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Exponential-Based Similarity Measure for Pythagorean Fuzzy Sets: a Comparative Numerical Study

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Abstract. In this research article, we address the issue of uncertainty by employing Pythagorean Fuzzy Sets (PFS), which serve as a generalized form of Interval Fuzzy Sets (IFS) of type 2, providing a more flexible and comprehensive framework for modeling imprecise information. PFS extends the capability of traditional fuzzy sets (FS) and IFS by incorporating a broader range of membership and non-membership degrees, constrained by the condition that the square sum of these degrees does not exceed one. To effectively capture the relationships and fundamental connectives among FS, IFS, and PFS, we utilize a system of three-sided standards and co-norms, which serve as mathematical tools for aggregating and comparing fuzzy information. In our exploration, special emphasis is placed on using S-norms that exhibit invariance across different standard settings, ensuring consistency and reliability in operations involving fuzzy constructs. The implications of this work extend to decision support systems, artificial intelligence, and complex data analysis, where robust uncertainty quantification is essential. By refining PFS operations and similarity assessment, our approach contributes to more interpretable and reliable outcomes in real-world applications. Moreover, recognizing the limitations in existing similarity measures, we propose an alternative method for evaluating the similarity between two PFSs, aimed at enhancing the accuracy and interpretability of results in applications involving uncertainty modeling, decision-making, and data analysis within complex and ambiguous environments.

2020 Mathematics Subject Classifications: AMS classification codes

Key Words and Phrases: Similarity measures, pythagorean fuzzy sets, exponential functions, s-norm.

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1. Introduction

Atanassov [1] expanded Zadeh's idea of fuzzy sets by introducing Intuitionistic Fuzzy Sets. Yager [2] further generalized this idea and introduced PFSs. PFSs and Intuitionistic Fuzzy Sets have proven to be effective tools for capturing ambiguity in MADM problems [3–6]. PFSs, being more comprehensive than IFSs, provide an additional approach to represent uncertainty and ambiguity. PFSs have found significant applications

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in various domains, including the extension of the TOPSIS model for MCDM [4, 7, 8],the MABAC approach based on Pythagorean fuzzy Choquet integrals [9], the utilization of the Process of AHP [10], and the multi-objective optimization based on proportion investigation (MOORA) strategy [11].Researchers have also developed techniques to assess the relationship between PFSs. Garg [12] proposed novel correlation coefficients to degree the association between two PFSs. Garg [13] introduced a dynamic model with stochastic data and prompt probabilities under the PF environment. Similarity measures (SMs) play a vital role in determining the resemblance between entities. Similarity metrics for Pythagorean fuzzy sets have been extensively studied and applied in several domains such as MADM problems, remedial findings, dynamic systems, design acknowledgment, AI, image processing, and other relevant fields. Xue [14] employed the LP tool for multi-dimensional analysis in Pythagorean fuzzy situations to address MAGDM problems. Zhang [15] established a decision strategy that depends on similarity measures within the Pythagorean fuzzy context. Peng [16] recently proposed a novel similarity measure and a Pythagorean fuzzy distance metric. Wei and Wei [17] introduced SMs constructed on the cosine function, specifically designed for dynamic challenges. Ejegwa [18] and others [19–21] proposed axiomatic explanations for SMs and distances for PFSs. Asif [22] developed Hamacher Aggregation Operators for Pythagorean Fuzzy Sets and has applied these operators to solve the Multi-Attribute Decision-Making Problem. Dağıstanlı [23] developed facility location selection for Ammunition depots based on GIS, and Pythagorean Fuzzy WASPAS. Das [24] introduced an FP-intuitionistic multi-fuzzy N-soft set and its induced FP-Hesitant N-soft set in decision-making. Das [25] developed an approach for solving group decision-making problems (DMPs) with intuitionistic fuzzy parameterized (IFP) intuitionistic multi-fuzzy N-soft set of dimension q (briefly, IFPIMFNSS) by introducing its induced IFP-hesitant N-soft set (IFPHNSS) as an extension of the multi-fuzzy N-soft set (MFNSS) based decision-making method (DMM). Das [26] employed an approach for solving real-life group decision-making problems (DMPs) with fuzzy parameterized intuitionistic fuzzy soft multisets (p-sets) by extending the fuzzy soft multiset (FSMS) based decision-making method (DMM). Das [27] proposed an efficient water quality evaluation model using weighted hesitant fuzzy soft sets for the water pollution rating problem. Khan [28] investigated some threats to data states in Data Loss Prevention Using the Concept of Complex Linear Diophantine Fuzzy Relations. Petrović [29] introduced some normalization techniques and also investigated the influence of these techniques on MCDM ranking. WANG [23] worked on the method of selecting sustainable food suppliers considering iterative factors. Sarfraz [30] introduced aggregation operators for selecting an Eco-friendly transportation mode for interval-valued Pythagorean Fuzzy Sets. Sarfraz [31] evaluated medical college projects with Hamacher Aggregation operators under the interval-valued complex T-spherical fuzzy environment. We provide illustrative examples to demonstrate the effectiveness of these measures. The main concepts of SMs and PFSs are studied in the "preliminaries" section. We then define and analyze the properties of the planned SMs in the section titled "Applications of the proposed SMs." Finally, we conclude our findings in the concluding section.

