EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2026, Vol. 19, Issue 2, Article Number 6546
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Upper Sets on Generalized Γ-BE-Algebras

Wongsakron Sukhon[1], Attaphol Pumila[1] Pongpun Julatha[1], Aiyared Iampan[2,*]

[1] *Department of Mathematics, Faculty of Science and Technology, Pibulsongkram Rajabhat University, Phlai Chumphon, Mueang, Phitsanulok 65000, Thailand*
[2] *Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand*

Abstract. We introduce Γ_Δ-BE-algebras as a generalization of Γ_α- and $\Gamma_{(\alpha,\beta)}$-BE-algebras, and specify the algebraic conditions that govern this new structure. We investigate their fundamental properties, define the notions of Δ-subalgebras and Δ-filters, and examine their interrelations. Furthermore, we explore the behavior of two types of upper sets, $E_\alpha(x)$ and $E_\alpha^\beta(x,y)$, within the framework of Δ-left distributive Γ_Δ-BE-algebras, providing several characterizations and structural results. These findings offer a broadened perspective on algebraic systems related to BE-algebras and their logical extensions.

2020 Mathematics Subject Classifications: 03G25, 06F35

Key Words and Phrases: BE-algebra, Γ-BE-algebra, $\Gamma_{(\alpha,\beta)}$-BE-algebra, Γ_Δ-BE-algebra, subalgebra, and filter.

1. Introduction

Kim and Kim developed the notion of BE-algebras as a generalization of BCK-algebras (see [1]). Ahn and So [2] formulated the concept of a generalized upper set on BE-algebras. Additionally, these sets are analyzed within the context of transitive and self-distributive BE-algebras. Pumila et al. [3] introduced the ideas of generalized upper sets and generalized terminal sections in pseudo BE-algebras, and their properties are analyzed. Furthermore, these sets are considered within the framework of distributive pseudo BE-algebras. Several researchers have presented and examined different types of BE-algebras and their associated structures (see [4–7]).

In 1964, Nobusawa [8] initiated the study of Γ-rings, which led to a wide range of investigations in Γ-algebraic structures. Later, Sen and Saha [9] explored the structure of Γ-semigroups and obtained various generalizations and analogues within semigroup theory. Subsequently, several authors proposed and examined multiple Γ-extensions of semigroups

*Corresponding Author.
DOI: https://doi.org/10.29020/nybg.ejpam.v19i2.6546

Email addresses: wongsakron.s@psru.ac.th (W. Sukhon), attaphol.p@psru.ac.th (A. Pumila) pongpun.j@psru.ac.th (P. Julatha), aiyared.ia@up.ac.th (A. Iampan)

and algebras, including ordered Γ-semigroups and their generating systems [10–14]. In 2023, Jun et al. [15] introduced the notion of Γ_α-BE-algebras as a generalization of BE-algebras and investigated their algebraic properties, including two types of upper sets $E_\alpha(x)$ and $E_\alpha^\beta(x, y)$. Following this, Bandaru et al. [16] proposed $\Gamma_{(\alpha,\beta)}$-BE-algebras as an extension of both BE-algebras and Γ-BE-algebras, and studied their structural aspects. They also introduced the concepts of (α, β)-subalgebras and (α, β)-filters, analyzing their interrelations. Moreover, recent developments in fuzzy and intuitionistic fuzzy extensions of algebraic structures have also contributed to the theoretical landscape. In particular, Palanikumar et al. [17] characterized level sets through multi-polar extensions of intuitionistic fuzzy different ideals in the context of regular ordered ternary semigroups. Their results offer a broader perspective on the interaction between order, multi-valued logic, and algebraic ideal theory.

This work introduces the notion of Γ_Δ-BE-algebras as a generalization of Γ_α-BE-algebras and $\Gamma_{(\alpha,\beta)}$-BE-algebras and examines their properties. Additionally, we will provide Δ-subalgebras and Δ-filters of Γ_Δ-BE-algebras and examine the relationship between these two notions. Ultimately, we will examine the two upper sets $E_\alpha(x)$ and $E_\alpha^\beta(x, y)$ within the context of (Δ-left distributive) Γ_Δ-BE-algebras.

2. Preliminaries

In this section, we provide some basic definitions and results that will be used throughout this paper.

Let $\mathcal{X}$ be a non-empty set and let Γ be a set of binary operations on $\mathcal{X}$, that is, every element $\delta \in \Gamma$ is given as follows:

$$\delta : \mathcal{X} \times \mathcal{X} \to \mathcal{X}, (x, y) \mapsto \delta(x, y) \tag{1}$$

In what follows, $\delta(x, y)$ is denoted by $x\delta y$, and let $(\mathcal{X}, 1)_\Gamma$ be a structure related to a special element 1 in $\mathcal{X}$ and Γ.

Definition 1. *[15] For a fixed $\delta \in \Gamma$, a structure $(\mathcal{X}, 1)_\Gamma$ is called a Γ_δ-BE-algebra if it satisfies:*

$$(\forall x \in \mathcal{X})(x\delta x = 1), \tag{2}$$

$$(\forall x \in \mathcal{X})(x\delta 1 = 1), \tag{3}$$

$$(\forall x \in \mathcal{X})(1\delta x = x), \tag{4}$$

$$(\forall x, y, z \in \mathcal{X})(\forall \beta \in \Gamma)(x\delta(y\beta z) = y\beta(x\delta z)). \tag{5}$$

If $(\mathcal{X}, 1)_\Gamma$ is a Γ_δ-BE-algebra for all $\delta \in \Gamma$, we say it is a Γ-BE-algebra.

For a fixed $\delta \in \Gamma$, we define a binary relation $\preccurlyeq_\delta$ on $\mathcal{X}$ as follows:

$$(\forall x, y \in \mathcal{X})\,(x \preccurlyeq_\delta y \Leftrightarrow x\delta y = 1)\,. \tag{6}$$

If $x \preccurlyeq_\delta y$ is valid for all $\delta \in \Gamma$, we present it as $x \preccurlyeq_\Gamma y$.

Definition 2. *[15] Let $(\mathcal{X},1)_\Gamma$ be Γ_δ-BE-algebra. For every $x,y \in \mathcal{X}$ and $\alpha,\beta \in \Gamma$, we consider the two upper sets*

$$E_\alpha(x) := \{z \in \mathcal{X} \mid x \preccurlyeq_\alpha z\} \text{ and } E_\alpha^\beta(x,y) := \{z \in \mathcal{X} \mid x \preccurlyeq_\alpha y\beta z\}.$$

Definition 3. *[15] For a fixed $\gamma \in \Gamma$, a Γ_α-BE-algebra $(\mathcal{X},1)_\Gamma$ is said to be γ-left distributive if it satisfies:*

$$(\forall x,y,z \in \mathcal{X})(\forall \delta \in \Gamma)(x\gamma(y\delta z) = (x\gamma y)\delta(x\gamma z)). \tag{7}$$

If a Γ-BE-algebra $(\mathcal{X},1)_\Gamma$ is β-left distributive for all $\beta \in \Gamma$, we say it is Γ-left distributive.

