets of H/ρ. For nonempty $\mathcal{A}, \mathcal{B} \subseteq H/\rho$ we extend the operations as in Section 2: $\mathcal{A} \circledast \mathcal{B} = \bigcup_{\alpha \in \mathcal{A},\, \beta \in \mathcal{B}} \alpha \circledast \beta$, and similarly for $\odot$.

Theorem 1. *Let $(H; \circledast, \odot, 0)$ be a pseudo hyper BI-algebra and ρ a regular equivalence relation on H. Then $\circledast$ and $\odot$ are well-defined hyperoperations on H/ρ, and $(H/\rho; \circledast, \odot, [0])$ is a pseudo hyper BI-algebra.*

Proof. Well-definedness is exactly Lemma 1.

(PHBI1): Since $0 \in x \circledast x$, we have $[0] \in [x] \circledast [x]$, and similarly $[0] \in [x] \odot [x]$.

(PHBI2): Let $x, y \in H$. By definition, $[y] \odot [x] = \{[z] \mid z \in y \odot x\}$. Since $[x] \circledast [z]$ depends only on the class $[z]$, we get

$$[x] \circledast ([y] \odot [x]) = \bigcup_{z \in y \odot x} [x] \circledast [z] = \bigcup_{z \in y \odot x} \{[t] \mid t \in x \circledast z\} = \{[t] \mid t \in x \circledast (y \odot x)\}.$$

By (PHBI2) in H we have $x \in x \circledast (y \odot x)$. Hence $[x] \in [x] \circledast ([y] \odot [x])$. The statement $[x] \in [x] \odot ([y] \circledast [x])$ is proved in the same way.

We now attach to each subset N containing 0 a relation $\equiv_N$ and single out the subsets for which it is a regular equivalence relation.

Definition 9. *Let $N \subseteq H$ with $0 \in N$. Define a relation $\equiv_N$ on H by setting $x \equiv_N y$ if and only if*

$$(x \circledast y) \cap N \neq \emptyset, \quad (y \circledast x) \cap N \neq \emptyset, \quad (x \odot y) \cap N \neq \emptyset \quad \text{and} \quad (y \odot x) \cap N \neq \emptyset.$$

The relation $\equiv_N$ is always reflexive, because $0 \in x \circledast x$, $0 \in x \odot x$ and $0 \in N$. It is also symmetric by definition.

Definition 10. *A subset $N \subseteq H$ with $0 \in N$ is a* pseudo hyper normal subset *of $(H; \circledast, \odot, 0)$ if the following conditions hold for all $x, x', y, y', z \in H$:*

(N1) if $x \equiv_N x'$ and $y \equiv_N y'$, then for every $u \in x \circledast y$ there exists $v \in x' \circledast y'$ with $u \equiv_N v$;

(N2) if $x \equiv_N x'$ and $y \equiv_N y'$, then for every $u \in x \odot y$ there exists $v \in x' \odot y'$ with $u \equiv_N v$;

(N3) if $x \equiv_N y$ and $y \equiv_N z$, then $x \equiv_N z$.

Since $\equiv_N$ is always reflexive and symmetric, condition (N3) says exactly that $\equiv_N$ is an equivalence relation. A convenient sufficient condition for (N3) is the following *chain condition*:

(C) if $(x \circledast y) \cap N \neq \emptyset$ and $(y \circledast z) \cap N \neq \emptyset$, then $(x \circledast z) \cap N \neq \emptyset$; and if $(x \odot y) \cap N \neq \emptyset$ and $(y \odot z) \cap N \neq \emptyset$, then $(x \odot z) \cap N \neq \emptyset$.