Table 1: Abbreviations used in the paper

Abbreviations	*Fullname*
FSs	$FuzzySets$
$IFSs$	$IntuitionisticFuzzySets$
MDs	$MembershipDegrees$
$NMDs$	$Non-MembershipDegrees$
SMs	$SimilarityMeasures$
$PFSs$	$Pythagoreanfuzzysets$
$MADM$	$Multipleattributedecision-making$
LP	$LinearProgramming$
AHP	$PythagoreanFuzzyAnalyticHierarchy$
$MCDM$	$Multipleattributedecision-making$

2. Preliminaries

In the context of set theory, the notion of vagueness can be defined in terms of the grades of membership function and non-membership function. These terms and concepts can be precisely defined as follows.

Definition 2.1 [1, 2] Consider a finite universe of discourse $X = \{x_1,\ x_2,\ ...,\ x_n\}$. A PFS K on $X = \{x_1,\ x_2,\ ...,\ x_n\}$ is mathematically expressed as

$K = \{(x_t, \mu_K(x_t), \nu_k(x_t)) : x_t \in X\}$,

where $\mu_k : X \to [0, 1]$ denotes the membership degree and $\nu_k : X \to [0, 1]$ signifies the non-membership degree of an element x_t to K in X, satisfying that $0 \leq \mu_k(x_t), \nu_k(x_t) \leq 1$ and $0 \leq \mu_k^2(x_t) + \nu_k^2(x_t)) \leq 1, \forall x_t \in X$. The hesitancy degree is defined as $\pi_k(x_t) = \sqrt{1 - \mu_k^2(x_t) - \nu_k^2(x_t)}$. For simplicity, the term " (μ_k, ν_k)" is defined as the 'Pythagorean fuzzy number (PFN)' and can be denoted as $\wedge = (\mu, \nu)$

Recently, Yager[2] introduced Pythagorean Fuzzy Sets as a variation of IFSs.

Definition 2.2 [4] Let $X = \{x_1,\ x_2,\ ...,\ x_n\}$ denote the conversation set. A PFS K is defined by a true-membership function $\mu_K : X \to [0, 1]$ and a non-true membership function $\nu_K : X \to [0, 1]$, in X, where $\mu_k(x_t)$ are grades of the membership x_t in K and $\nu_k(x_t)$are the negation of x_t in K, and $\mu_k^2(x_t) + \nu_k^2(x_t)) \leq 1$. Simply stated, $\mu_{kt} = \mu_k(x_t)$ and $\nu_{kt} = \nu_k(x_t)$, t = 1,2,. . . ,n. P(X) represents the entire collection of PF subsets in X, and for $K \in P(X)$. It is given as

$$K = \{(x_t, P_K(x_t)) : \ x_t \in X\} \tag{2.1}$$

Where $P_K(x_t) = (\mu_{K_t}, \nu_{K_t})$ is PF value.

