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Geometric Bonferroni Aggregation Operators for Complex Interval-Valued Intuitionistic Fuzzy Aggregation Operators and Their Applications to Cybersecurity Tools Selection

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Abstract. Cybersecurity tools selection is a specialized and efficient approach that deals with protecting computer networks, systems, data, and digital infrastructures from harm, malicious assaults, and unauthorized access. This paper aims to develop an innovative selection method for cybersecurity problems using proposed aggregation operators (AOs) and distance measures (DMs) in the environment of complex interval-valued intuitionistic fuzzy sets (CIVIFSs). First, we introduce novel Bonferroni operational laws for CIVIFSs. Then, some novel aggregation operators, namely complex interval-valued intuitionistic fuzzy geometric Bonferroni aggregation operators (CIVIFGBAOs) and complex interval-valued intuitionistic fuzzy weighted geometric Bonferroni aggregation operators (CIVIFWGBAOs), are derived. The essential properties such as monotonicity, idempotency, and boundedness are investigated. Furthermore, new distance measures for CIVIFSs are proposed and their properties are analyzed. A novel decision-making framework is developed based on the proposed AOs and DMs and applied to cybersecurity tool selection. Comparative analysis demonstrates the superiority and robustness of the proposed methods over existing approaches. The results indicate that the alternative $\breve{A}_3$ is the most suitable option according to the proposed framework.

2020 Mathematics Subject Classifications: 03E72, 90B50, 68M12

Key Words and Phrases: Complex interval-valued intuitionistic fuzzy sets, aggregation operators, bonferroni operators, decision making, cybersecurity tools selection

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1. Introduction

We establish the introduction by offering a brief overview of cybersecurity and its uses. After that, we present brief explanations of fuzzy sets (FSs), interval-valued FSs (IVFSs), complex FSs (CFSs), intuitionistic FSs (IFSs), complex IFSs (CIFSs), and CIVIFSs. These reviews need to support the applications of the theory of CIVIFSs. The research gap is then demonstrated, and the motivation for this study is presented. Finally, we provide a summary of the paper's framework and identify the primary advantages and contributions of our proposed work.

1.1. Analysis of Cybersecurity

Cybersecurity is an essential and ever-evolving discipline that deals with protecting computer networks, systems, data, and digital infrastructures from harm, malicious assaults, and unauthorized access. It incorporates an extensive number of procedures and tools, such as encryption, firewalls, intrusion detection systems, zero-trust architectures, and multi-factor authentication that aim to protect digital assets. These safety procedures are important for protecting against threats that could cause major financial and reputational harm, like malware, ransomware, data breaches, phishing, and advanced persistent threats. Smart networked devices, Cloud computing, and the rapid expansion of digital transformation make cybersecurity more crucial and organizations more vulnerable to cyber threats. The purpose of cybersecurity is to provide a safe and stable digital environment. To evaluate and enhance cybersecurity performance, researchers have developed a number of fuzzy models and frameworks in recent years. Soner employed fuzzy cognitive maps to model and analyze cybersecurity risk transmission in ports [1]. Alsughayyir and Alsager developed an innovative technique for evaluating cybersecurity risks using multi-Q valued bipolar picture fuzzy model [2]. Mahadik et al. [3] presented a D-making approach based on fuzzy logic systems preventing cybersecurity risk management. Al-Emran et al. [4] used FSs and PLS-SEM to evaluate the challenges affecting cybersecurity behavior in the Metaverse. Zhang discussed enhancing Cybersecurity awareness using threat intelligence techniques based on hybrid graph net-bipolar fuzzy rough sets [5]. Jan et al. [6] presented mathematical technique of network security and communication using interval-valued complex spherical fuzzy information. Ali and Yang utilized newly defined AOs for bipolar complex fuzzy soft inference systems to enhance the cybersecurity risk assessment model [7]. Dutta et al. [8] developed an interval picture fuzzy matrix game-based strategy to address cyber threats in the medical field. Qiao presented a comprehensive framework for computer network security assessment using interval neutrosophic fuzzy information [9]. Alotaibi et al. [10] designed a hybrid framework of fuzzy soft expert sets and deep learning for securing industrial cyber-physical networks against threats.