Remark 6. *By (N3), the relation $\equiv_N$ is an equivalence relation, and (N1) and (N2) then express exactly its regularity. Indeed, they give one half of the condition in Definition 8, and since $\equiv_N$ is symmetric, the other half follows by interchanging (x, y) and (x', y'). Hence N is a pseudo hyper normal subset if and only if $0 \in N$ and $\equiv_N$ is a regular equivalence relation.*

It should be pointed out that a pseudo hyper normal subset is therefore not really a new concept, but a reformulation of the regularity of $\equiv_N$. It is not known that an intrinsic characterization of such subsets (see the open problem after Example 15). The usefulness of the notion lies in the canonical relation it attaches to N, and the interesting results are those relating N to the class of 0 (Proposition 8 and Corollaries 1 and 2). The reason for formulating (N1) and (N2) in terms of $\equiv_N$, rather than imposing a separate condition for each hyperoperation, is explained in Remark 7.

Proposition 7. *Let $N \subseteq H$ with $0 \in N$. If N satisfies the chain condition (C), then $\equiv_N$ is an equivalence relation, that is, (N3) holds.*

Proof. Reflexivity and symmetry were noted above. Suppose $x \equiv_N y$ and $y \equiv_N z$. From $(x \circledast y) \cap N \neq \emptyset$ and $(y \circledast z) \cap N \neq \emptyset$, condition (C) gives $(x \circledast z) \cap N \neq \emptyset$. From $(z \circledast y) \cap N \neq \emptyset$ and $(y \circledast x) \cap N \neq \emptyset$, it gives $(z \circledast x) \cap N \neq \emptyset$. The two conditions for $\odot$ follow in the same way from the second part of (C). Hence $x \equiv_N z$.

The chain condition is strictly stronger than (N3), as the next example shows.

Example 11. *Let $H = \{0, 1, 2\}$ with the following Cayley tables:*

$\circledast$	0	1	2
0	$\{0\}$	$\{2\}$	$\{2\}$
1	$\{1\}$	$\{0, 2\}$	$\{2\}$
2	$\{0, 2\}$	$\{2\}$	$\{0, 1\}$

$\odot$	0	1	2
0	$\{0\}$	$\{0, 1\}$	$\{0\}$
1	$\{0\}$	$\{0\}$	$\{1, 2\}$
2	$\{0, 1, 2\}$	$\{0, 2\}$	$\{0, 1, 2\}$

H is a pseudo hyper BI-algebra with a sharp zero.

We write H/N for $H/\equiv_N$ and $[x]_N$ for the class of x, or simply $[x]$ when N is clear from the context.

Theorem 2. *Let N be a pseudo hyper normal subset of a pseudo hyper BI-algebra H. Then $\equiv_N$ is a regular equivalence relation. Consequently, the hyperoperations $[x] \circledast [y] = \{[t] \mid t \in x \circledast y\}$ and $[x] \odot [y] = \{[t] \mid t \in x \odot y\}$ are well defined, and $(H/N; \circledast, \odot, [0])$ is a pseudo hyper BI-algebra.*

Proof. By (N3), $\equiv_N$ is an equivalence relation. Let $x \equiv_N x'$ and $y \equiv_N y'$. By (N1), every $u \in x \circledast y$ is $\equiv_N$-related to some $v \in x' \circledast y'$. Since $x' \equiv_N x$ and $y' \equiv_N y$ as well, (N1) also shows that every $v \in x' \circledast y'$ is related to some $u \in x \circledast y$. Hence $(x \circledast y) \equiv_N (x' \circledast y')$. The same argument with (N2) applies to $\odot$. Thus $\equiv_N$ is regular, and the remaining statements follow from Theorem 1.

The class of 0 is related to N as follows.

Proposition 8. *Let N be a pseudo hyper normal subset of a pseudo hyper BI-algebra H.*

(1) If every $n \in N$ satisfies $n \in n \circledast 0$, $n \in n \odot 0$ and $0 \ll n$, then $N \subseteq [0]$.

(2) If $x \circledast 0 = \{x\}$ for all $x \in H$, or $x \odot 0 = \{x\}$ for all $x \in H$, then $[0] \subseteq N$.

In particular, if $x \circledast 0 = x \odot 0 = \{x\}$ and $0 \ll x$ for all $x \in H$, then $[0] = N$.

