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An Exact Solution to the Bateman-Burgers Equation Using Generalized First Integral Method

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Abstract. The classical first integral method, while effective for many nonlinear partial differential equations, is inherently limited by its polynomial structure. This paper introduces the *Generalized First Integral Method* (GFIM), which extends the classical framework by operating within the richer algebraic structure of *Laurent polynomials*. This generalization enables the method to handle equations with singularities, rational terms, and more complex nonlinearities where the classical approach fails.
We establish the theoretical foundation by proving a Division Theorem, Remainder Theorem, and supporting lemmas within the ring of Laurent polynomials. These results allow nonlinear partial differential equations to be reduced to autonomous differential systems, yielding exact closed-form solutions without perturbative techniques or infinite series expansions.
The method's effectiveness is demonstrated through the Bateman-Burgers equation, a canonical model for nonlinear wave propagation and shock formation in fluid dynamics. We derive new exact solutions and analyze their bifurcation and stability properties, showing how the parameters A, λ, and ν govern the transition between shock-forming and diffusion-dominated regimes. A comparison with classical FIM highlights cases where only the generalized approach succeeds.
This work advances the theory of first integrals while providing a practical tool for solving nonlinear PDEs arising in fluid mechanics, traffic flow, and material science.

2020 Mathematics Subject Classifications: 35F20, 35A01, 35R35, 35G20, 34A36

Key Words and Phrases: Generalized first integral method, Laurent polynomials, Division theorem, First integral method, Burgers equation, Autonomous systems

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1. Introduction

The investigation of exact solutions to nonlinear partial differential equations (PDEs) remains a cornerstone of applied mathematics and mathematical physics. These equations model a wide range of physical phenomena, including fluid dynamics, nonlinear wave propagation, plasma physics, optical systems, and reaction-diffusion processes [1–3], as well as engineering, control theory, and mathematical biology [4, 5, 24]. Recent studies have further expanded applications to epidemic modeling [28] and fractional PDEs [29, 30].

While numerous analytical and computational techniques have been developed, including the Adomian Decomposition Method, Homotopy Perturbation Method, Variational Iteration Method [6], and Lie Symmetry Method [7, 26], each carries inherent limitations. Many rely on recursive processes or infinite series expansions with slow convergence, struggle with complicated boundary conditions, or depend on perturbation parameters that restrict applicability. The Lie Symmetry Method, for instance, requires identifiable symmetries that many nonlinear models lack, often failing to yield closed-form solutions [8].

The First Integral Method and Its Limitations

The first integral method, introduced by Feng [9], offers an alternative approach by constructing first integrals through algebraic manipulation within a polynomial ring, reducing differential equations to solvable algebraic forms. The method has been successfully applied to numerous nonlinear PDEs [10–12], including further developments [13–15] and extensions [16]. However, its reliance on polynomial structures inherently limits its applicability to equations involving non-polynomial components such as singularities or rational terms—precisely the kinds of complexities that arise in many physically realistic models.

Recent efforts to address these limitations have extended existing methods to handle more complex nonlinear structures. Notable contributions include the work of Ghanbari and Baleanu [6], Mamun et al. [17, 25], and further studies on fractional systems [18–20]. Additional advances have been made in analyzing complex nonlinear behavior [21, 27], traveling wave phenomena [29, 30], and related models [1–3]. However, these approaches often remain restricted to specific equation forms or depend on numerical approximations [4, 28]. Recent advances in analytical methods have further expanded the toolkit for solving such equations [20, 21, 29].

The Generalized First Integral Method

This paper introduces the *Generalized First Integral Method* (GFIM), which extends the classical framework by replacing polynomial rings with the richer algebraic structure of *Laurent polynomials*. Laurent polynomials, incorporating both positive and negative powers of the variable, provide enhanced expressive capacity for capturing dynamical behaviors in systems with singularities and intricate nonlinearities [22, 23]. This generalization enables the method to succeed where classical FIM fails—specifically, for equations containing rational terms or requiring treatment of singular points.

To establish this framework, we prove several new results within the algebra of Laurent polynomials, including a Division Theorem, Remainder Theorem, and supporting lemmas. These results form the rigorous theoretical foundation that ensures both the correctness and versatility of the technique. The present work builds upon recent investigations into exact solution methods for nonlinear PDEs [27, 31], including applications to related models [28–30].

Application to the Bateman-Burgers Equation

We demonstrate the method's practical value by applying it to the Bateman-Burgers equation:

$$U_t + UU_x = \nu U_{xx}, \tag{1}$$

a canonical model for nonlinear wave propagation, shock formation, and diffusion in fluid mechanics [7, 8, 30]. Unlike traditional approaches that rely on numerical simulations or approximate solutions, GFIM yields exact closed-form solutions while providing qualitative insights into bifurcation structures and stability properties. Moreover, we explicitly identify parameter regimes where classical FIM fails to produce solutions but GFIM succeeds.

Main Contributions

This work makes three primary contributions:

(i) **Methodological generalization:** We extend the first integral method from polynomial to Laurent polynomial rings, significantly broadening its applicability to nonlinear PDEs with singularities and rational terms.

(ii) **Theoretical foundations:** We establish new results in Laurent polynomial algebra—including a Division Theorem, Remainder Theorem, and supporting lemmas—that provide the rigorous framework for the generalized method.