$(\mu_1, \nu_1)$and$(\mu_2, \nu_2)$are two PF values, for these PF values, we have

$$(\mu_1, \nu_1) = (\mu_2, \nu_2) \Leftrightarrow \mu_1 = \mu_2 \ and \ \nu_1 = \nu_2$$

And $(\mu_1, \nu_1) \leq (\mu_2, \nu_2) \Leftrightarrow \mu_1 \leq \mu_2 \ and \ \nu_1 \geq \nu_2$

And $(\mu_1, \nu_1) < (\mu_2, \nu_2) \Leftrightarrow \mu_1 < \mu_2 \ and \ \nu_1 > \nu_2$

Definition 2.3 [32] For two PFSs $K, L \in P(X)$, we can assert $K \subseteq L \Leftrightarrow \mu_{K_t} \leq \mu_{L_t}$ and $\nu_{K_t} \geq \nu_{L_t}$ for each $x_t \in X; t = 1, 2, ..., n$.

In this study, we use S-norm as a technique for putting SMs into practice. As a result, we provide the definitions below:

Definition 2.4 [33] A mapping $s : PFSs(X) \times PFSs(X) \rightarrow [0, 1]$. $s(K, L)$ is called the degree of similarity between two PFSs K and L if $s(K, L)$ satisfies the following axioms:

(1) $0 \leq s(K, L) \leq 1$

(2) $s(K, L) = s(L, K)$

(3) $s(K, L) = if\ and\ only\ if\ K = L$

(4) $If\ K \subseteq L$ and $L \subseteq M,$ then $s(L, M) \leq s(K, L)\ and\ s(K, M) \leq s(L, M)$

Definition 2.5 [34] A binary operation defined on unit interval, $S : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is continuous S-norm if it satisfying the subsequent requirements:

(S-1) $S(k, 0) = k$ for all $k \in [0, 1]$

(S-2) If $k \leq q$ and $l \leq m$ implies $S(k, l) \leq S(q, m)$

(S-3) $S(k, q) = S(q,\ k)$

(S-4) $S\left(k,\ S(q, m)\right) \leq S\left(S(k, q), m\right)$ foreach $k, l, m, q\ \in [0, 1]$

Here are some examples of basic S-norms:

(I). Standard Union: $s(K, L) = max(K, L)$

(II). Algebraic Sum: $s(K, L) = K + L - KL$

(III). Bounded Sum: $s(K, L) = min\ (1,\ K + L)$

(IV) Drastic Union: $s(K, L) = \begin{cases} K, & when\ L = 0 \\ L, & when\ K = 0 \\ 1, & otherwise \end{cases}$

In addition, we introduce several novel similarity measures (SMs) for Pythagorean fuzzy sets based on the notable features of the grades of membership functions and non-membership functions. The motivation behind this study is the development of a new exponential similarity metric for PFSs using S-norm, which offers the advantage of maintaining consistency across different standards. Furthermore

Theorem 2.1: [35] There is a continuous and strictly increasing function for each continuous s-norm $g_s : [0, 1] \rightarrow [0,\ +\infty)$ such that $g_s(0) = 0,\ g_s(1) = 1$ and for any $x, y\ \in [0, 1]:$

$x\ S\ y = \begin{cases} g_s^{-1}(x) + g_s(y)], & g_s(x) + g_s(y)\ \in [0, 1] \\ 1, & otherwise \end{cases}$

Here, function g_s is called a generator of s-norm.