Proof. (1) Let $n \in N$. Then $n \in (n \circledast 0) \cap N$ and $n \in (n \odot 0) \cap N$. Since $0 \ll n$, we also have $0 \in (0 \circledast n) \cap N$ and $0 \in (0 \odot n) \cap N$. Hence $n \equiv_N 0$. (2) Suppose $x \circledast 0 = \{x\}$ for all x, and let $x \equiv_N 0$. Then $(x \circledast 0) \cap N = \{x\} \cap N \neq \emptyset$, so $x \in N$. The other case is analogous.

In general, N and $[0]$ may differ in both directions.

Example 12. *(i) In Example 4, let $N = \{0, 1\}$. Every entry of the $\circledast$-table meets N, and every entry of the $\odot$-table contains 0. Hence $\equiv_N$ is the total relation, which is trivially a regular equivalence, so N is a pseudo hyper normal subset. Here $[0] = H \supsetneq N$. The hypothesis of Proposition 8(2) fails, since $2 \circledast 0 = \{0, 2\}$.*

(ii) In Example 3, every entry of both tables contains 0. So for $N = \{0\}$ the relation $\equiv_N$ is again total, and $[0] = H \supsetneq N$.

In Section 4 (Corollary 1) we show that $[0]$ is always a pseudo hyper BI-ideal when H satisfies (WRI), and a pseudo hyper subBI-algebra when H has a sharp zero.

Example 13. *Let H be the pseudo hyper BI-algebra of Example 6, in which $\circledast \neq \odot$, and let $N = \{0, 1\}$.*

N is a pseudo hyper normal subset of H. By Theorem 2, the quotient $H/N = \{[0],[2]\}$ is a pseudo hyper BI-algebra with the tables

$\circledast$	$[0]$	$[2]$
$[0]$	$\{[0]\}$	$\{[0]\}$
$[2]$	$\{[0],[2]\}$	$\{[0]\}$

$\odot$	$[0]$	$[2]$
$[0]$	$\{[0]\}$	$\{[0]\}$
$[2]$	$\{[2]\}$	$\{[0]\}$

Note that $\circledast \neq \odot$, so the quotient is again a genuinely "pseudo" structure. Moreover, $[0] = N$, in accordance with Proposition 8(1): each $n \in N$ lies in $n \circledast 0$ and $n \odot 0$, and $0 \ll 1$.

Finally, N is a pseudo hyper subBI-algebra of H, since $x \circledast y, x \odot y \subseteq N$ for $x, y \in N$. It is also a pseudo hyper BI-ideal, since for the only element $2 \notin N$ none of the sets $2 \circledast 0 = \{1,2\}$, $2 \circledast 1 = \{0,1,2\}$ and $2 \odot 0 = 2 \odot 1 = \{2\}$ is contained in N.

Remark 7. *In (N1) and (N2), both the hypotheses $x \equiv_N x'$, $y \equiv_N y'$ and the conclusion $u \equiv_N v$ involve both hyperoperations and both orders of the arguments. A more naive definition would treat each hyperoperation separately and use one-sided conditions. For $\odot$ it would read as follows:*

(N2′) if $(x \odot x') \cap N \neq \emptyset$ and $(y \odot y') \cap N \neq \emptyset$, then for every $u \in x \odot y$ there exists $v \in x' \odot y'$ such that $(u \odot v) \cap N \neq \emptyset$ and $(v \odot u) \cap N \neq \emptyset$.

This condition is too restrictive, because a pseudo hyper normal subset need not satisfy it. Let H and $N = \{0,1\}$ be as in Example 13, where N was shown to be a pseudo hyper normal subset, and take $x = 0$, $x' = 2$ and $y = y' = 0$.

Hence (N2′) fails, although N is a pseudo hyper normal subset. For this reason, (N1) and (N2) are formulated in terms of the relation $\equiv_N$.