1.2. Analysis of Fuzzy Set Theory and Their Extensions

Selecting the best or most ideal choice among a number of options is often a very difficult, complex, and sometimes confusing task. This complexity is due to the importance of taking into consideration a number of criteria, overlapping objectives, and the unpredictability of different events. In such cases, D-making methods have become widely accepted as useful tools for systematically assessing and resolving such kinds of complex problems, providing organized approaches to reach logical and knowledgeable decisions. Elbanna proposed the procedure perspective for strategic D-making, based on binary numbers 0 and 1. Sometimes, we require more possibilities than just zero and one. For this, Zadeh demonstrated FS theory in 1965. It is a significant mathematical structure that addresses the uncertainty, imprecision, and ambiguity that often exist in real-world circumstances. Unlike classical set theory, FS theory enables elements to have distinct degrees of membership ranging between 0 and 1. FSs have been mostly employed across multiple fields, including economics, engineering, D-making, medical diagnosis, control systems, cybersecurity, and artificial intelligence. Barlybayev and Turgibbayeva discussed the construction and application of a fuzzy expert system for evaluating information security threats [11]. Upadhyay et al. [12] utilized a fuzzy logic-based framework to predict cybersecurity risk. Yang [13] used multi-criteria D-making and fuzzy off logic approaches for the evaluation of intrusion detection systems. Akram et al. [14] proposed Bonferroni mean AOs for T-spherical FSs and employed them in MADM problems. Ruan et al. [15] demonstrated Bonferroni AOs within the framework of Fermatean FSs. They employed them to deal with MCDM problems. Wang et al. [16] developed a new MCDM approach using Hesitant Fermatean fuzzy Bonferroni AOs. Ma et al. [17] discussed a real-life problem with D-making approach using Bonferroni AOs within the framework of picture FSs. Basuri et al. [18] used neutrosophic CRITIC-COPRAS assessment approach for sustainable location problems based on decision analytics.

Since truth and falsity grades are important to each other, FS theory's information has been proposed utilizing only truth grades. Therefore, Atanassov [19] presented the notion of IFSs, based on both the truth and falsity grades. IFS are more advanced and comprehensive framework than the FS model. IFSs are crucial for addressing hesitation and uncertainty in a D-making environment. Hussain et al. [20] developed an innovative MCDM procedure using prioritized IF operators to evaluate sustainable digital security. Xie et al. [21] proposed a novel algorithm for cybersecurity using dynamic IFSs. Xu and Lan demonstrated an innovative MCDM technique using IF entropy measures to solve real-life problems [22]. Seikh and Chatterjee discussed a novel approach for selection and assessment of E-learning platforms based on IF confidence level-based Dombi AOs with uncertain weight information [23]. Ashraf et al. [24] introduced Einstein AOs based on IFSs. They employed the proposed approaches to solve MCDM problems. Unver discussed classification problems using IF Gaussian operators [25]. Seikh and Mandal developed an innovative D-making approach employing IFSs with Dombi AOs [26].

1.3. Analysis of Complex Fuzzy Set Theory and Their Extensions

Traditional FSs and IFSs cannot capture uncertainty in domains where the type of data is not only ambiguous in terms of its magnitude but also in terms of its timing or direction. Classical FSs present uncertainty utilizing truth grade with values in the closed interval [0,1]. However, they cannot handle a challenge that is affected by time, frequency, oscillation, or directional variation, all of which are prevalent in many real-world situations. To handle advanced forms of uncertainty, Ramot et al. [27] demonstrated the notions of CFSs by integrating both a membership degree and a phase component. The CFS model is very useful in dynamic, oscillatory, and time-dependent phenomena. Ullah et al. [28] designed a novel D-making technique for signal processing based on newly defined AOs of CFSs. Khan et al. [29] introduced new complex fuzzy DMs for the D-making procedure. They utilized them successfully in real-life applications. Anis et al. [30] discussed signal processing problems using CFSs with dynamical graphs. When analyzing two-dimensional data in the manner of truth and falsity grade values, the complex-valued truth grade is insufficient. Therefore, Alkouri and Salleh [31] proposed an advanced extension of classical CFSs, called CIFSs. They are constructed to represent uncertainty by integrating complex-valued truth and false grades. CIFSs are widely utilized in D-making problems. Azam et al. [32] established a novel MCDM procedure for cybersecurity problems based on CIFSs. Azam et al. [33] proposed new AOs for CIFSs and used them for choosing computer network security systems. Akram et al. [34] constructed a novel MCDM approach using CIF Hamacher AOs.