Example 1. For the s-norm $x\ S\ y = \max\{x, y\}$ and any $x, y\ \in [0, 1]$, the generator is $g_s(x) = x$

Example 2. For the s-norm $x\ S\ y = \min\left\{1,\ \left(x^p + y^p\right)^{\frac{1}{p}}\right\},\ \ p \geq 1$ and any $x, y\ \in [0, 1]$, the generator is $g_s(x) = x^p$.

3. Improved S-similarity measure based on the exponential function for Pythagorean Fuzzy Sets

In this section, we will establish a similarity metric between two PFSs. It is crucial to first propose a method for handling a similarity measure between two PF values and then extend it to encompass two PFSs.

Zhang and Xu [4] offer a SM between two PF values, which is as follows:

D $=(\mu_D,\nu_D)$, E$=(\mu_E,\nu_E)$ be two PF values, The degree of resemblance between D and E is defined as sm (D,E) $= \frac{\rho(\mathrm{D,E})}{\rho(\mathrm{D,E})+\rho(\mathrm{D,E}^c)}$ (3.1)

Where $E^c = (\nu_E,\mu_E)$ and

$\rho\,(\mathrm{D,R}) = \frac{1}{2}\left(\left|\mu_D^2 - \mu_E^2\right| + \left|\nu_D^2 - \nu_E^2\right|\right)$ (3.2)

Here, Consider d and e be two PF values, such that $D = (\mu_D, \nu_D)$and $E = (\mu_E, \nu_E)$, where ${\mu_D}^2 + \nu_D^2 \leq 1$ and $\mu_E^2 + \nu_E^2 \leq 1$. The Improved s-SM for the PF values d and e symbolized by $s(D,E)_S$ is given by

$$s(D,E)_S = \sqrt{S\left(e^{-\left|\mu_D^2-\mu_E^2\right|}, e^{-\left|\nu_D^2-\nu_E^2\right|}\right)} \tag{3.3}$$

where $S(.,.)$ is a continuous S-norm.

Theorem 3.1: The given function (3.3) is a valid similarity measure for Pythagorean fuzzy sets.

Consider $D = (\mu_D, \nu_D)$, $E = (\mu_E, \nu_E)$ and $F = (\mu_F, \nu_F)$are three PF standards. Consequently, $s(.,.)_S$ equivalent to the formula (3.1) is fulfilled by the ensuing attribute.

(1) $0 \leq s(D,E)_S \leq 1$.

(2) $s(D,E)_S = s(E,D)_S$.

(3) $s(D,E)_S = 1 \leftrightarrow D = E$.

(4) If $\mu_D \leq \mu_E \leq \mu_F$, and $\nu_D \geq \nu_E \geq \nu_F$, then $s(D,F)_S \leq \min\{s(D,E)_S, s(E,F)_S\}$.

Proof: We can easily acquire 1, 2, and 3. For axiom 4th:

If $\mu_D \leq \mu_E \leq \mu_F$, and $\nu_D \geq \nu_E \geq \nu_F$, then

$$(\mu_D - \mu_E)^2 \leq (\mu_D - \mu_F)^2,$$

$(\mu_F - \mu_E)^2 \leq (\mu_D - \mu_F)^2,$

$$(\nu_D - \nu_E)^2 \leq (\nu_D - \nu_F)^2,$$

$(\nu_F - \nu_E)^2 \leq (\nu_D - \nu_F)^2.$

By third postulate of above mentioned Definition 2.5

$$S\left(e^{-\left|\mu_D^2-\mu_E^2\right|}, e^{-\left|\nu_D^2-\nu_E^2\right|}\right) \leq S\left(e^{-\left|\mu_D^2-\mu_F^2\right|}, e^{-\left|\nu_D^2-\nu_F^2\right|}\right),$$

$S\left(e^{-\left|\mu_F^2-\mu_E^2\right|}, e^{-\left|\nu_F^2-\nu_E^2\right|}\right) \leq S\left(e^{-\left|\mu_D^2-\mu_F^2\right|}, e^{-\left|\nu_D^2-\nu_F^2\right|}\right).$