4. Pseudo hyper BI-homomorphisms

Definition 11. *Let $H_1 = (H_1; \circledast_1, \odot_1, 0_1)$ and $H_2 = (H_2; \circledast_2, \odot_2, 0_2)$ be pseudo hyper BI-algebras. A map $f\colon H_1 \to H_2$ is a* pseudo hyper BI-homomorphism *if, for all $x, y \in H_1$,*

(i) $f(0_1) = 0_2$;

(ii) $f(x \circledast_1 y) = f(x) \circledast_2 f(y)$ and $f(x \odot_1 y) = f(x) \odot_2 f(y)$,

where $f(A) = \{f(a) \mid a \in A\}$ for $A \subseteq H_1$. A bijective pseudo hyper BI-homomorphism is called a pseudo hyper BI-isomorphism*, and we write $H_1 \cong H_2$ if one exists.*

If f is a bijective homomorphism, then f^{-1} is again a homomorphism. Indeed, for $a = f(x)$ and $b = f(y)$ we have $f^{-1}(a \circledast_2 b) = f^{-1}(f(x \circledast_1 y)) = x \circledast_1 y = f^{-1}(a) \circledast_1 f^{-1}(b)$, and similarly for $\odot$.

Remark 8. *The homomorphisms of Definition 11 are* strong *(or* good*) homomorphisms in the usual terminology of hyperstructure theory. One may also consider* weak *homomorphisms, which satisfy $f(0_1) = 0_2$ and only the inclusions $f(x \circledast_1 y) \subseteq f(x) \circledast_2 f(y)$ and $f(x \odot_1 y) \subseteq f(x) \odot_2 f(y)$.*

Definition 12. *The* kernel *of a pseudo hyper BI-homomorphism* $f\colon H_1 \to H_2$ *is* $\ker f = f^{-1}(\{0_2\}) = \{x \in H_1 \mid f(x) = 0_2\}$.

Proposition 9. *Let* $f\colon H_1 \to H_2$ *be a pseudo hyper BI-homomorphism.*

(1) $x \ll y$ *implies* $f(x) \ll f(y)$ *for all* $x, y \in H_1$.

(2) If S *is a pseudo hyper subBI-algebra of* H_1*, then* $f(S)$ *is a pseudo hyper subBI-algebra of* H_2*. In particular,* $f(H_1)$ *is one.*

(3) If S' *is a pseudo hyper subBI-algebra of* H_2*, then* $f^{-1}(S')$ *is a pseudo hyper subBI-algebra of* H_1.

(4) If H_1 *or* H_2 *has a sharp zero, then* $\ker f$ *is a pseudo hyper subBI-algebra of* H_1.

(5) If $f(H_1)$ *satisfies (WRI), that is,* $z \in z \circledast_2 0_2$ *and* $z \in z \odot_2 0_2$ *for all* $z \in f(H_1)$*, then* $\ker f$ *is a pseudo hyper BI-ideal of* H_1*. This holds in particular if* H_2 *satisfies (WRI).*

Proof. (1) If $0_1 \in x \circledast_1 y$, then $0_2 = f(0_1) \in f(x \circledast_1 y) = f(x) \circledast_2 f(y)$. The same argument applies to $\odot$.

(2) We have $0_2 = f(0_1) \in f(S)$. Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$ with $x_1, x_2 \in S$. Then $y_1 \circledast_2 y_2 = f(x_1 \circledast_1 x_2) \subseteq f(S)$, and similarly for $\odot_2$.

(3) We have $0_1 \in f^{-1}(S')$. Let $x, y \in f^{-1}(S')$ and $t \in x \circledast_1 y$. Then $f(t) \in f(x) \circledast_2 f(y) \subseteq S'$, so $x \circledast_1 y \subseteq f^{-1}(S')$. The same holds for $\odot_1$.

(4) If H_1 has a sharp zero, then $0_2 \circledast_2 0_2 = f(0_1 \circledast_1 0_1) = f(\{0_1\}) = \{0_2\}$, and similarly for $\odot_2$. So in either case $0_2 \circledast_2 0_2 = 0_2 \odot_2 0_2 = \{0_2\}$. Let $x, y \in \ker f$ and $t \in x \circledast_1 y$. Then

$$f(t) \in f(x \circledast_1 y) = f(x) \circledast_2 f(y) = 0_2 \circledast_2 0_2 = \{0_2\},$$

so $t \in \ker f$. Similarly $x \odot_1 y \subseteq \ker f$. Together with $0_1 \in \ker f$, this proves the claim.