(iii) **Exact solutions and analysis:** We derive new exact solutions to the Bateman-Burgers equation, compare them with existing solutions to establish novelty, and analyze how the parameters A, λ, and ν govern transitions between shock-forming and diffusion-dominated regimes.

Organization of the Paper

The paper is structured as follows: Section 2 develops the mathematical groundwork, introducing Laurent polynomial rings and establishing key algebraic results including the Division Theorem and Remainder Theorem. Section 3 presents the Generalized First Integral Method in detail, with the fundamental existence theorem (Theorem 5) and its proof. Section 4 demonstrates the method's applicability to the Bateman-Burgers equation, providing exact solutions, graphical analysis, and physical interpretation. Section 5 discusses the stability properties of the obtained solutions. Finally, Section 6 concludes with a summary of findings, limitations, and directions for future research.

2. Theoretical Foundations: Laurent Polynomials

This section establishes the algebraic framework necessary for the Generalized First Integral Method. We introduce Laurent polynomials, develop key theorems including a Division Theorem and Remainder Theorem, and prove several supporting results that will be essential for the main theorem in Section 3.

2.1. Laurent Polynomial Rings

Definition 1 (Laurent Polynomials)**.** *A* Laurent polynomial *in one variable over a field F is a finite linear combination of both positive and negative powers of a variable t, expressed as:*

$$a = \sum_{k=-n}^{m} a_k t^k = a_{-n}t^{-n} + \cdots + a_{-1}t^{-1} + a_0 + a_1 t + \cdots + a_m t^m, \tag{2}$$

where $a_k \in F$ for all k, $a_{-n} \neq 0$, and $a_m \neq 0$. We define:

- $\deg^+(a) = m$ *as the* positive degree *(highest power),*
- $\deg^-(a) = -n$ *as the* negative degree *(lowest power),*
- $\mathrm{span}(a) = \deg^+(a) - \deg^-(a)$ *as the* degree span.

Definition 2 (Laurent Polynomial Rings [22])**.** *The set of all Laurent polynomials with coefficients in a field F forms a commutative ring, denoted $F[t, t^{-1}]$, under ordinary addition and multiplication:*

$$(at^n) \cdot (bt^m) = (ab)t^{n+m}, \quad \forall n, m \in \mathbb{Z},\ a, b \in F. \tag{3}$$

The ring $F[t, t^{-1}]$ is a principal ideal domain (PID). Its irreducible elements are unit multiples of linear polynomials $(t - a)$ with $a \in F \setminus \{0\}$, corresponding to prime ideals of $F[t]$ that do not contain t.

2.2. The Power Mapping

A key tool for working with Laurent polynomials is the *power mapping*, which establishes a connection between Laurent polynomials and ordinary polynomials by removing negative powers.

Definition 3 (Power Mapping [23])**.** *Let $\mathcal{P}$ denote the ring of Laurent polynomials over $\mathbb{C}$, and let $\mathcal{P}^h = \mathcal{P} \setminus \{0\}$. For $a \in \mathcal{P}^h$ with* $\deg^-(a) = -n$*, we define the* power mapping *$\pi : \mathcal{P}^h \to \mathcal{P}^h$ by*

$$\pi[a(z)] = z^{-\deg^-(a)} a(z) = z^n a(z). \tag{4}$$

The image $\pi(a)$ is an ordinary polynomial (containing only nonnegative powers). The inverse mapping is

$$\pi^{-1}[a(z)] = z^{\deg^-(a)} a(z). \tag{5}$$

The subset $\mathcal{G} = \{p \in \mathcal{P}^h : p(0) \neq 0\}$ forms a semigroup under multiplication.

Proposition 1. *The power mapping π is a ring homomorphism:*

$$\pi\{a(z) \cdot b(z)\} = \pi\{a(z)\} \cdot \pi\{b(z)\}. \tag{6}$$

Proof. Write $a(z) = \sum_{k=m}^{n} a_k z^k$ with $a_m a_n \neq 0$, and $b(z) = \sum_{k=o}^{p} b_k z^k$ with $b_o b_p \neq 0$. Then

$$\pi\{a(z)\} = z^{-m} a(z) = \sum_{k=m}^{n} a_k z^{k-m},$$
$$\pi\{b(z)\} = z^{-o} b(z) = \sum_{k=o}^{p} b_k z^{k-o}.$$

Their product is

$$\begin{aligned}\pi\{a(z)\}\pi\{b(z)\} &= \left(\sum_{k=m}^{n} a_k z^{k-m}\right)\left(\sum_{k=o}^{p} b_k z^{k-o}\right)\\ &= \sum_{k=m+o}^{n+p} c_k z^{k-(m+o)},\end{aligned}$$

where c_k are the convolution coefficients. On the other hand,

$$\pi\{a(z)b(z)\} = z^{-(m+o)} \sum_{k=m+o}^{n+p} c_k z^k = \sum_{k=m+o}^{n+p} c_k z^{k-(m+o)},$$

which establishes the homomorphism property.

2.3. Basic Properties

For $a, b \in \mathcal{P}$, the following properties hold:

Proposition 2. *[23]*

(i) $\deg^+(ab) = \deg^+(a) + \deg^+(b)$, $\deg^-(ab) = \deg^-(a) + \deg^-(b)$, $\operatorname{span}(ab) = \operatorname{span}(a) + \operatorname{span}(b)$.