1.4. Analysis of Interval-valued Fuzzy Set Theory and Their Extensions

It is frequently challenging to assign an element a precise membership value in real-world situations. Classical FS theory fails when presenting incomplete information. For this, Gehrke et al. [35] formulated the framework of IVFSs to address limitations in the FS model. IVFS model offers a more adaptable and practical approach by representing truth grade with intervals rather than single numbers. Therefore, IFSs provide a more comprehensive structure for D-making problems. Vo and Nguyen proposed an innovative ranking method using IVFSs and rough sets [36]. Madi and Aziz utilized interval-valued numbers to demonstrate an enhanced fuzzy D-making approach [37]. Liu and Shu established Hamacher AOs using interval-valued Fermatean FSs [38]. They employed them to solve real-life applications. Further, Atanassov extends the notions of IFSs to IVIFSs [39]. The IVIFS model is a more generalized and powerful development of the IFS model, constructed to present higher extent of ambiguity, uncertainty, and hesitation in complex D-making tasks. Both truth grade and false grade in the IVIFS model are intervals rather than single values. IVIFSs are very useful in MCDM, expert systems, risk assessment, medical diagnosis, and economics. Imanov et al. [40] discussed the simulation of the Azerbaijan national cybersecurity index utilizing an IVIF approach. Imran et al. [41] developed an innovative approach of D-making using Aczel-Alsina Bonferroni AOs based on IVIFSs. Dong et al. [42] studied the framework of defense and attack games in infrastructure networks based on IVIF payoffs. Xiong et al. [43]

developed Hamy mean AOs based on IVIFSs. They used them to propose the D-making procedure. Zhang et al. [44] solved MCDM problems using Heronian mean AOs based on IVIFSs. Ali and Rasool established interval-valued q-rung orthopair fuzzy Aczel-Alsina operations using Bonferroni mean AOs and employed them in real-world applications [45].

Later on, Garg and Rani [46] generalized the notions of the IVIFS model to the CIVIFS model; provided a robust and more comprehensive mathematical structure to address phase-sensitive, multi-dimensional, and highly uncertain information. CIVIFS model integrates complex numbers, intervals, and IFS. They are useful in MCDM problems. Khan et al. [47] proposed an innovative D-making process for medical diagnosis problems using CIVIFSs. Nasir et al. [48] utilized CIVIF relations to cybersecurity problems. Su et al. [49] proposed an innovative representation of quaternion numbers and correlation coefficients based on CIVIFSs and discussed their real-life applications. Zindani et al. [50] demonstrated a novel TODIM method using CIVIFSs. They utilized the proposed techniques to solve D-making problems.

1.5. Research Gaps and Motivations

A CIVIFS, which integrates complex numbers, intervals, and IF data, is a well-known and dependable model for presenting ambiguous and uncertain data. Based on long-term observation, analytical assessment, and expert assessment highlighted a substantial research gap in the integration of advanced aggregation approaches with CIVIF information. Specifically, there is no established method for developing geometric Bonferroni operators within the CIVIFS framework to address and handle complex, uncertain data. Furthermore, previous literature fails to effectively address the construction of weighted geometric Bonferroni operators capable of dealing with two-dimensional interval-valued data, which is essential for addressing both magnitude and phase-related uncertainties. Finally, there is still a need to integrate these newly defined operators into practical D-making procedures, as present models do not handle the multifaceted and interrelated nature of real-world decision issues.

In D-making approaches such as TOPSIS, VIKOR, AHP, and BWM where the judgment of extremely vague and incomplete information is crucial, these findings offer serious challenges for specialists. To overcome these difficulties, a model based on geometric Bonferroni operators or weighted geometric Bonferroni operators for CIVIF proves to be very beneficial and reliable. The strengths of FS, IFS, IVFS, IVIFS, and CIFS frameworks were demonstrated in the preceding subsections. But there are still a number of limitations. However, CIVIFSs have not been deeply applied or investigated in the literature, despite their complex capacity to represent uncertainty in both magnitude and phase. According to our analysis, CIVIFSs are still not widely utilized, especially when they apply to proposed geometric Bonferroni AOs. Furthermore, a considerable research gap is observed regarding the incorporation of geometric Bonferroni AOs within the framework of CIVIFSs. A framework that simultaneously integrates geometric Bonferroni AOs and weighted geometric Bonferroni AOs for this kind of fuzzy environment has not yet been proposed. Moreover, Furthermore, while Bonferroni geometric aggregation operators have shown potential in representing attribute interactions and improving decision accuracy, their integration with CIVIFSs for cybersecurity tool selection has not received sufficient consideration in the current literature. There is currently no comprehensive MCDM framework that evaluates and ranks cybersecurity technologies by combining the modeling capacity of CIVIFSs with the interaction-aware and correlation-sensitive capabilities of Bonferroni geometric AOs.

The Bonferroni operator improves real-life modeling by capturing interrelationships between criteria that are generally neglected by traditional aggregation approaches. This is especially useful in MCDM problems, when the criteria are not mutually independent and their interactions affect the final outcome. The Bonferroni mean's flexibility, as defined by its configurable parameters, enables decision-makers to fine-tune the influence of these interdependencies based on the problem's specific context. Furthermore, integrating Bonferroni aggregation and CIVIFS strengthens and improves the D-making process in uncertain contexts. It also offers a more comprehensive evaluation by considering not only the magnitude and direction (phase) of information but also the ambiguity and uncertainty represented through intervals. This hybrid method presents a significant tool for integrating expert information while maintaining the complexity of human reasoning. These advanced proposed AOs with framework of CIVIFSs are well-equipped to solve complex D-making problems.