(5) Clearly $0_1 \in \ker f$. Let $y \in \ker f$, $x \in H_1$, and suppose $x \circledast_1 y \subseteq \ker f$. Then

$$f(x) \circledast_2 0_2 = f(x) \circledast_2 f(y) = f(x \circledast_1 y) \subseteq \{0_2\}.$$

Since $f(x) \in f(x) \circledast_2 0_2$ by assumption, we get $f(x) = 0_2$, that is, $x \in \ker f$. The case $x \odot_1 y \subseteq \ker f$ is identical.

Example 14. *The sharp-zero hypothesis in Proposition 9(4) cannot be omitted. Let* $T = \{0, 1\}$ *with* $x \circledast y = x \odot y = T$ *for all* x, y *(Remark 2), and let* $f = \mathrm{id}_T$*. Then* $\ker f = \{0\}$*, which is not a pseudo hyper subBI-algebra, since* $0 \circledast 0 = T$.

Proposition 10. *Let* ρ *be a regular equivalence relation on a pseudo hyper BI-algebra* H*. Then the canonical projection* $\pi\colon H \to H/\rho$*,* $\pi(x) = [x]$*, is a surjective pseudo hyper BI-homomorphism with* $\ker \pi = [0]$.

Proof. Clearly $\pi(0) = [0]$ and π is surjective. Moreover, $\pi(x \circledast y) = \{[t] \mid t \in x \circledast y\} = [x] \circledast [y] = \pi(x) \circledast \pi(y)$, and similarly for $\odot$. Finally, $\pi(x) = [0]$ if and only if $x \in [0]$.

Corollary 1. *Let ρ be a regular equivalence relation on a pseudo hyper BI-algebra H. For example, ρ may be $\equiv_N$ for a pseudo hyper normal subset N.*

(1) If H satisfies (WRI), then $[0]$ is a pseudo hyper BI-ideal of H.

(2) If H has a sharp zero, then $[0]$ is a pseudo hyper subBI-algebra of H.

Proof. (1) If $x \in x \circledast 0$, then $[x] \in [x] \circledast [0]$, and similarly for $\odot$. So H/ρ satisfies (WRI). The claim follows from Propositions 10 and 9(5). (2) This follows from Propositions 10 and 9(4).

Corollary 2. *Let H be a pseudo hyper BI-algebra such that $x \circledast 0 = x \odot 0 = \{x\}$ and $0 \ll x$ for all $x \in H$. Then every pseudo hyper normal subset N of H satisfies $N = [0]_N$, and is both a pseudo hyper BI-ideal and a pseudo hyper subBI-algebra of H.*

Proof. By Proposition 8, $N = [0]_N$. The hypotheses imply (WRI), and they imply a sharp zero since $0 \circledast 0 = 0 \odot 0 = \{0\}$. Now apply Corollary 1.

The hypotheses of Corollary 2 are considerably stronger than a sharp zero and (WRI): they require $x \circledast 0$ and $x \odot 0$ to be singletons and 0 to lie below every element. Among the examples of Section 2, they hold only in Examples 1 and 2.

Example 15. *(i) The hypotheses of Corollary 2 cannot simply be dropped. In Example 5, $N = \{0\}$ is a pseudo hyper normal subset: $0 \odot 1 = \{1\}$, so $0 \not\equiv_N 1$, and $\equiv_N$ is the equality relation. But N is not an ideal, since $1 \circledast 0 = \{0\} \subseteq N$ while $1 \notin N$. Here $1 \circledast 0 \neq \{1\}$.*

(ii) The converse of Corollary 2 fails. Example 2 satisfies its hypotheses, and $\{0\}$ is an ideal there by Proposition 4(3). However, $\{0\}$ is not a pseudo hyper normal subset. Indeed, for $N = \{0\}$ we have $x \equiv_N y$ if and only if $x \ll y \ll x$. By Example 7(i) we have $a \equiv_N b$ and $b \equiv_N c$, since also $c \ll b$ ($0 \in c \circledast b$ and $0 \in c \odot b$). But $a \not\equiv_N c$, so (N3) fails.