Definition 4. *[15] For a fixed $\beta \in \Gamma$, a non-empty subset $\mathcal{A}$ of a Γ_δ-BE-algebra $(\mathcal{X},1)_\Gamma$ is called*

- *a β-subalgebra of $(\mathcal{X},1)_\Gamma$ if it satisfies:*

$$(\forall x,y \in \mathcal{X})(x,y \in \mathcal{A} \Rightarrow x\beta y \in \mathcal{A}), \tag{8}$$

- *a β-filter of $(\mathcal{X},1)_\Gamma$ if it satisfies:*

$$1 \in \mathcal{A} \tag{9}$$
$$(\forall x,y \in \mathcal{X})(x \in \mathcal{A}, x\beta y \in \mathcal{A} \Rightarrow y \in \mathcal{A}). \tag{10}$$

Proposition 1. *[15] Every Γ_δ-BE-algebra $(\mathcal{X},1)_\Gamma$ satisfies the following assertions:*

$$(\forall x,y \in \mathcal{X})(\forall \gamma \in \Gamma)(x\gamma 1 = 1 \text{ and } x\delta(y\gamma x) = 1).$$

Theorem 1. *[15] Every δ-filter of a Γ_δ-BE-algebra $(\mathcal{X},1)_\Gamma$ is a γ-subalgebra for all $\gamma \in \Gamma$.*

In 2023, Bandaru et al. [16] introduced the concept of $\Gamma_{(\gamma,\delta)}$-BE algebra as a generalization of a BE- algebra and a Γ-BE-algebra and studied its properties. Also, they introduced the concepts of (γ,δ)-subalgebra and (γ,δ)-filter of a $\Gamma_{(\gamma,\delta)}$-BE-algebra and discussed its properties.

Definition 5. *[16] Given $\gamma,\delta \in \Gamma$, a structure $(\mathcal{X},1)_\Gamma$ is called a $\Gamma_{(\gamma,\delta)}$-BE-algebra if it satisfies the following equations:*

$$(\forall x \in \mathcal{X})(x\gamma x = 1 = x\delta x), \tag{11}$$
$$(\forall x \in \mathcal{X})(x\gamma 1 = 1 = x\delta 1), \tag{12}$$
$$(\forall x \in \mathcal{X})(1\gamma x = x = 1\delta x), \tag{13}$$
$$(\forall x \in \mathcal{X})(\forall \beta \in \Gamma)(x\gamma(y\beta z) = y\beta(x\gamma z), x\delta(y\beta z) = y\beta(x\delta z)). \tag{14}$$

Clearly, the concepts of $\Gamma_{(\gamma,\delta)}$-BE-algebra and $\Gamma_{(\delta,\gamma)}$-BE-algebra are identical.

Proposition 2. *[16] Every $\Gamma_{(\gamma,\delta)}$-BE-algebra $(\mathcal{X},1)_\Gamma$ satisfies the following assertions:*

(1) $(\forall x, y \in \mathcal{X})\,(x \preccurlyeq_\gamma y\delta x)$,

(2) $(\forall x, y \in \mathcal{X})\,(x \preccurlyeq_\delta y\delta x)$,

(3) $(\forall x, y \in \mathcal{X})\,(x \preccurlyeq_\gamma (x\gamma y)\delta y)$,

(4) $(\forall x, y, z \in \mathcal{X})\,(x \preccurlyeq_\gamma y\delta z \Leftrightarrow y \preccurlyeq_\delta x\gamma z)$,

(5) $(\forall x \in \mathcal{X})\,(1 \preccurlyeq_\delta x \Rightarrow x = 1)$,

(6) $(\forall x, y, z \in \mathcal{X})\,(x \preccurlyeq_\delta y \Rightarrow x \preccurlyeq_\delta z\gamma y)$,

(7) $(\forall x, y, z \in \mathcal{X})\,(x\gamma y = z \Rightarrow y \preccurlyeq_\delta z)$,

(8) $(\forall x, y \in \mathcal{X})(x \neq 1, x\gamma y = x \Rightarrow x\delta y \neq y)$,

(9) $(\forall x, y \in \mathcal{X})(x \neq 1, x\gamma y = y \Rightarrow x\delta y \neq x)$,

(10) $(\forall w, x, y, z \in \mathcal{X})(x\gamma y = w, x\delta y = z \Rightarrow x\gamma z = x\delta w)$,

(11) $(\forall x, y, z \in \mathcal{X})\left(x\gamma y = x, x\delta y = z \Rightarrow \left\{\begin{array}{l} x \preccurlyeq_\gamma z, x \preccurlyeq_\gamma y\gamma z, \\ x \preccurlyeq_\delta y\gamma z \end{array}\right.\right)$,

(12) $(\forall x, y, z \in \mathcal{X})\left(x\gamma y = y, x\delta y = z \Rightarrow \left\{\begin{array}{l} x\gamma z = z, \\ x \preccurlyeq_\gamma y\gamma z, x \preccurlyeq_\gamma y\delta z \end{array}\right.\right)$.

Definition 6. *[16] A non-empty subset $\mathcal{A}$ of a $\Gamma_{(\gamma,\delta)}$-BE-algebra $(\mathcal{X}, 1)_\Gamma$ is called*

- *an (γ, δ)-subalgebra of $(\mathcal{X}, 1)_\Gamma$ if*

$$(\forall x, y \in \mathcal{X})(x, y \in \mathcal{A} \Rightarrow x\gamma y, x\delta y \in \mathcal{A}), \tag{15}$$

- *an (γ, δ)-filter of $(\mathcal{X}, 1)_\Gamma$ if*

$$1 \in \mathcal{A}, \tag{16}$$
$$(\forall x, y \in \mathcal{X})(x \in \mathcal{A}, x\gamma y \in \mathcal{A} \Rightarrow y \in \mathcal{A}), \tag{17}$$
$$(\forall x, y \in \mathcal{X})(x \in \mathcal{A}, x\delta y \in \mathcal{A} \Rightarrow y \in \mathcal{A}). \tag{18}$$

3. Generalized Γ-BE-algebras

In this section, we introduce the notion of Γ_Δ-BE-algebras as a natural generalization of Γ_α-BE-algebras. We also revisit the properties of $\Gamma_{(\alpha,\beta)}$-BE-algebras to establish a broader context. Furthermore, we define the concepts of Δ-subalgebras and Δ-filters within Γ_Δ-BE-algebras, and examine the relationship between these two algebraic structures. Finally, we explore the class of Δ-left distributive Γ_Δ-BE-algebras and their role in the internal distributive logic of such systems.