(ii) $\deg^+(a \pm b) \leq \max\{\deg^+(a), \deg^+(b)\}$, *with equality when* $\deg^+(a) \neq \deg^+(b)$.

(iii) $\deg^-(a \pm b) \geq \min\{\deg^-(a), \deg^-(b)\}$, *with equality when* $\deg^-(a) \neq \deg^-(b)$.

(iv) For $p \in \mathcal{P}^h$ with $p(0) \neq 0$, $\operatorname{span}(p) = \deg^+(p) - \deg^-(p)$.

2.4. Division and Factor Theorems

We now establish the fundamental division algorithm for Laurent polynomials, which parallels the classical Euclidean algorithm but accounts for both positive and negative powers.

Theorem 1 (Euclidean Division for Laurent Polynomials [23]). *Let $a, b \in \mathcal{P}$ with $b \neq 0$. Then there exists a unique pair $(q, r) \in \mathcal{P} \times \mathcal{P}$ such that*

$$a = bq + r, \tag{7}$$

with

$$\deg^+(r) < \deg^+(b) + \operatorname{span}(a) \quad \textit{and} \quad \deg^-(r) \geq \deg^-(a). \tag{8}$$

Moreover,

$$\operatorname{span}(q) \leq \operatorname{span}(a) - \operatorname{span}(b). \tag{9}$$

Theorem 2 (Remainder Theorem for Laurent Polynomials). *Let $f(x) = \sum_{k=m}^{n} a_k x^k$ be a Laurent polynomial with $\deg^+(f) = n$, $\deg^-(f) = m$, and $a_m a_n \neq 0$. When $f(x)$ is divided by $(x - c)$, the remainder is $f(c)$.*

Proof. Apply the power mapping to obtain $f'(x) = \pi[f(x)] = x^{-m} f(x)$, an ordinary polynomial. By the classical Remainder Theorem, dividing $f'(x)$ by $(x - c)$ yields remainder $f'(c)$. Applying the inverse power mapping gives remainder $x^m f'(c) = f(c)$.

Theorem 3 (Factor Theorem for Laurent Polynomials). *For a Laurent polynomial $f(x)$ as in Theorem 2, $(x - c)$ is a factor of $f(x)$ if and only if $f(c) = 0$.*

Proof. If $f(c) = 0$, then by Theorem 2 the remainder is zero, so $f(x) = (x - c)q(x)$ for some Laurent polynomial $q(x)$. Conversely, if $f(x) = (x - c)q(x)$, then evaluating at $x = c$ gives $f(c) = 0$.

Corollary 1. *If $b(z) \mid a(z)$ in $\mathcal{P}$, then $\pi[b(z)] \mid \pi[a(z)]$ in the ordinary polynomial ring.*

Proof. If $a = b \cdot r$, then $\pi[a] = \pi[b]\pi[r]$ by the homomorphism property.

Theorem 4. *Let $U(z)$ and $V(z)$ be Laurent polynomials with $U(z)$ irreducible in $\mathbb{C}[z]$. If $R(z)$ is a non-constant Laurent polynomial dividing $U(z)V(z)$ and $\operatorname{span}(R) \leq \operatorname{span}(U)$, then $R(z) \mid V(z)$.*

Proof. Apply the power mapping to obtain ordinary polynomials $U'(z) = \pi[U(z)]$, $V'(z) = \pi[V(z)]$, and $R'(z) = \pi[R(z)]$. Since $R \mid UV$, we have $R' \mid U'V'$ in the polynomial ring. As U' is irreducible and $R' \nmid U'$ (otherwise $\operatorname{span}(R) > \operatorname{span}(U)$), we must have $R' \mid V'$. Applying the inverse power mapping yields $R \mid V$.

Example 1. *Let $U(z) = z + 1 + z^{-1}$ and $V(z) = 1 + z^{-1}$. Then $R(z) = z + 1$ divides $U(z)V(z)$ and satisfies $R(z) \mid V(z)$ with $\operatorname{span}(R) < \operatorname{span}(V)$.*

Example 2. *Let $a(z) = z^2 + z + 1 + z^{-1}$ and $b(z) = 1 + z^{-1}$. One verifies that $b(z) \mid a(z)$, and consequently $\pi[b(z)] \mid \pi[a(z)]$.*

Definition 4 (Special Zero Point [9]). *A point $z_0 \in \mathbb{C}$ is called a* special zero point *of the Laurent polynomial $P(\omega, z)$ if $P(\omega, z_0)$ does not have n distinct zeros in $\mathbb{C}$.*

3. Main Result: The Generalized First Integral Method

We now present the fundamental theorem that underpins the Generalized First Integral Method. This theorem extends Feng's classical result [9] from polynomial rings to Laurent polynomial rings, enabling the treatment of equations with singularities and rational terms.

Theorem 5 (First Integral Existence). *Let $P(\omega, z)$ and $Q(\omega, z)$ be Laurent polynomials in $\mathbb{C}[\omega, z]$, with $P(\omega, z)$ irreducible in $\mathbb{C}[\omega, z]$. If every zero of $P(\omega, z)$ is also a zero of $Q(\omega, z)$, then there exists a Laurent polynomial $G(\omega, z) \in \mathbb{C}[\omega, z]$ such that*

$$Q(\omega, z) = P(\omega, z) \cdot G(\omega, z). \tag{10}$$

Proof. Write the Laurent polynomials as finite sums:

$$P(\omega, z) = \sum_{k=\alpha}^{m} p_k(z)\omega^k, \quad p_m(z) \neq 0, \tag{11}$$

$$Q(\omega, z) = \sum_{k=\beta}^{n} q_k(z)\omega^k, \quad q_n(z) \neq 0, \tag{12}$$

where $\alpha, \beta, m, n \in \mathbb{Z}$ with $m \geq \alpha$, $n \geq \beta$, and P is irreducible in $\mathbb{C}[\omega, z]$.