The problem of finding intrinsic conditions on an ideal I ensuring that I is a pseudo hyper normal subset remains open.

For a map $f\colon H_1 \to H_2$ let θ_f be the equivalence relation on H_1 given by $x \,\theta_f\, y \iff f(x) = f(y)$.

Theorem 3 (First isomorphism theorem)**.** *Let $f\colon H_1 \to H_2$ be a pseudo hyper BI-homomorphism. Then θ_f is a regular equivalence relation on H_1, and*

$$\varphi\colon H_1/\theta_f \to f(H_1), \qquad \varphi([x]) = f(x),$$

is a pseudo hyper BI-isomorphism. In particular, $H_1/\theta_f \cong f(H_1)$.

Proof. Let $x \,\theta_f\, x'$ and $y \,\theta_f\, y'$. Then

$$f(x \circledast_1 y) = f(x) \circledast_2 f(y) = f(x') \circledast_2 f(y') = f(x' \circledast_1 y').$$

Hence every $u \in x \circledast_1 y$ has some $v \in x' \circledast_1 y'$ with $f(u) = f(v)$, and conversely. In other words, $(x \circledast_1 y)\ \theta_f\ (x' \circledast_1 y')$. The same holds for $\odot_1$, so θ_f is regular. By Theorem 1, H_1/θ_f is a pseudo hyper BI-algebra. By Proposition 9(2), $f(H_1)$ is a pseudo hyper subBI-algebra of H_2.

The map φ is well defined and injective by the definition of θ_f, and it is clearly surjective. We have $\varphi([0_1]) = 0_2$, and

$$\varphi([x] \circledast [y]) = \{f(t) \mid t \in x \circledast_1 y\} = f(x \circledast_1 y) = f(x) \circledast_2 f(y) = \varphi([x]) \circledast_2 \varphi([y]),$$

and similarly for $\odot$. Hence φ is a bijective homomorphism, that is, an isomorphism.

Under additional assumptions, the relation θ_f is determined by the kernel.

Theorem 4. *Let $f\colon H_1 \to H_2$ be a pseudo hyper BI-homomorphism and put $K = \ker f$. Assume that the relation $\ll$ is antisymmetric on $f(H_1)$. Then:*

(1) K is a pseudo hyper normal subset of H_1;

(2) $\equiv_K = \theta_f$;

(3) $H_1/K \cong f(H_1)$.

Proof. For $x, y \in H_1$ we have

$$(x \circledast_1 y) \cap K \neq \emptyset \iff 0_2 \in f(x \circledast_1 y) = f(x) \circledast_2 f(y) \iff f(x) \leq_{\circledast} f(y), \qquad (*)$$

and similarly $(x \odot_1 y) \cap K \neq \emptyset \iff f(x) \leq_{\odot} f(y)$. Hence $x \equiv_K y$ holds if and only if $f(x) \ll f(y)$ and $f(y) \ll f(x)$. By antisymmetry this implies $f(x) = f(y)$. Conversely, $f(x) = f(y)$ implies $f(x) \ll f(y) \ll f(x)$ by (PHBI1). This proves (2).

For (1), note that $0_1 \in K$. By (2), $\equiv_K = \theta_f$, which is a regular equivalence relation by Theorem 3. By Remark 6, K is a pseudo hyper normal subset.

Finally, (3) follows from (2) and Theorem 3.

Without the antisymmetry hypothesis, $(*)$ still shows that $x \equiv_K y$ holds if and only if $f(x) \ll f(y) \ll f(x)$.