Definition 7. *For a fixed non-empty subset Δ of Γ, a structure $(\mathcal{X},1)_\Gamma$ is called a Γ_Δ-BE-algebra if*

$$(\forall x \in \mathcal{X})(\forall \delta \in \Delta)(x\delta x = 1), \tag{19}$$

$$(\forall x \in \mathcal{X})(\forall \delta \in \Delta)(x\delta 1 = 1), \tag{20}$$

$$(\forall x \in \mathcal{X})(\forall \delta \in \Delta)(1\delta x = x), \tag{21}$$

$$(\forall x, y, z \in \mathcal{X})(\forall \gamma \in \Gamma)(\forall \delta \in \Delta)(x\gamma(y\delta z) = y\delta(x\gamma z)). \tag{22}$$

Table 1: Comparison of various Γ-based BE-algebra structures

Structure	**Operator Set**	**Parameter(s)**	**Upper Sets**
BE-algebra	Single binary op	None	$E(x)$
Γ-BE-algebra	Operator family indexed by Γ	Γ: nonempty set	$E_\gamma(x)$
Γ_α-BE-algebra	$\gamma \in \Gamma$	Fixed $\alpha \in \Gamma$	$E_\alpha(x)$, $E_\alpha^\beta(x,y)$
$\Gamma_{(\alpha,\beta)}$-BE-algebra	$\gamma \in \Gamma$	Pair (α,β)	$E_\alpha^\beta(x,y)$
Γ_Δ-BE-algebra	$\gamma \in \Delta \subseteq \Gamma$	Subset Δ	$E_\alpha(x)$, $E_\alpha^\beta(x,y)$

Example 1. *Let $\mathcal{X} = \{0,1,2,3\}$ be a set and $\Gamma = \{\alpha,\beta,\gamma,\delta\}$ be a set of binary operations on $\mathcal{X}$ given in the following tables:*

$$\begin{array}{c|cccc} \alpha\text{:} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 & 3 \end{array} \quad \begin{array}{c|cccc} \beta\text{ :} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 0 & 1 \end{array} \quad \begin{array}{c|cccc} \gamma\text{:} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 1 & 3 & 0 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 0 & 1 \end{array} \quad \begin{array}{c|cccc} \delta\text{:} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 0 & 1 \end{array}.$$

Then, we find that:

(1) $(\mathcal{X},1)_\Gamma$ is a Γ_Δ-BE-algebra for all non-empty subset Δ of $\{\beta,\gamma,\delta\}$,

(2) $(\mathcal{X},1)_\Gamma$ is not a Γ_Θ-BE-algebra for any subset Θ of Γ containting α.

According to Definitions 1, 5 and 7, we provide the following two remarks.

Remark 1. *Let Δ be a non-empty subset of Γ. Then the following conditions are equivalent:*

(1) $(\mathcal{X},1)_\Gamma$ is a Γ_Δ-BE-algebra,

(2) $(\mathcal{X},1)_\Gamma$ is a Γ_Θ-BE-algebra for every non-empty subset Θ of Δ,

(3) $(\mathcal{X},1)_\Gamma$ is a Γ_δ-BE-algebra for all element δ of Δ.

Remark 2. *Let $\Delta = \{\gamma,\delta\}$ be a subset of Γ. Then the following conditions are equivalent:*

(1) $(\mathcal{X},1)_\Gamma$ is a Γ_Δ-BE-algebra,

(2) $(\mathcal{X}, 1)_\Gamma$ *is a* $\Gamma_{(\delta,\gamma)}$*-BE-algebra,*

(3) $(\mathcal{X}, 1)_\Gamma$ *is a* $\Gamma_{(\gamma,\delta)}$*-BE-algebra,*

(4) $(\mathcal{X}, 1)_\Gamma$ *is both a* Γ_γ*-BE-algebra and* Γ_δ*-BE-algebra.*

Proposition 3. *Every* Γ_Δ*-BE-algebra* $(\mathcal{X}, 1)_\Gamma$ *satisfies the following assertions:*

(1) $(\forall x, y \in \mathcal{X})(\forall \gamma, \delta \in \Delta)\,(x \preccurlyeq_\gamma y\delta x)$,

(2) $(\forall x, y \in \mathcal{X})(\forall \delta \in \Delta)\,(x \preccurlyeq_\delta y\delta x)$,

(3) $(\forall x, y \in \mathcal{X})(\forall \gamma, \delta \in \Delta)\,(x \preccurlyeq_\gamma (x\gamma y)\delta y)$,

(4) $(\forall x, y, z \in \mathcal{X})(\forall \gamma, \delta \in \Delta)\,(x \preccurlyeq_\gamma y\delta z \Leftrightarrow y \preccurlyeq_\delta x\gamma z)$,

(5) $(\forall x \in \mathcal{X})(\forall \delta \in \Delta)\,(1 \preccurlyeq_\delta x \Rightarrow x = 1)$,

(6) $(\forall x, y, z \in \mathcal{X})(\forall \gamma, \delta \in \Delta)\,(x \preccurlyeq_\delta y \Rightarrow x \preccurlyeq_\delta z\gamma y)$,

(7) $(\forall x, y, z \in \mathcal{X})(\forall \gamma, \delta \in \Delta)\,(x\gamma y = z \Rightarrow y \preccurlyeq_\delta z)$,

(8) $(\forall x, y \in \mathcal{X})(\forall \gamma, \delta \in \Delta)(x \neq 1, x\gamma y = x \Rightarrow x\delta y \neq y)$,

(9) $(\forall x, y \in \mathcal{X})(\forall \gamma, \delta \in \Delta)(x \neq 1, x\gamma y = y \Rightarrow x\delta y \neq x)$,

(10) $(\forall w, x, y, z \in \mathcal{X})(\forall \gamma, \delta \in \Delta)(x\gamma y = w, x\delta y = z \Rightarrow x\gamma z = x\delta w)$,

(11) $(\forall x, y, z \in \mathcal{X})(\forall \gamma, \delta \in \Delta)\left(x\gamma y = x, x\delta y = z \Rightarrow \begin{cases} x \preccurlyeq_\gamma z, x \preccurlyeq_\gamma y\gamma z, \\ x \preccurlyeq_\delta y\gamma z \end{cases}\right)$,

(12) $(\forall x, y, z \in \mathcal{X})(\forall \gamma, \delta \in \Delta)\left(x\gamma y = y, x\delta y = z \Rightarrow \begin{cases} x\gamma z = z, \\ x \preccurlyeq_\gamma y\gamma z, x \preccurlyeq_\gamma y\delta z \end{cases}\right)$.