Hence

$$\sqrt{1 - S\left((\mu_D - \mu_E)^2, (\nu_D - \nu_E)^2\right)}$$

$$\geq \sqrt{1 - S\left((\mu_D - \mu_F)^2, (\nu_D - \nu_F)^2\right)},$$

$$\sqrt{1 - S\left((\mu_F - \mu_E)^2, (\nu_F - \nu_E)^2\right)}$$

$$\geq \sqrt{1 - S\left((\mu_D - \mu_F)^2, (\nu_D - \nu_F)^2\right)}.$$

with definition 3.1

$$s(D,E)_S \geq s(D,F)_S, s(E,F)_S \geq s(D,F)_S,$$

then

$s(D,F)_S \prec \min\{s(D,E)_S, s(E,F)_S\}$.

As a result, the evidence is complete.

Definition 3.1. Wei and Wei [17] have offered:

Let $X = \{x_1, x_2, x_3, ..., x_n\}$ denote the conversation set. Then operator $\vee$ can be defined as :

$$\bigvee_{i=1}^{n} x_i = \max(x_1,\ x_2, \ldots, x_n)$$

1. The weighted similarity metric based on cosine computes similarity depending on squared membership, non-membership, and hesitation terms, normalized by their magnitudes. This measure is useful when the relative proportions of these terms are more important than their absolute differences. It is applied to problems where directionality matters, such as classification tasks or clustering. The following is a weighted similarity metric based on the cosine between PFSs K and L:

$$WPFC^1(K,L) = \sum_{i=1}^{n} \omega_i \frac{\mu_K^2(d_i)\mu_L^2(d_i) + \nu_K^2(d_i)\nu_L^2(d_i) + \pi_K^2(d_i)\pi_L^2(d_i)}{\sqrt{\mu_K^4(d_i) + \nu_K^4(d_i) + \pi_K^4(d_i)}\sqrt{\mu_L^4(d_i) + \nu_L^4(d_i) + \pi_L^4(d_i)}} \tag{3.4}$$

2. The following are two weighted similarity estimates based on cosine between PFSs K and L:

$$WPFCS^1(K,L) = \sum_{i=1}^{n} \omega_i \cos\left[\frac{\pi}{2}\begin{pmatrix} \left|\mu_K^2(d_i) - \mu_L^2(d_i)\right| \\ \vee \left|\nu_K^2(d_i) - \nu_L^2(d_i)\right| \\ \vee \left|\pi_K^2(d_i) - \pi_L^2(d_i)\right| \end{pmatrix}\right] \tag{3.5}$$

$$WPFCS^2(K,L) = \sum_{i=1}^{n} \omega_i \cos\left[\frac{\pi}{4}\begin{pmatrix} \left|\mu_K^2(d_i) - \mu_L^2(d_i)\right| \\ + \left|\nu_K^2(d_i) - \nu_L^2(d_i)\right| \\ + \left|\pi_K^2(d_i) - \pi_L^2(d_i)\right| \end{pmatrix}\right] \tag{3.6}$$

These measures use the maximum deviation across membership/non-membership/hesitation terms, penalizing the largest discrepancy and offering a smoother penalty. It is preferable for balanced scenarios where all deviations contribute more equally.

3. The following are two weighted similarity metrics based on cotangent between PFSs K and L:

$$WPFCT^1(K,L) = \sum_{i=1}^{n} \omega_i \cot \left[\frac{\pi}{4} + \frac{\pi}{4} \begin{pmatrix} \left|\mu_K^2(d_i) - \mu_L^2(d_i)\right| \\ \vee \left|\nu_K^2(d_i) - \nu_L^2(d_i)\right| \\ \vee \left|\pi_K^2(d_i) - \pi_L^2(d_i)\right| \end{pmatrix} \right] \tag{3.7}$$

$$WPFCT^2(K,L) = \sum_{i=1}^{n} \omega_i \cot \left[\frac{\pi}{4} + \frac{\pi}{8} \begin{pmatrix} \left|\mu_K^2(d_i) - \mu_L^2(d_i)\right| \\ \vee \left|\nu_K^2(d_i) - \nu_L^2(d_i)\right| \\ \vee \left|\pi_K^2(d_i) - \pi_L^2(d_i)\right| \end{pmatrix} \right] \tag{3.8}$$

These measures use cotangent functions to map deviations into similarity scores, with different sensitivity parameters. It is effective for high-confidence matching, such that strict thresholds in pattern recognition.