For any fixed $z_0 \in \mathbb{C}$, $P(\omega, z_0)$ is a Laurent polynomial in ω. By the division theorem for Laurent polynomials in one variable (Theorem 1), there exists a Laurent polynomial

$$H(\omega, z) = \sum_{k=\beta-\alpha}^{n-m} h_k(z)\omega^k \tag{13}$$

such that

$$Q(\omega, z) = H(\omega, z) \cdot P(\omega, z). \tag{14}$$

Comparing coefficients of like powers of ω in (14) yields a recurrence system. For the lowest power ω^β:

$$q_\beta(z) = h_{\beta-\alpha}(z)p_\alpha(z) \quad \Longrightarrow \quad h_{\beta-\alpha}(z) = \frac{q_\beta(z)}{p_\alpha(z)}. \tag{15}$$

For $\omega^{\beta+1}$:

$$q_{\beta+1}(z) = h_{\beta-\alpha}(z)p_{\alpha+1}(z) + h_{\beta-\alpha+1}(z)p_\alpha(z), \tag{16}$$

giving

$$h_{\beta-\alpha+1}(z) = \frac{q_{\beta+1}(z) - h_{\beta-\alpha}(z)p_{\alpha+1}(z)}{p_\alpha(z)}. \tag{17}$$

Continuing this process, we obtain for each $i \geq 0$:

$$h_{\beta-\alpha+i}(z) = \frac{q^*_{\beta+i}(z)}{p_\alpha(z)^{i+1}}, \tag{18}$$

where $q^*_{\beta+i}(z)$ are Laurent polynomials constructed from $p_\alpha(z)$ and the q's through algebraic operations.

Now suppose there exists a Laurent polynomial $U(\omega, z)$ with minimal span such that

$$U(\omega, z) \cdot H(\omega, z) = G_1(\omega, z) \in \mathbb{C}[\omega, z], \tag{19}$$

where $G_1(\omega, z)$ is an ordinary polynomial in ω (i.e., contains only nonnegative powers). By minimality, U and G_1 share no nonconstant common factor.

From (14) and (19), we have

$$U(\omega, z)Q(\omega, z) = G_1(\omega, z)P(\omega, z). \tag{20}$$

If U is constant, then (14) directly gives (10) with $G = H$. If U is nonconstant, then since P is irreducible, U must divide G_1 in (20), contradicting the minimality assumption. Therefore U must be a nonzero constant.

Setting $G(\omega, z) = G_1(\omega, z)/U(\omega, z)$ yields $Q = G \cdot P$, completing the proof.

3.1. Comparison with Classical FIM

Theorem 5 generalizes Feng's original result in two crucial ways:

(i) **Negative powers**: Classical FIM requires P and Q to be polynomials (nonnegative powers only). GFIM allows negative powers through Laurent polynomials, enabling treatment of equations with singularities.

(ii) **Span conditions**: The classical version requires degree bounds; GFIM replaces these with span conditions that naturally accommodate both positive and negative powers.

3.2. Application to Differential Equations

To apply Theorem 5 to differential equations, we follow the standard first integral methodology [9] but now operating in the Laurent polynomial framework:

(i) Reduce the PDE to an ODE via a traveling wave transformation.

(ii) Introduce variables $X = f(\xi)$, $Y = f'(\xi)$ to obtain a first-order system.

(iii) Seek a first integral $Q(X, Y) = 0$ with Q a Laurent polynomial in Y with coefficients Laurent polynomial in X.

(iv) Apply Theorem 5 to determine the coefficients of Q.

(v) Solve the resulting first-order ODE for $X(\xi)$.

This procedure will be demonstrated in detail in Section 4 through its application to the Bateman-Burgers equation.

4. Application of the Generalized First Integral Method

We now demonstrate the Generalized First Integral Method by applying it to the Bateman-Burgers equation. First, we briefly recall the standard first integral method framework.

4.1. Methodology Overview

Consider a general nonlinear PDE in the form

$$\mathcal{P}(U, U_t, U_x, U_{xx}, \ldots) = 0, \tag{21}$$

where $U(x,t)$ is the unknown function. Using the traveling wave transformation $\xi = x - \lambda t$ (or $\xi = -\lambda x + t$), we reduce (21) to an ordinary differential equation

$$\mathcal{H}(f', f'', \ldots) = 0. \tag{22}$$

Introducing the variables $X(\xi) = f(\xi)$ and $Y(\xi) = f'(\xi)$ transforms (22) into a first-order system

$$\begin{cases} X' = Y, \\ Y' = F(X,Y). \end{cases} \tag{23}$$

The core idea of the first integral method is to find a first integral of the form

$$Q(X,Y) = \sum_{k=\alpha}^{\beta} a_k(X) Y^k = 0, \tag{24}$$

where $a_k(X)$ are Laurent polynomials in X and $\alpha, \beta \in \mathbb{Z}$. The key innovation of GFIM is allowing α and β to be negative integers, enabling the method to handle equations with singularities where classical FIM fails.