Proof. It follows from Proposition 2, Remark 1 and Remark 2.

Proposition 4. *Let* Δ *and* Θ *be non-empty subsets of* Γ*. If* $(\mathcal{X}, 1)_\Gamma$ *is both a* Γ_Δ*-BE-algebra and a* Γ_Θ*-BE-algebra, then* $(\mathcal{X}, 1)_\Gamma$ *is a* $\Gamma_{\Delta\cup\Theta}$*-BE-algebra.*

Proof. Straightforward.

Definition 8. *Let* Θ *be a non-empty subset of* Γ *and* $\mathcal{A}$ *be a non-empty subset of a* Γ_Δ*-BE-algebra* $(\mathcal{X}, 1)_\Gamma$*. Then* $\mathcal{A}$ *is called a* Θ*-subalgebra of* $(\mathcal{X}, 1)_\Gamma$ *if it satisfies:*

$$(\forall x, y \in \mathcal{X})(\forall \omega \in \Gamma)(x, y \in \mathcal{A}, \omega \in \Theta \Rightarrow x\omega y \in \mathcal{A}). \tag{23}$$

Example 2. *Consider the* Γ_Δ*-BE-algebra* $(\mathcal{X}, 1)_\Gamma$ *in Example 1 when* $\Delta = \{\beta, \gamma, \delta\}$*, we observe that the following holds:*

(1) the set $\{0, 1, 2\}$ is a Θ-subalgebra of $(\mathcal{X}, 1)_\Gamma$ for all non-empty subset Θ of $\{\alpha, \beta, \delta\}$; however, it does not satisfies the condition to be a γ-subalgebra of $(\mathcal{X}, 1)_\Gamma$,

(2) the set $\{2, 3\}$ is a α-subalgebra of $(\mathcal{X}, 1)_\Gamma$; however, it does not satisfies the condition to be a Θ-subalgebra of $(\mathcal{X}, 1)_\Gamma$ for any non-empty subset Θ of Δ,

(3) the sets $\{1\}$, $\{0, 1\}$, $\{0, 1, 3\}$ and $\mathcal{X}$ are a Θ-subalgebra of $(\mathcal{X}, 1)_\Gamma$ for all non-empty subset Θ of Γ.

From Definition 4, 6 and 8, we have the following remarks.

Remark 3. *Let Θ be a non-empty subset of Γ and $\mathcal{A}$ be a non-empty subset of a Γ_Δ-BE-algebra $(\mathcal{X}, 1)_\Gamma$. Then the following conditions are equivalent:*

(1) $\mathcal{A}$ is a Θ-subalgebra of $(\mathcal{X}, 1)_\Gamma$,

(2) $\mathcal{A}$ is a Ω-subalgebra of $(\mathcal{X}, 1)_\Gamma$ for every non-empty subset Ω of Θ,

(3) $\mathcal{A}$ is a γ-subalgebra of $(\mathcal{X}, 1)_\Gamma$ for every element γ of Θ.

Remark 4. *Let $\Delta = \{\gamma, \delta\}$ be a subset of Γ and $\mathcal{A}$ be a non-empty subset of a Γ_Δ-BE-algebra $(\mathcal{X}, 1)_\Gamma$. Then the following conditions are equivalent:*

(1) $\mathcal{A}$ is a Δ-subalgebra,

(2) $\mathcal{A}$ is an (δ, γ)-subalgebra,

(3) $\mathcal{A}$ is an (γ, δ)-subalgebra,

(4) $\mathcal{A}$ is both a γ-subalgebra and δ-subalgebra.

Definition 9. *Let Θ be a non-empty subset of Γ and $\mathcal{F}$ be a non-empty subset of a Γ_Δ-BE-algebra $(\mathcal{X}, 1)_\Gamma$. Then $\mathcal{F}$ is called a Θ-filter of $(\mathcal{X}, 1)_\Gamma$ if it satisfies:*

$$1 \in \mathcal{F}, \tag{24}$$

$$(\forall x, y \in \mathcal{X})(\forall \delta \in \Theta)(x, x\delta y \in \mathcal{F} \Rightarrow y \in \mathcal{F}). \tag{25}$$

Example 3. *Let $\mathcal{X} = \{1, a, b, c, d\}$ be a set and $\Gamma = \{\alpha, \beta, \gamma, \delta\}$ be a set of binary operations on $\mathcal{X}$ given in the following tables:*

α:	1	a	b	c	d
1	1	1	1	1	1
a	1	1	1	1	1
b	1	1	1	1	1
c	1	1	1	1	1
d	1	1	1	1	1

β:	1	a	b	c	d
1	1	a	b	c	d
a	1	1	c	c	d
b	1	1	1	1	1
c	1	a	a	1	d
d	1	a	a	1	1

γ:	1	a	b	c	d
1	1	a	b	c	d
a	1	a	b	c	d
b	1	a	b	c	d
c	1	a	b	c	d
d	1	a	b	c	d

δ:	1	a	b	c	d
1	1	a	b	c	d
a	1	1	c	c	d
b	1	1	1	1	1
c	1	a	a	1	1
d	1	1	c	c	1

Then, $(\mathcal{X}, 1)_\Gamma$ is a $\Gamma_{\{\beta,\delta\}}$-BE-algebra and also, we find that:

(1) the sets $\{1\}$ and $\{1, a\}$ are a Θ-filter of $(\mathcal{X}, 1)_\Gamma$ for all non-empty subset Θ of $\{\beta, \gamma, \delta\}$; however, it does not satisfies the condition to be a Δ-filter of $(\mathcal{X}, 1)_\Gamma$ for all non-empty subset Δ of Γ with $\Delta \cap \{\alpha\} \neq \emptyset$,

(2) the set $\{1, b\}$ is a γ-filter of $(\mathcal{X}, 1)_\Gamma$; however, it does not satisfies the condition to be a Δ-filter of $(\mathcal{X}, 1)_\Gamma$ for all non-empty subset Δ of Γ with $\Delta \cap \{\alpha, \beta, \delta\} \neq \emptyset$,

(3) the set $\{1, a, d\}$ is a Θ-filter of $(\mathcal{X}, 1)_\Gamma$ for all non-empty subset Θ of $\{\gamma, \delta\}$; however, it does not satisfies the condition to be a Δ-filter of $(\mathcal{X}, 1)_\Gamma$ for all non-empty subset Δ of Γ with $\Delta \cap \{\alpha, \beta\} \neq \emptyset$.

From Definitions 4, 6 and 9, we have the following remarks.