Where $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$is the weighted vector of d_i with $\omega_i \in [0,1]$, $where\ i = 1, 2, ..., n$, $\sum_{i=1}^{n} \omega_i = 1$ and sign " $\vee$ "is the used for maximum.

Definition 3.2 Let $X = \{x_1, x_2, ..., x_n\}$ signify the conversation set and $D, E \in P(X)$ be two PFSs. The s-similarity measure for the PFSs D and E denoted as

$$s(D,E)_S = \sum_{t=1}^{n} \omega_j s\left(P_D(x_t), P_E(x_t)\right)_S, \tag{3.9}$$

Where $\omega_j > 0$is the weight of the component $x_t \in X, t = 1, 2, ...n$, where $\sum_{t=1}^{n} \omega_t = 1$ and $s(.,,)_S$ is well-defined in the eq. (3.3).

Theorem 3.2: Consider $D, E, F \in P(X)$ be three PFSs. Therefore, $s(.,.)_S$ analogous to formula (3.9) is fulfilled in the ensuing properties:

(i) $0 \leq s(D,E)_S \leq 1$.

(ii) $s(D,E)_S = s(E,D)_S$.

(iii) $s(D,E)_S = 1 \leftrightarrow D = E$.

(iv) If $D \subseteq E \subseteq F$, then $s(D,F)_S \leq \min\{s(D,F)_S, s(E,F)_S\}$.

Proof. Basically we can attain (i), (ii) and (iii). For (iv):

If $D \subseteq E \subseteq F$,hen $P_D(x_t) \leq P_E(x_t) \leq P_F(x_t)$; then, with 4$^{\text{th}}$ axiom of theorem 3.1

$s\left(P_D(x_t), P_F(x_t)\right)_S \leq \min\left\{s\left(P_D(x_t), P_E(x_t)\right)_S, s\left(P_E(x_t), P_F(x_t)\right)_S\right\}$.

Then

$\sum_{t=1}^{n} \omega_t s\left(P_D(x_t), P_F(x_t)\right)_S \leq \sum_{t=1}^{n} \omega_t s\left(P_D(x_t), P_E(x_t)\right)_S$,

$\sum_{t=1}^{n} \omega_j s\left(P_D(x_t), P_F(x_t)\right)_S \leq \sum_{t=1}^{n} \omega_t s\left(P_E(x_t), P_F(x_t)\right)_S$,

In conclusion, we obtain

$s(D,F)_S \leq s(D,E)_S, s(D,E)_S \leq s(E,F)_S$.

Finally

$s(D,F)_S \leq \min\{s(D,E)_S, s(E,F)_S\}$.

Therefore, $s(D,F)_S \leq \min\{s(D,E)_S, s(E,F)_S\}$ & the evidence is completed.

Table 2: SMs between $\mathbf{K_t}$ $(\mathbf{t = 1, 2, 3})$ and $\mathbf{K}$

SMs	(K_1, K)	(K_2, K)	(K_3, K)
$s(K_t, K)_{S_1}$	0.7852	0.8204	0.6803
$s(K_t, K)_{S_2}$	1.0000	1.0000	1.0000
$s(K_t, K)_{S_3}$	0.8768	0.9040	0.8228

4. Numerical Experimentation

In order to offer further details on the exponential similarity measure for PFSs, we present three numerical examples. These illustrations are meant to improve comprehension and practical use of the suggested similarity measure.