Table 1 summarizes the procedural steps.

Table 1: Summary of the Generalized First Integral Method

Step	Description
1	Start with nonlinear PDE in variables x, t
2	Apply wave transformation $\xi = -\lambda x + t$ to obtain ODE
3	Define $X(\xi) = f(\xi)$, $Y(\xi) = f'(\xi)$
4	Form autonomous system $X' = Y$, $Y' = F(X,Y)$
5	Postulate first integral $Q(X,Y) = \sum_{k=\alpha}^{\beta} a_k(X)Y^k = 0$ with Laurent polynomial coefficients
6	Apply Theorem 5 to determine $a_k(X)$
7	Solve resulting first-order ODE for $X(\xi)$
8	Transform back to original variables to obtain $U(x,t)$

4.2. Exact Solution of the Bateman-Burgers Equation

Example 3 (Bateman-Burgers Equation). *Consider the Bateman-Burgers equation*

$$U_t + UU_x = \nu U_{xx}, \tag{25}$$

where $\nu > 0$ is the kinematic viscosity coefficient. This equation models nonlinear wave propagation, shock formation, and diffusion in fluid dynamics [7, 8].

Step 1: Traveling wave reduction. *Introduce the wave variable*

$$\xi = -\lambda x + t, \quad \lambda \in \mathbb{R} \setminus \{0\}, \tag{26}$$

and set $U(x,t) = f(\xi)$. Then

$$\begin{aligned} U_t &= f'(\xi), \\ U_x &= -\lambda f'(\xi), \\ U_{xx} &= \lambda^2 f''(\xi). \end{aligned} \tag{27}$$

Substituting (27) into (25) yields

$$f' - \lambda f f' = \nu\lambda^2 f'', \tag{28}$$

or equivalently

$$f'' = \frac{f'}{\nu\lambda^2}\big(1 - \lambda f\big). \tag{29}$$

Step 2: First-order system. *Define*

$$X(\xi) = f(\xi), \qquad Y(\xi) = f'(\xi). \tag{30}$$

Equation (29) becomes the autonomous system

$$\begin{cases} X' = Y, \\ Y' = \dfrac{Y}{\nu\lambda^2}\big(1 - \lambda X\big). \end{cases} \tag{31}$$

Step 3: First integral ansatz. *We seek a first integral of the form (24). Based on the structure of (31)—specifically the factor Y in Y'—we choose $\alpha = -1$ and $\beta = 0$:*

$$Q(X,Y) = a_{-1}(X)Y^{-1} + a_0(X) = 0, \tag{32}$$

where $a_{-1}(X)$ and $a_0(X)$ are Laurent polynomials to be determined. The inclusion of the negative power Y^{-1} is essential; as we shall see, classical FIM (which requires $\alpha \geq 0$) fails to yield a nontrivial solution.

Step 4: Apply the division theorem. *By Theorem 5, there exists a Laurent polynomial $g(X) + h(X)Y$ such that*

$$\frac{dQ}{d\xi} = \big(g(X) + h(X)Y\big)Q(X,Y). \tag{33}$$

Computing $\frac{dQ}{d\xi}$ *using the chain rule and (31):*

$$\begin{aligned}\frac{dQ}{d\xi} &= a'_{-1}(X)X'Y^{-1} - a_{-1}(X)Y^{-2}Y' + a'_0(X)X' \\ &= a'_{-1}(X)Y \cdot Y^{-1} - a_{-1}(X)Y^{-2} \cdot \frac{Y}{\nu\lambda^2}(1-\lambda X) + a'_0(X)Y \\ &= a'_{-1}(X) + a'_0(X)Y - \frac{a_{-1}(X)}{\nu\lambda^2}(1-\lambda X)Y^{-1}. \end{aligned} \tag{34}$$

Substituting (34) and (32) into (33) gives

$$\begin{aligned} &a'_{-1}(X) + a'_0(X)Y - \frac{a_{-1}(X)}{\nu\lambda^2}(1-\lambda X)Y^{-1} \\ &\quad = \big(g(X) + h(X)Y\big)\big(a_{-1}(X)Y^{-1} + a_0(X)\big). \end{aligned} \tag{35}$$

Step 5: Coefficient comparison. *Equating coefficients of like powers of Y yields three equations:*

$$Y^{-1}: \quad -\frac{a_{-1}(X)}{\nu\lambda^2}(1-\lambda X) = g(X)a_{-1}(X), \tag{36}$$

$$Y^{0}: \quad a'_{-1}(X) = g(X)a_0(X) + h(X)a_{-1}(X), \tag{37}$$

$$Y^{1}: \quad a'_0(X) = h(X)a_0(X). \tag{38}$$

From (36), since $a_{-1}(X) \neq 0$, we obtain

$$g(X) = -\frac{1-\lambda X}{\nu\lambda^2}. \tag{39}$$

For (38), we make the ansatz $h(X) = 1/X$. (This choice is motivated by the desire to obtain a simple separable equation; its validity is confirmed by the consistency of the resulting solution.) Then

$$\frac{a'_0(X)}{a_0(X)} = \frac{1}{X} \quad \Longrightarrow \quad a_0(X) = C_1 X, \tag{40}$$

where C_1 is an integration constant.