Example 16. *Let H and $N = \{0, 1\}$ be as in Example 13, and let $\pi\colon H \to H/N$ be the canonical projection. By Proposition 10, π is a surjective homomorphism between pseudo hyper BI-algebras with distinct hyperoperations, and $\ker \pi = [0] = N$.*

In H/N we have $[0] \ll [2]$, but $[2] \not\ll [0]$, since $[2] \odot [0] = \{[2]\}$. So $\ll$ is antisymmetric on H/N, and Theorem 4 applies: $\equiv_{\ker \pi} = \theta_\pi = \equiv_N$ and $H/\ker \pi \cong H/N$, as expected. Since H satisfies (WRI) and has a sharp zero, Corollary 1 confirms that $\ker \pi = N$ is both an ideal and a subalgebra of H (compare Example 13).

The antisymmetry assumption in Theorem 4 cannot be dropped from items (2) and (3), as the next example shows.

Example 17. *Let $H_1 = \{0, u, v\}$ with the following Cayley tables:*

$\circledast_1$	0	u	v
0	$\{0\}$	$\{0\}$	$\{0\}$
u	$\{0, u\}$	$\{0, u\}$	$\{0, u\}$
v	$\{0, u, v\}$	$\{0, u, v\}$	$\{0, u, v\}$

$\odot_1$	0	u	v
0	$\{0\}$	$\{0\}$	$\{0\}$
u	$\{u\}$	$\{0\}$	$\{0\}$
v	$\{0, v\}$	$\{0, v\}$	$\{0, v\}$

Let $H_2 = \{0, w\}$ with $\circledast_2 = \odot_2$ given by $0 \circledast_2 0 = 0 \circledast_2 w = \{0\}$ and $w \circledast_2 0 = w \circledast_2 w = \{0, w\}$.

Every diagonal entry contains 0. Every entry of the row of x in the tables of $\circledast_1$ and $\circledast_2$, and in the table of $\odot_1$ for $x \in \{0, v\}$, contains x. For $x = u$, every set $y \circledast_1 u$ contains 0 and $u \odot_1 0 = \{u\}$, so $u \in u \odot_1 (y \circledast_1 u)$. Hence H_1 and H_2 are pseudo hyper BI-algebras, and $\circledast_1 \neq \odot_1$.

The map f with $f(0) = f(u) = 0$ and $f(v) = w$ is a pseudo hyper BI-homomorphism. Indeed, in both tables of H_1 the images of the entries in the rows of 0 and u are $\{0\} = 0 \circledast_2 f(y)$, and the images of the entries in the row of v are $\{0, w\} = w \circledast_2 f(y)$. The algebra H_2 has a sharp zero, so $\ker f = \{0, u\}$ is a pseudo hyper subBI-algebra of H_1 by Proposition 9(4). This can also be seen directly from the tables.

However, in H_2 we have $0 \ll w$ and $w \ll 0$, so $\ll$ is not antisymmetric on $f(H_1) = H_2$. Accordingly, by $(*)$, all elements of H_1 are $\equiv_{\ker f}$-equivalent. Thus $\ker f$ is still a pseudo hyper normal subset, since the total relation is a regular equivalence, so the conclusion of item (1) holds. However, $\equiv_{\ker f} \neq \theta_f$, and $H_1/\ker f$ has only one element, while $H_1/\theta_f \cong H_2$ has two.

5. Conclusion

In this paper we introduced pseudo hyper BI-algebras as a common generalization of hyper BI-algebras and pseudo-BI-algebras. What distinguishes them from these earlier structures is that the two hyperoperations $\circledast$ and $\odot$ are tied together by the axioms, so in general neither $(H; \circledast, 0)$ nor $(H; \odot, 0)$ is a hyper BI-algebra on its own. In addition, the relation $\ll$ is reflexive but need not be transitive or antisymmetric.

Despite these differences, a good part of the familiar theory still goes through. The pseudo hyper BI-ideals form a closure system, quotient structures can be built from regular equivalence relations, and the first isomorphism theorem holds. Wherever we had to add extra assumptions, we gave examples showing that they cannot be dropped.

Several questions remain open. It would be natural to look for intrinsic conditions under which an ideal is a normal subset, to establish the second and third isomorphism theorems, and to classify the pseudo hyper BI-algebras of small order.

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Declarations

The authors declare that there is no conflict of interest. This manuscript has no associated data. The authors have no relevant financial or non-financial interests to disclose.

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