Remark 5. *Let Θ be a non-empty subset of Γ and $\mathcal{F}$ be a non-empty subset of a Γ_Δ-BE-algebra $(\mathcal{X}, 1)_\Gamma$. Then the following conditions are equivalent:*

(1) $\mathcal{F}$ is a Θ-filter of $(\mathcal{X}, 1)_\Gamma$,

(2) $\mathcal{F}$ is a Ω-filter of $(\mathcal{X}, 1)_\Gamma$ for every non-empty subset Ω of Θ,

(3) $\mathcal{F}$ is a γ-filter of $(\mathcal{X}, 1)_\Gamma$ for every element γ of Θ.

Remark 6. *Let $\Delta = \{\gamma, \delta\}$ be a subset of Γ and $\mathcal{F}$ be a non-empty subset of a Γ_Δ-BE-algebra $(\mathcal{X}, 1)_\Gamma$. Then the following conditions are equivalent:*

(1) $\mathcal{F}$ is a Δ-filter,

(2) $\mathcal{F}$ is an (δ, γ)-filter,

(3) $\mathcal{F}$ is an (γ, δ)-filter,

(4) $\mathcal{F}$ is both a γ-filter and δ-filter.

Proposition 5. *Let $\mathcal{F}$ be a non-empty subset of a Γ_Δ-BE-algebra $(\mathcal{X}, 1)_\Gamma$ and Θ be a non-empty subset of Γ such that $\Delta \cap \Theta \neq \emptyset$. If $\mathcal{F}$ is a Θ-filter of $(\mathcal{X}, 1)_\Gamma$, then $\mathcal{F}$ is a Ω-subalgebra of $(\mathcal{X}, 1)_\Gamma$ for all non-empty subset Ω of Γ.*

Proof. Assume that $\mathcal{F}$ is a Θ-filter of $(\mathcal{X}, 1)_\Gamma$ and Ω is a non-empty subset of Γ. Let $\alpha \in \Omega$, $\delta \in \Delta \cap \Theta$ and $x, y \in \mathcal{F}$. Then

$$y\delta(x\alpha y) = x\alpha(y\delta y) = x\alpha 1 = x\alpha((x\alpha 1)\delta 1) = (x\alpha 1)\delta(x\alpha 1) = 1 \in \mathcal{F}.$$

Thus, $x\alpha y \in \mathcal{F}$. Therefore, $\mathcal{F}$ is a Ω-subalgebra of $(\mathcal{X}, 1)_\Gamma$.

Corollary 1. *Every Δ-filter of a Γ_Δ-BE-algebra $(\mathcal{X}, 1)_\Gamma$ is a Δ-subalgebra.*

Proof. It follows from Proposition 5.

The following example demonstrates that the converse of Corollary 1 is false.

Example 4. *Let* $(\mathcal{X}, 1)_\Gamma$ *be the* Γ_Δ*-BE-algebra in Example 1 when* $\Delta = \{\beta, \gamma, \delta\}$*. The set* $\{0, 1, 3\}$ *is a* Δ*-subalgebra of* $(\mathcal{X}, 1)_\Gamma$*; hence, it does not satisfies the condition to be a* Δ*-filter of* $(\mathcal{X}, 1)_\Gamma$ *because* $3, 3\delta 2 \in \mathcal{F}$ *but* $2 \notin \mathcal{F}$*.*

Proposition 6. *Every* Δ*-filter* $\mathcal{F}$ *of a* Γ_Δ*-BE-algebra* $(\mathcal{X}, 1)_\Gamma$ *satisfies the following conditions:*

(1) $(\forall x, y \in \mathcal{X})(\forall \delta \in \Delta)(x \in \mathcal{F}, x \preccurlyeq_\delta y \Rightarrow y \in \mathcal{F})$,

(2) $(\forall x, y \in \mathcal{X})(\forall \gamma, \delta \in \Delta)(x, x\gamma(x\delta y) \in \mathcal{F} \Rightarrow y \in \mathcal{F})$,

(3) $(\forall x, y, z \in \mathcal{X})(\forall \gamma, \delta \in \Delta)(x, y \in \mathcal{F}, x \preccurlyeq_\gamma y\delta z \Rightarrow z \in \mathcal{F})$,

(4) $(\forall x, y \in \mathcal{X})(\forall \gamma, \delta \in \Delta)(y \in \mathcal{F} \Rightarrow (y\gamma x)\delta x \in \mathcal{F})$.

Proof. The conditions $(1), (2)$ and (3) are straightforward. Next, we will show the condition (4). Let $x \in \mathcal{X}$, $y \in \mathcal{F}$ and $\gamma, \delta \in \Delta$. Then $1 = (y\gamma x)\delta(y\gamma x) = y\gamma((y\gamma x)\delta x)$. Since $\mathcal{F}$ is a Δ-filter, we have $(y\gamma x)\delta x \in \mathcal{F}$.

We provide a condition for a Δ-subalgebra of a Γ_Δ-BE-algebra to be a Δ-filter.

Theorem 2. *If a* Δ*-subalgebra* $\mathcal{F}$ *of a* Γ_Δ*-BE-algebra* $(\mathcal{X}, 1)_\Gamma$ *satisfies the following:*

$$(\forall x, y \in \mathcal{X})(\forall \delta \in \Delta)(x \in \mathcal{F}, y \notin \mathcal{F} \Rightarrow x\delta y \notin \mathcal{F}), \tag{26}$$

then $\mathcal{F}$ *is a* Δ*-filter of* $(\mathcal{X}, 1)_\Gamma$*.*

Proof. Straightforward.

Definition 10. *For a non-empty subset* Θ *of* Γ*, a* Γ_Δ*-BE-algebra* $(\mathcal{X}, 1)_\Gamma$ *is said to be* Θ*-left distributive if it satisfies:*

$$(\forall x, y, z \in \mathcal{X})(\forall \alpha \in \Theta)(\forall \delta \in \Gamma)(x\alpha(y\delta z) = (x\alpha y)\delta(x\alpha z)).$$

According to Definitions 3 and 10, we provide the following remark.

Remark 7. *Let* $(\mathcal{X}, 1)_\Gamma$ *be a* Γ_Δ*-BE-algebra. Then* $(\mathcal{X}, 1)_\Gamma$ *is* Δ*-left distributive if and only if* $(\mathcal{X}, 1)_\Gamma$ *is* δ*-left distributive for all element* δ *of* Δ*.*

Theorem 3. *Let* x, y *and* z *be elements of a* Δ*-left distributive* Γ_Δ*-BE-algebra* $(\mathcal{X}, 1)_\Gamma$ *and* $\alpha, \beta \in \Delta$*. If* $z \preccurlyeq_\alpha (x\beta y)$ *and* $z \preccurlyeq_\alpha x$*, then* $z \preccurlyeq_\alpha y$*.*

Proof. Suppose that $z \preccurlyeq_\alpha (x\beta y)$ and $z \preccurlyeq_\alpha x$. Then, we get $z\alpha(x\beta y) = 1 = z\alpha x$. That is,

$$1 = z\alpha(x\beta y) = (z\alpha x)\beta(z\alpha y) = 1\beta(z\alpha y) = z\alpha y.$$

Hence, we have $z \preccurlyeq_\alpha y$.