Example 4.1

Let the feature space $D = \{d_1, d_2, d_3\}$ of PFSs depict the following three well-known patterns $K_t(t = 1, 2, 3)$:

$$K_1 = \{(d_1, 0.7, 0.6), (d_2, 0.7, 0.7), (d_3, 0.8, 0.5)\}$$

$$K_2 = \{(d_1, 0.63, 0.67), (d_2, 0.9, 0.4), (d_3, 0.8, 0.53)\}$$

$$K_3 = \{(d_1, 0.83, 0.4), (d_2, 0.5, 0.7), (d_3, 0.6, 0.71)\}$$

Let an unknown pattern $K \in PFSs(D)$ that will be prominent, where
$K = \{(d_1, 1, 0), (d_2, 1, 0), (d_3, 1, 0)\}$.

When we contemplate the size of $d_t(t = 1, 2, 3)$, to be 0.5, 0.33, and 0.17, separately. While the specific weights in the numerical example were hypothetical, their choice aligns with common practices in fuzzy multi-criteria decision-making (MCDM) literature. Weights were normalized (summing to 1) to ensure comparability across examples. The assigned weights (e.g., 0.5, 0.3, 0.2) reflect a deliberate hierarchy of criteria importance. Similar arbitrary weights are often used in foundational SM papers (e.g., Wei & Wei [17], Garg [12]) to isolate the impact of the SM itself rather than weight sensitivity. [36] proposed the idea of weighted fuzzy soft multiset (WFSMS) as a generalization of FSMS-based decision-making for solving decision-making in an uncertain situation.[37] developed an advanced approach to fuzzy soft group decision making using weighted average ratings. [38] introduced a weighted hesitant bipolar-valued fuzzy soft set in decision making problems.

The suggested weighted SMs are then based on exponential functions generated from K to $K_t(t = 1, 2, 3)$ are presented in Table 2.

Example 4.2 (Pattern identification) [12, 17] Consider the following three well-known patterns. $K_t(t = 1, 2, 3)$ which the PFSs symbolize $K_t(t = 1, 2, 3)$ in the feature space as $D = \{d_1, d_2, d_3\}$

$$K_1 = \{(d_1, 1, 0), (d_2, 0.8, 0), (d_3, 0.7, 0.1)\}$$

$$K_2 = \{(d_1, 0.8, 0.1), (d_2, 1.0, 0), (d_3, 0.9, 0.1)\}$$

Table 3: SMs based on exponential function between (K_t ($\mathbf{t = 1, 2, 3}$) and $\mathbf{K}$

SMs	(K_1, K)	(K_2, K)	(K_3, K)
$WPFC^1(K_t, K)$	0.9621	0.9640	0.9823
$WPFCS^1(K_t, K)$	0.6573	0.7199	0.9450
$WPFCS^2(K_t, K)$	0.8717	0.8987	0.9800
$WPFCT^1(K_t, K)$	0.4321	0.4523	0.7376
$WPFCT^2(K_t, K)$	0.6323	0.6408	0.8274
$s(K_t, K)_{S_1}$	0.7842	0.7937	0.9190
$s(K_t, K)_{S_2}$	0.7421	0.7587	0.8953
$s(K_t, K)_{S_3}$	0.7622	0.7718	0.8990

Table 4: SMs between $\mathbf{K_t}$ ($\mathbf{t = 1, 2, 3, 4, 5}$) and $\mathbf{K}$

SMs	(K_1, K)	(K_2, K)	(K_3, K)	(K_4, K)	(K_5, K)
$WPFC^1(K_t, K)$	0.6633	0.7318	**0.7610**	0.2944	0.2397
$WPFCS^1(K_t, K)$	**0.9253**	0.8858	0.8904	0.7897	0.6392
$WPFCS^2(K_t, K)$	**0.9374**	0.9250	0.9256	0.8132	0.7244
$WPFCT^1(K_t, K)$	**0.7180**	0.6524	0.6909	0.5393	0.4393
$WPFCT^2(K_t, K)$	**0.7484**	0.7124	0.7481	0.5690	0.5000
$s(K_t, K)_{S_1}$	**0.8995**	0.8742	0.8750	0.8285	0.7838
$s(K_t, K)_{S_2}$	**0.8560**	0.8295	0.8397	0.6504	0.5564
$s(K_t, K)_{S_3}$	**0.8660**	0.8422	0.8519	0.7356	0.6787

$K_3 = \{(d_1, 0.6, 0.2), (d_2, 0.8, 0), (d_3, 0.6, 0.2)\}$.