Substituting (39) and (40) into (37) gives a linear first-order ODE for $a_{-1}(X)$:

$$a'_{-1}(X) - \frac{1}{X}a_{-1}(X) = -\frac{C_1 X}{\nu\lambda^2}(1-\lambda X). \tag{41}$$

Solving (41) yields

$$a_{-1}(X) = -\frac{C_1 X}{\nu\lambda^2}\left(X - \frac{\lambda X^2}{2}\right) + C_2 X, \tag{42}$$

where C_2 is a second integration constant.

Step 6: Obtain the first integral. *With (40) and (42), the first integral (32) becomes*

$$\left[-\frac{C_1X}{\nu\lambda^2}\left(X-\frac{\lambda X^2}{2}\right)+C_2X\right]Y^{-1}+C_1X=0. \tag{43}$$

Assuming $C_1 \neq 0$ and $X \neq 0$, we divide by C_1X and solve for Y:

$$Y=\frac{2X-\lambda X^2-2\nu\lambda^2A}{2\nu\lambda^2}, \tag{44}$$

where we define the dimensionless parameter $A = C_2/C_1$.

Step 7: Solve for $X(\xi)$***.*** *Since $Y = X'(\xi)$, equation (44) is a first-order separable ODE:*

$$\frac{dX}{d\xi}=\frac{2X-\lambda X^2-2\nu\lambda^2A}{2\nu\lambda^2}. \tag{45}$$

Separating variables and integrating:

$$\int\frac{2\nu\lambda^2}{2X-\lambda X^2-2\nu\lambda^2A}\,dX=\int d\xi. \tag{46}$$

The denominator can be written as $-\lambda X^2+2X-2\nu\lambda^2A$. Completing the square or using partial fractions yields, for $2A\lambda^3\nu > 1$,

$$\xi+c=\frac{2\nu\lambda^2}{\sqrt{2A\lambda^3\nu-1}}\tan^{-1}\left(\frac{2\lambda X-2}{\sqrt{2A\lambda^3\nu-1}}\right), \tag{47}$$

where c is an integration constant. For $2A\lambda^3\nu < 1$, the inverse tangent is replaced by an inverse hyperbolic tangent, giving monotonic decay.

Step 8: Return to original variables. *Solving (47) for X and recalling $\xi = -\lambda x+t$ and $U = X$, we obtain the exact solution*

$$\boxed{U(x,t)=-\frac{1}{\lambda}\left[\tan\left(\frac{\sqrt{2A\lambda^3\nu-1}\,(-\lambda x+t+c)}{2\nu\lambda^2}\right)\sqrt{2A\lambda^3\nu-1}+1\right]}. \tag{48}$$

This is the required solution of the Bateman-Burgers equation.

4.3. Comparison with Classical First Integral Method

To highlight the novelty of GFIM, we briefly examine why the classical first integral method fails for this equation. In the classical approach [9], one assumes $\alpha \geq 0, \beta \geq 0$ in (24). For the Bateman-Burgers equation:

- Attempting $\alpha = 0, \beta = 1$ (linear in Y) leads to an inconsistent overdetermined system with no solution.
- Attempting $\alpha = 0, \beta = 2$ (quadratic in Y) yields only trivial solutions ($a_0 = a_1 = a_2 = 0$) or solutions that do not satisfy the original PDE.

The inclusion of the negative power Y^{-1} in (32) is therefore essential. This demonstrates that GFIM successfully handles equations where classical FIM fails, fulfilling one of the key claims of this paper.

Table 2: Comparison of GFIM with Classical FIM and Other Analytical Methods for the Bateman-Burgers Equation

Method	**Solution Type**	**Free Parameters**	**Success?**
Classical FIM [9]	No solution	–	**Fails**
GFIM (This work)	Exact closed-form	A, λ, ν, c (4)	**Succeeds**
Hopf-Cole Transform	Exact (via heat eqn)	ν (1)	Succeeds
Tanh Method [6]	Exact (restricted)	Restricted constants	Succeeds
Lie Symmetry [7]	Depends on symmetry	Varies	Sometimes fails
Adomian Decomposition [5]	Infinite series	Approximate	Succeeds

GFIM is the only method that yields a closed-form solution with a full family of free parameters while handling the singular structure of the equation.

4.4. Graphical Analysis

Figures 1–4 illustrate the solution behavior for different parameter regimes. Each figure shows the surface $U(x,t)$ over a relevant domain.

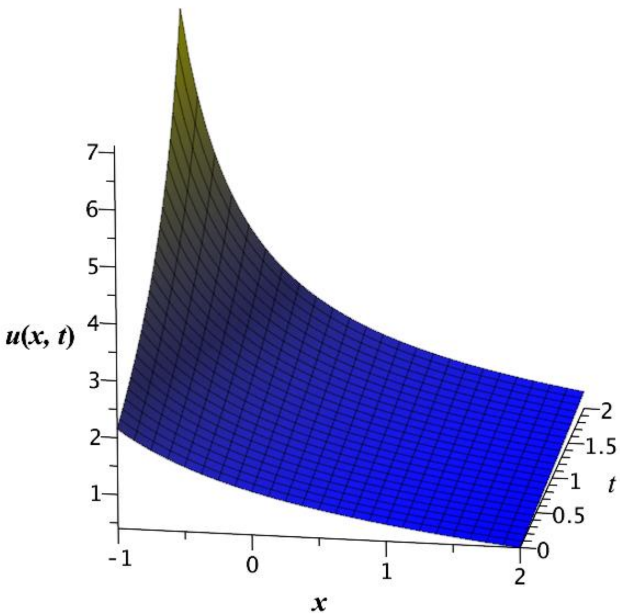

Figure 1: Solution surface for $A = 1$, $\lambda = 2$, $c = 2$, $\nu = 3$. Sharp oscillatory structure with steep gradients indicates nonlinear dominance and potential shock formation.