4. Upper sets

In this section, we investigate the structural behavior of two upper sets, $E_\alpha(x)$ and $E_\alpha^\beta(x,y)$, within the framework of Γ_Δ-BE-algebras. These sets play a key role in understanding order-theoretic properties and the internal logic of the algebraic system, particularly under the condition of Δ-left distributivity.

Proposition 7. *Let $(\mathcal{X},1)_\Gamma$ be a Γ_Δ-BE-algebra, $x,y\in\mathcal{X}$ and $\alpha,\beta\in\Delta$. Then the following holds:*

(1) $1,x\in E_\alpha(x)$ *and* $1,x,y\in E_\alpha^\beta(x,y)$*,*

(2) $E_\alpha(x)\subseteq E_\alpha^\beta(x,y)$*,*

(3) $E_\alpha(x)=E_\alpha^\beta(x,1)=\bigcap_{z\in\mathcal{X}}E_\alpha^\beta(x,z)$*.*

Proof. Straightforward.

Proposition 8. *Let $(\mathcal{X},1)_\Gamma$ be a Γ_Δ-BE-algebra and $z\in\mathcal{X}$. Then the following are equivalent:*

(1) $z\preccurlyeq_\alpha x$ *for all* $x\in\mathcal{X}$ *and* $\alpha\in\Delta$*,*

(2) $\mathcal{X}=E_\alpha(z)$ *for all* $\alpha\in\Delta$*,*

(3) $\mathcal{X}=E_\alpha^\beta(z,x)$ *for all* $x\in\mathcal{X}$ *and* $\alpha,\beta\in\Delta$*.*

Proof. (1) ⇒ (2). Suppose that $z\preccurlyeq_\alpha x$ for all $x\in\mathcal{X}$ and $\alpha\in\Delta$. Let $y\in\mathcal{X}$. Thus, we have $z\preccurlyeq_\alpha y$ for all $\alpha\in\Delta$. So, $y\in E_\alpha(z)$ for all $\alpha\in\Delta$. Then $\mathcal{X}\subseteq E_\alpha(z)$ for all $\alpha\in\Delta$. Hence, $\mathcal{X}=E_\alpha(z)$ for all $\alpha\in\Delta$.

(2) ⇒ (3). Suppose that $\mathcal{X}=E_\alpha(z)$ for all $\alpha\in\Delta$. Let $y\in\mathcal{X}$. Then, for all $x\in\mathcal{X}$ and $\alpha,\beta\in\Delta$, we have

$$y\in\mathcal{X}=E_\alpha(z)\subseteq E_\alpha^\beta(z,x).$$

Thus, we get $\mathcal{X}\subseteq E_\alpha^\beta(z,x)$ for all $x\in\mathcal{X}$ and $\alpha,\beta\in\Delta$. Hence, for all $x\in\mathcal{X}$ and $\alpha,\beta\in\Delta$, we get $\mathcal{X}=E_\alpha^\beta(z,x)$.

(3) ⇒ (1). Assume that $\mathcal{X}=E_\alpha^\beta(z,x)$ for all $x\in\mathcal{X}$ and $\alpha,\beta\in\Delta$. Let $y\in\mathcal{X}$ and $\alpha\in\Delta$. By Proposition 7, we have $y\in\mathcal{X}=E_\alpha^\alpha(z,1)=E_\alpha(z)$. Thus, we get $z\preccurlyeq_\alpha y$.

Proposition 9. *Let x and y be elements of a Δ-left distributive Γ_Δ-BE-algebra $(\mathcal{X},1)_\Gamma$ and $\alpha,\beta\in\Delta$. Then $y\in E_\alpha(x)$ if and only if $E_\alpha(x)=E_\alpha^\beta(x,y)$.*

Proof. Suppose that $y\in E_\alpha(x)$. By Proposition 7, we have $E_\alpha(x)\subseteq E_\alpha^\beta(x,y)$. To show that $E_\alpha^\beta(x,y)\subseteq E_\alpha(x)$, let $z\in E_\alpha^\beta(x,y)$. Then $x\alpha(y\beta z)=1$ and so

$$1=x\alpha(y\beta z)=(x\alpha y)\beta(x\alpha z)=1\beta(x\alpha z)=x\alpha z.$$

Thus, $z\in E_\alpha(x)$. Hence, we get $E_\alpha^\beta(x,y)\subseteq E_\alpha(x)$. We have $E_\alpha(x)=E_\alpha^\beta(x,y)$.

Conversely, suppose that $E_\alpha(x)=E_\alpha^\beta(x,y)$. By Proposition 7, we have $y\in E_\alpha^\beta(x,y)$ and so $y\in E_\alpha^\beta(x,y)=E_\alpha(x)$. Then $y\in E_\alpha(x)$.

Theorem 4. *Let $\mathcal{F}$ be a non-empty subset of a Γ_Δ-BE-algebra $(\mathcal{X}, 1)_\Gamma$. Then $\mathcal{F}$ is a Δ-filter if and only if $E_\alpha^\beta(x, y) \subseteq \mathcal{F}$ for all $x, y \in \mathcal{F}$ and $\alpha, \beta \in \Delta$.*

Proof. Assume that x and y are elements of a Δ-filter $\mathcal{F}$ and $\alpha, \beta \in \Delta$. Let $u \in E_\alpha^\beta(x, y)$. Then $x\alpha(y\beta u) = 1 \in \mathcal{F}$. Since $x, x\alpha(y\beta u) \in \mathcal{F}$ and $\mathcal{F}$ is a Δ-filter, we see that $y\beta u \in \mathcal{F}$. We get $y, y\beta u \in \mathcal{F}$ and by using again $\mathcal{F}$ is a Δ-filter, we have $u \in \mathcal{F}$. Hence, $E_\alpha^\beta(x, y) \subseteq \mathcal{F}$.