Consider an unknown forms $K \in PFSs(D)$ that will be known, where

$K = \{(d_1, 0.5, 0.3), (d_2, 0.6, 0.2), (d_3, 0.8, 0.1)\}$.

When we contemplate the size of $d_t(t = 1, 2, 3)$, to be 0.5, 0.3, and 0.2, correspondingly, then we employ the provided weighted SMs depend on exponential functions generated from K to $K_t(t = 1, 2, 3)$ are specified in table 3.

Example 4.3 (medicinal analysis) [12, 17] Consider the following set of diagnoses:

$K = \{K_1, K_2, K_3, K_4, K_5\}$,

in which K_1 = viral infection, K_2 = Malaria, K_3 = typhoid fever, K_4 = abdominal issue and K_5 = chest problem and a collection of signs

$D = \{D_1, D_2, D_3, D_4, D_5\}$,

in which D_1= temperature, D_2= headache, D_3= stomachache, D_4= cough, D_5= chest discomfort

Consider that a patient, with reference to all indications, can be exemplified by the subsequent Pythagorean Fuzzy Sets:

$K = \{(D_1, 0.8, 0.1), (D_2, 0.6, 0.1), (D_3, 0.2, 0.8), (D_4, 0.6, 0.1), (D_5, 0.1, 0.6)\}$.

Our reason is to categorize the pattern K in one of K_t $(t = 1, 2, 3, 4, 5)$. The suggested similarity metrics are based on exponential functions that have been listed from K to $K_t(t = 1, 2, 3, 4, 5)$are specified in Table 4. We examined the weight of $D_t(t = 1, 2, 3, 4, 5)$to be 0.15, 0.25, 0.20, 0.15 and 0.25 respectively.

Resulting from Table 4's findings, the status for SMs with diverse S-norms is as follows: $K_5 < K_4 < K_2 < K_3 < K_1$. The inferences are conditionally valid based on the chosen weights. This is inherent to most fuzzy MCDM studies unless weights are derived from empirical data. However, the primary contribution of this work lies in the SM's mathematical properties (Theorem 3.1–3.2) and its comparative advantages (Table 2–4), which hold irrespective of specific weights.

5. Conclusion

This paper introduces a novel set of similarity measures (SMs) specifically designed for Pythagorean Fuzzy Sets (PFS), utilizing S-norms applied to both membership degrees (MDs) and non-membership degrees (NMDs) to evaluate similarity with greater precision and adaptability. These measures aim to enhance the expressiveness and analytical strength of fuzzy modeling in environments characterized by uncertainty and vagueness. Building upon this theoretical foundation, we further demonstrate the practical utility of the proposed SMs by applying them to pattern identification and medical analysis tasks, using numerical experimentation to validate their performance. The experimental results affirm that the SMs, formulated using exponential functions within the PFS framework, are highly effective in managing real-world dynamic problems, where traditional fuzzy approaches often fall short. Their ability to capture subtle differences in patterns makes them particularly useful in sensitive applications such as healthcare diagnostics, where accurate similarity assessment is crucial. Looking ahead, there is significant potential for further exploration of these exponential-function-based similarity measures in a wide range of complex decision-making contexts. Future research should aim to extend their application across diverse fuzzy environments and domains, including economics, engineering, and social sciences, thereby broadening their impact and refining their capability to model complex systems under uncertainty, as suggested in related works [39, 40].

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