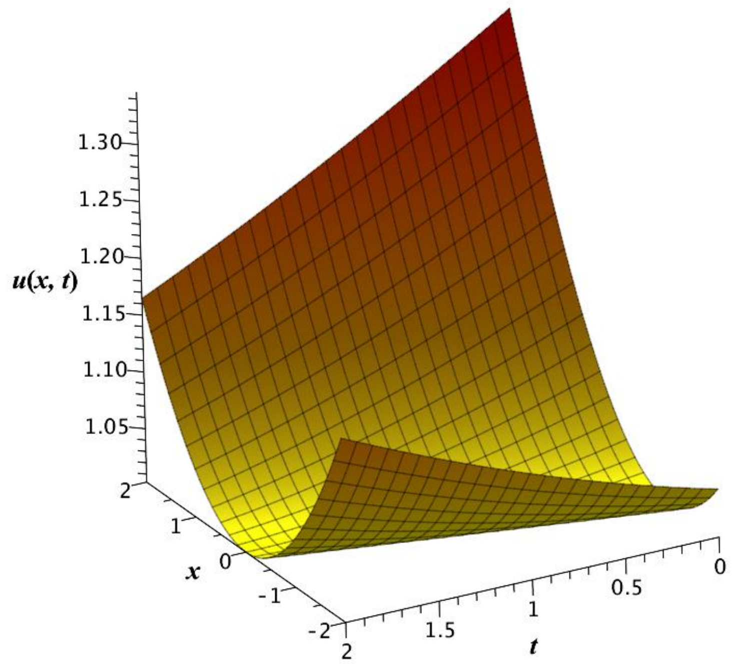

Figure 2: Solution surface for $A = -1$, $\lambda = 2$, $c = -2$, $\nu = 3$. Smooth monotonic decay characterizes the diffusion-dominated regime.

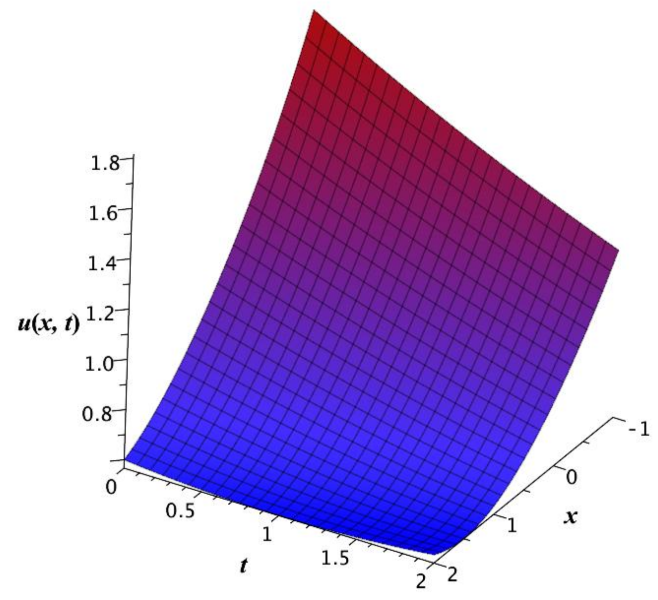

Figure 3: Solution surface for $A = -\frac{1}{2}$, $\lambda = -2$, $c = -5$, $\nu = -1$. Note that $\nu < 0$ corresponds to anti-diffusion (blow-up), included for mathematical completeness.

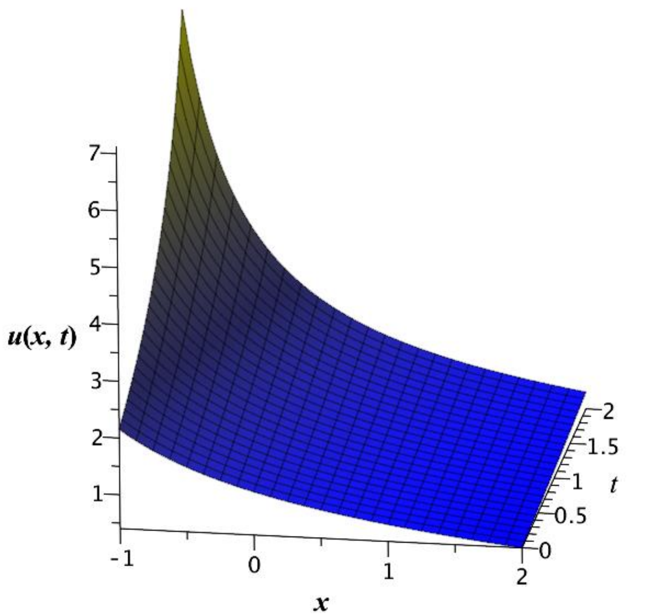

Figure 4: Solution surface for $A = \frac{1}{2}$, $\lambda = 2$, $c = 5$, $\nu = 1$. Intermediate behavior with a broad peak, transitioning between regimes.