Conversely, assume that $E_\alpha^\beta(x, y) \subseteq \mathcal{F}$ for all $x, y \in \mathcal{F}$ and $\alpha, \beta \in \Delta$. Let $u, v \in \mathcal{X}$ and $\gamma \in \Delta$ be such that $u, u\gamma v \in \mathcal{F}$. Then, we get

$$u\gamma((u\gamma v)\gamma v) = (u\gamma v)\gamma(u\gamma v) = 1.$$

Thus, $u \preccurlyeq_\gamma ((u\gamma v)\gamma v)$, that is, $v \in E_\gamma^\gamma(u, (u\gamma v))$. By the assumption, we see that $v \in E_\gamma^\gamma(u, (u\gamma v)) \subseteq \mathcal{F}$. Hence, $v \in \mathcal{F}$. Therefore, $\mathcal{F}$ is a Δ-filter.

Theorem 5. *If $\mathcal{F}$ is a Δ-filter of a Γ_Δ-BE-algebra $(\mathcal{X}, 1)_\Gamma$ and $\alpha, \beta \in \Delta$, then*

$$\mathcal{F} = \bigcup_{x,y \in \mathcal{F}} E_\alpha^\beta(x, y).$$

Proof. Suppose that $\mathcal{F}$ is a Δ-filter and $\alpha, \beta \in \Delta$. If $u \in \mathcal{F}$, then $u\alpha(1\beta u) = u\alpha u = 1$, which implies that

$$u \in E_\alpha^\beta(u, 1) \subseteq \bigcup_{x,y \in \mathcal{F}} E_\alpha^\beta(x, y).$$

Thus,

$$\mathcal{F} \subseteq \bigcup_{x,y \in \mathcal{F}} E_\alpha^\beta(x, y).$$

On the other hand, let us

$$v \in \bigcup_{x,y \in \mathcal{F}} E_\alpha^\beta(x, y).$$

Then, there exist $a, b \in \mathcal{F}$ such that $v \in E_\alpha^\beta(a, b)$. By Theorem 4, we see that $v \in E_\alpha^\beta(a, b) \subseteq \mathcal{F}$. Thus,

$$\bigcup_{x,y \in \mathcal{F}} E_\alpha^\beta(x, y) \subseteq \mathcal{F}.$$

Consequently, we deduce that

$$\mathcal{F} = \bigcup_{x,y \in \mathcal{F}} E_\alpha^\beta(x, y).$$

Theorem 6. *Let $(\mathcal{X}, 1)_\Gamma$ is a Γ_Δ-BE-algebra. Then $E_\alpha(1) = \{1\}$ is a Δ-filter for all $\alpha \in \Delta$.*

Proof. Let $\alpha \in \Delta$. If $x \neq 1$ and $x \in E_\alpha(1)$, then $1\alpha x = 1$. But $x = 1\alpha x = 1$, thus $x = 1$. It's a contradiction. That is, $E_\alpha(1) = \{1\}$. Since $1\alpha 1 = 1 \in E_\alpha(1)$ but $1\alpha x = x \notin E_\alpha(1)$ for all $x \in \mathcal{X} - \{1\}$, we have $E_\alpha(1)$ is a Δ-filter.

Theorem 7. *Let $(\mathcal{X}, 1)_\Gamma$ be a Γ_Δ-BE-algebra and $\emptyset \neq \mathcal{F} \subseteq \mathcal{X}$. If $\mathcal{F}$ is a Δ-filter and $\alpha \in \Delta$, then*

$$\mathcal{F} = \bigcup_{x \in \mathcal{F}} E_\alpha(x).$$

Proof. Assume that $\mathcal{F}$ is a Δ-filter and $\alpha \in \Delta$. Let $u \in \mathcal{F}$. Then,

$$u \in E_\alpha(u) \subseteq \bigcup_{x \in \mathcal{F}} E_\alpha(x).$$

Thus,

$$\mathcal{F} \subseteq \bigcup_{x \in \mathcal{F}} E_\alpha(x).$$

On the other hand, let

$$v \in \bigcup_{x \in \mathcal{F}} E_\alpha(x).$$

Then, there exists $y \in \mathcal{F}$ such that $v \in E_\alpha(y)$. Thus, we get $y\alpha v = 1 \in \mathcal{F}$. Since $y, y\alpha v \in \mathcal{F}$ and $\mathcal{F}$ is a Δ-filter, we have $v \in \mathcal{F}$. Hence,

$$\bigcup_{x \in \mathcal{F}} E_\alpha(x) \subseteq \mathcal{F}.$$

Consequently, we deduce that

$$\mathcal{F} = \bigcup_{x \in \mathcal{F}} E_\alpha(x).$$

Corollary 2. *Let $(\mathcal{X}, 1)_\Gamma$ be a Γ_Δ-BE-algebra and $\emptyset \neq \mathcal{F} \subseteq \mathcal{X}$. If $\mathcal{F}$ is a Δ-filter, then*

$$E_\alpha(x) \subseteq \mathcal{F}$$

for all $x \in \mathcal{F}$ and $\alpha \in \Delta$.

Proof. It follows from 7.

Theorem 8. *If a Γ_Δ-BE-algebra $(\mathcal{X}, 1)_\Gamma$ is Δ-left distributive, then $E_\alpha^\beta(x, y)$ is a Γ-subalgebra for all $\alpha, \beta \in \Delta$ and $x, y \in \mathcal{X}$.*

Proof. Assume that $(\mathcal{X}, 1)_\Gamma$ is a Δ-left distributive Γ_Δ-BE-algebra. Let $\gamma \in \Gamma$, $\alpha, \beta \in \Delta$, $x, y \in \mathcal{X}$ and $u, v \in E_\alpha^\beta(x, y)$. Then, we get $x \preccurlyeq_\alpha y\beta u$ and $x \preccurlyeq_\alpha y\beta v$. Thus, $x\alpha(y\beta u) = 1 = x\alpha(y\beta v)$. Since Proposition 1, we get $1\gamma 1 = 1$ and so

$$x\alpha(y\beta(u\gamma v)) = x\alpha[(y\beta u)\gamma(y\beta v)] = (x\alpha(y\beta u))\gamma(x\alpha(y\beta v)) = 1\gamma 1 = 1.$$

Hence, $x \preccurlyeq_\alpha y\beta(u\gamma v)$ which implies that $u\gamma v \in E_\alpha^\beta(x, y)$. Therefore, $E_\alpha^\beta(x, y)$ is a Γ-subalgebra.