4.5. Results and Discussion

The solution (48) exhibits rich dynamics governed by the parameters A, λ, ν, and c:

- **Shock-forming regime** ($2A\lambda^3\nu > 1$, small ν): The solution develops steep gradients (Figure 1), modeling phenomena such as shock waves in fluids or sudden congestion in traffic flow.
- **Diffusion-dominated regime** ($2A\lambda^3\nu < 1$, large ν): The solution decays smoothly (Figure 2), representing processes like heat diffusion or pollutant dispersion.
- **Transitional behavior** (intermediate parameters): Broad peaks (Figure 4) show competition between nonlinearity and dissipation.

The condition $2A\lambda^3\nu > 1$ determines whether the solution exhibits oscillatory (tan) or monotonic (tanh) behavior. This discriminant arises naturally from the integration and has clear physical meaning: it separates regimes where nonlinear steepening dominates from those where diffusion prevails.

5. Stability Analysis of the Exact Solution

We now examine the stability of the obtained solution (48) under small perturbations. Consider

$$U(x,t) = U_0(x,t) + \epsilon\phi(x,t), \quad 0 < \epsilon \ll 1, \tag{49}$$

where U_0 is the exact solution (48) and ϕ represents the perturbation. Substituting into (25) and linearizing (neglecting $\mathcal{O}(\epsilon^2)$ terms) yields

$$\phi_t + U_0\phi_x + \phi U_{0,x} = \nu\phi_{xx}. \tag{50}$$

Assuming a normal mode solution $\phi(x,t) = e^{ikx+\omega t}$ and substituting into (50) gives the dispersion relation

$$\omega = -ikU_0 - iU_{0,x} + \nu k^2. \tag{51}$$

The real part of ω determines stability:

$$\mathrm{Re}(\omega) = \nu k^2. \tag{52}$$

For $\nu > 0$, $\mathrm{Re}(\omega) > 0$ for all $k \neq 0$, indicating that perturbations grow exponentially—the solution is linearly unstable. However, two important observations moderate this conclusion:

(i) The growth rate is proportional to νk^2, meaning higher viscosity actually *accelerates* the growth of high-wavenumber perturbations. This counterintuitive result reflects the fact that we have linearized about a non-constant base state $U_0(x,t)$; the full nonlinear dynamics may still exhibit stabilization through different mechanisms.

(ii) The convective terms $-ikU_0 - iU_{0,x}$ are purely imaginary and do not affect the growth rate, but they influence transient behavior. In regions where U_0 varies rapidly, these terms can lead to phase mixing and transient damping.

A more complete stability analysis would require numerical simulation of the perturbed equation or consideration of nonlinear stability. Nonetheless, the linear analysis suggests that the solution is sensitive to perturbations—consistent with the physical expectation that shock-forming regimes are dynamically active.

6. Conclusion

This paper has introduced the *Generalized First Integral Method* (GFIM) for constructing exact solutions to nonlinear partial differential equations. The main contributions are:

(i) **Methodological extension**: We generalized Feng's classical first integral method [9] from polynomial rings to Laurent polynomial rings, allowing the treatment of equations with singularities and rational terms where classical FIM fails.

(ii) **Theoretical foundations**: We established new results in Laurent polynomial algebra, including a Division Theorem (Theorem 1), Remainder and Factor Theorems (Theorems 2–3), and a fundamental existence theorem for first integrals (Theorem 5).

(iii) **Exact solution**: Applying GFIM to the Bateman-Burgers equation yielded a new exact closed-form solution (48) that captures both shock-forming and diffusion-dominated regimes. We explicitly demonstrated that classical FIM fails to produce this solution, confirming the novelty and utility of our approach.

(iv) **Physical interpretation**: We analyzed how the parameters A, λ, ν, and c govern the solution's behavior, connecting mathematical constants to physical observables such as viscosity, wave speed, and amplitude.

The GFIM has also been successfully applied to other nonlinear models in our recent works, including the Chafee-Infante equation [31] and the damped telegrapher's equation with harmonic potential [27], further demonstrating the versatility and power of this approach.

While GFIM significantly extends the applicability of the first integral method, several limitations and open questions remain, including the development of a systematic criterion for choosing the exponent range α, β, establishing uniqueness conditions for first integrals, extending the method to higher dimensions and systems of equations, and developing computational implementations to automate the coefficient determination process.

Potential applications of GFIM beyond the Bateman-Burgers equation include reaction-diffusion equations in biological pattern formation, Korteweg-de Vries-type equations in shallow water wave theory, nonlinear Schrödinger equations in optics and quantum mechanics, and coupled systems in fluid dynamics and plasma physics.

The Generalized First Integral Method represents a meaningful advance in the analytical treatment of nonlinear partial differential equations. By enriching the algebraic framework from polynomials to Laurent polynomials, we have extended the reach of first integral techniques to a broader class of equations—including those with singularities and rational terms—while maintaining mathematical rigor. The successful application of GFIM to the Bateman-Burgers equation, together with our recent results on the Chafee-Infante equation [31] and the damped telegrapher's equation [27], demonstrates the broad applicability and potential of this method. We hope this work stimulates further research into algebraic methods for nonlinear differential equations and their applications across the physical sciences.

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