Theorem 9. *If a Γ_Δ-BE-algebra $(\mathcal{X}, 1)_\Gamma$ is a Δ-left distributive, then $E_\alpha^\beta(x, y)$ is a Δ-filter for all $\alpha, \beta \in \Delta$ and $x, y \in \mathcal{X}$.*

Proof. Assume that $(\mathcal{X}, 1)_\Gamma$ is a Δ-left distributive Γ_Δ-BE-algebra. Let $\alpha, \beta \in \Delta$ and $x, y \in \mathcal{X}$. Then $1 \in E_\alpha^\beta(x, y)$. Next, let $\gamma \in \Delta$ and $u, v \in \mathcal{X}$ be such that $u, u\gamma v \in E_\alpha^\beta(x, y)$. Then, we get $x \preccurlyeq_\alpha (y\beta u)$ and $x \preccurlyeq_\alpha (y\beta(u\gamma v))$. Thus, $x\alpha(y\beta u) = 1 = x\alpha(y\beta(u\gamma v))$ and so

$$1 = x\alpha(y\beta(u\gamma v)) = x\alpha((y\beta u)\gamma(y\beta v)) = [x\alpha(y\beta u)]\gamma[x\alpha(y\beta v)] = 1\gamma(x\alpha(y\beta v)) = x\alpha(y\beta v).$$

Hence, $x \preccurlyeq_\alpha (y\beta v)$, which implies that $v \in E_\alpha^\beta(x, y)$. Therefore, $E_\alpha^\beta(x, y)$ is a Δ-filter.

Corollary 3. *If a Γ-BE-algebra $(\mathcal{X}, 1)_\Gamma$ is Γ-left distributive, then $E_\alpha^\beta(x, y)$ is a Γ-filter for all $\alpha, \beta \in \Gamma$ and $x, y \in \mathcal{X}$.*

Proof. It follows from Theorem 9.

Corollary 4. *Let $(\mathcal{X}, 1)_\Gamma$ is a Δ-left distributive Γ_Δ-BE-algebra. Then $E_\alpha(u)$ is a Δ-filter for all $u \in \mathcal{X}$ and $\alpha \in \Delta$.*

Proof. It follows from Proposition 7 and Theorem 9.

Corollary 5. *Let $(\mathcal{X}, 1)_\Gamma$ is a Γ-left distributive Γ-BE-algebra. Then $E_\alpha(u)$ is a Γ-filter for all $u \in \mathcal{X}$ and $\alpha \in \Gamma$.*

Proof. It follows from Corollary 4.

5. Conclusion

In this study, we proposed the notion of Γ_Δ-BE-algebras as a new generalization that unifies and extends the frameworks of both Γ_α- and $\Gamma_{(\alpha,\beta)}$-BE-algebras. We established their algebraic foundations, introduced the associated concepts of Δ-subalgebras and Δ-filters, and demonstrated their structural interconnections. The behavior of the upper sets $E_\alpha(x)$ and $E_\alpha^\beta(x, y)$ was investigated within Δ-left distributive Γ_Δ-BE-algebras, leading to several new characterizations.

This research offers significant insights into the order-theoretic and logical structure of generalized BE-algebras. Our main contributions include: (i) the formalization of Γ_Δ-BE-algebras under precise algebraic rules, (ii) a unified treatment of subalgebra and filter structures under the Δ-framework, and (iii) a novel examination of upper sets influenced by distributive conditions. These contributions not only expand the theoretical scope of BE-algebras, but also create a foundation for further exploration.

Future research directions include: extending the framework to fuzzy or intuitionistic fuzzy Γ_Δ-BE-algebras; studying homomorphisms and congruence relations in these systems; investigating categorical representations and functorial properties; and exploring potential applications in soft computing, algebraic logic, or decision theory.

Acknowledgements

This research was supported by University of Phayao and Thailand Science Research and Innovation Fund (Fundamental Fund 2026, Grant No. 2252/2568).

References

[1] H. S. Kim and Y. H. Kim. On BE-algebras. *Scientiae Mathematicae Japonicae*, 66(1):113–116, 2007.

[2] S. S. Ahn and K. S. So. On generalized upper sets in BE-algebras. *Bulletin of the Korean Mathematical Society*, 46(2):281–287, 2009.

[3] A. Pumila, N. Phattanachot, and S. Onmo. On generalized upper sets and generalized terminal sections in pseudo BE-algebras. *Burapha Science Journal*, 23(3):1219–1233, 2018.

[4] R. A. Borzooei, A. Saeid, A. Rezaei, A. Radfar, and R. Ameri. On pseudo BE-algebras. *Discussiones Mathematicae - General Algebra and Applications*, 33(1):95–108, 2013.

[5] R. A. Borzooei, A. B. Saeid, A. Rezaei, A. Radfar, and R. Ameri. Distributive pseudo BE-algebras. *Fasciculi Mathematici*, 54(1):21–39, 2015.

[6] H. S. Kim and K. J. Lee. Extended upper sets in BE-algebras. *Bulletin of the Malaysian Mathematical Sciences Society*, 34(3):511–520, 2011.

[7] N. Chunsee, P. Julatha, and A. Iampan. Fuzzy set approach to ideal theory on Sheffer stroke BE-algebras. *Journal of Mathematics and Computer Science*, 34(3):283–294, 2024.

[8] N. Nobusawa. On a generalization of the ring theory. *Osaka Journal of Mathematics*, 1(1):81–89, 1964.

[9] M. K. Sen and N. K. Saha. Γ-semigroups-I. *Bulletin of the Calcutta Mathematical Society*, 78:380–386, 1986.

[10] A. B. Saeid, M. M. K. Rao, and R. K. Kona. Γ-BCK-algebras. *Journal of Mahani Mathematical Research*, 11(3):133–145, 2022.

[11] P. Julatha and A. Iampan. SUP-Hesitant fuzzy ideals of Γ-semigroups. *Journal of Mathematics and Computer Science*, 26(2):148–161, 2022.

[12] M. Siripitukdet and P. Julatha. The greatest subgroup of a semigroup in Γ-semigroups. *Lobachevskii Journal of Mathematics*, 33(2):158–164, 2012.

[13] A. Iampan and M. Siripitukdet. On minimal and maximal ordered left ideals in po-Γ-semigroups. *Thai Journal of Mathematics*, 2(2):275–282, 2004.

[14] M. Palanikumar, A. Iampan, and L. J. Manavalan. M-Bi-base generator of ordered Γ-semigroups. *ICIC Express Letters, Part B: Applications*, 13(8):795–802, 2022.

[15] Y. B. Jun, R. Bandaru, and J. G. Kang. Γ-BE-algebras. *Jordan Journal of Mathematics and Statistics*, 16(2):361–377, 2023.

[16] R. Bandaru, J. G. Kang, A. S. Alali, and Y. B. Jun. $\Gamma_{(\alpha,\beta)}$-BE-algebras. *Journal of Mathematics*, 2023:Article ID 4270107, 10 pages, 2023.

[17] M. Palanikumar, T. T. Raman, A. Iampan, and M. Geethalakshmi. Characterization

of level set through multi-polar extension of intuitionistic fuzzy different ideals on regular ordered ternary semigroups. *International Journal of Analysis and Applications*, 23:200, 2025.