1.6. Advantages and Main Contributions

In this subsection, we highlight the main advantages of CIVIFSs with the proposed approaches, such as geometric Bonferroni AOs and weighted geometric Bonferroni AOs. The proposed study provides numerous substantial advantages by integrating and expanding existing AOs within the CIVIFS framework. Specifically, it extends multiple well-known illustrations, including the conventional geometric operator, the complement geometric operator, and the weighted geometric operator, the standard geometric Bonferroni AO, all of which have been introduced for simpler fuzzy structures. By adapting the proposed operators from traditional FSs to the more expressive CIVIFS environment, the newly defined approach improves the accuracy and application of fuzzy D-making frameworks in uncertain and complex settings.

This study further explores the theory of CIVIFSs. We propose innovative Bonferroni operational laws for CIVIFSs. We develop novel geometric Bonferroni AOs and weighted geometric Bonferroni AOs using CIVIFSs. Essential properties of the newly defined AOs, such as idempotency, monotonicity, and boundness are demonstrated. We introduce innovative DMs using CIVIFSs. A new approach to D-making using proposed AOs and DMs is constructed. We solve cybersecurity problems based on the proposed D-making framework. Moreover, a comparative discussion is presented on the superiority of the proposed techniques and the existing approaches.

1.7. Manuscript Summary

Consequently, this study has been divided as follows. Section 2 revises the basic notions of CIVIFSs. In section 3, we demonstrate the basic Bonferroni operational laws in the CIVIF environment. Section 4 provides geometric Bonferroni AOs and weighted geometric Bonferroni AOs for CIVIFSs. Section 5 introduces innovative DMs for CIVIFSs. In section 6, we develop a novel D-making algorithm based on the proposed approaches. Section 7 discussed a case study using a newly demonstrated D-making approach. In section 8, we present a comparative study of the proposed approaches and existing approaches. Section 8 provides the conclusion of the proposed study.

The proposed work is presented in detail in Figure 1.

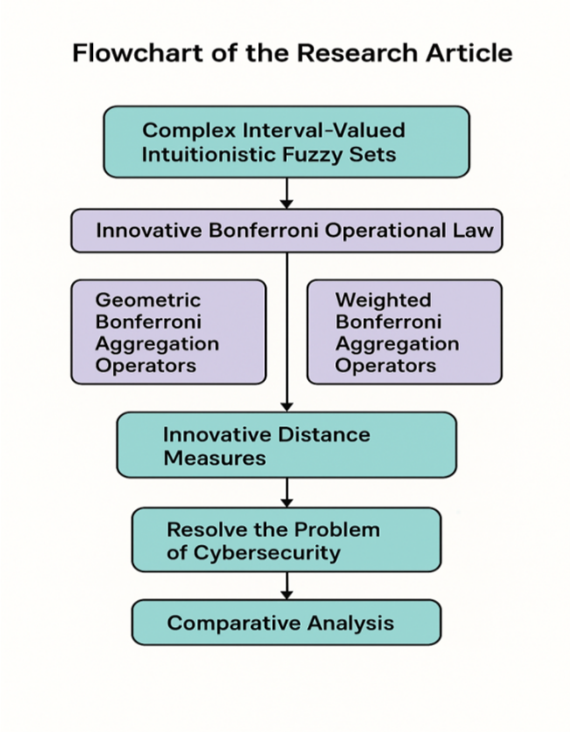

Figure 1: Detailed Diagram of the Proposed Study.

2. Preliminaries

This section provides a review of the fundamental concepts of CIVIFSs.

Definition 1 ([19]). *Let U represents a universal set (US). An IFS $\mathcal{I}$ defined over U is given as*

$$\mathcal{I} = \{(\upsilon, \langle T_{\mathcal{I}}(\upsilon), F_{\mathcal{I}}(\upsilon)\rangle) \mid \upsilon \in U\}$$

where $T_{\mathcal{I}}(\upsilon)$ and $F_{\mathcal{I}}(\upsilon)$ are identified as membership (MS) and non-membership (NMS) functions, respectively which satisfy the condition $0 \leq T_{\mathcal{I}}(\upsilon) + F_{\mathcal{I}}(\upsilon) \leq 1$.

Definition 2 ([39]). *Let U represents a US. An IVIFS $\ddot{I}$ demonstrated over U is formulated as*

$$\ddot{I} = \left\{ \left(v, \left\langle \left[\underset{\cdot}{a}_{\ddot{I}}, \dot{b}_{\ddot{I}} \right], \left[\dot{c}_{\ddot{I}}, \underset{\wedge}{d}_{\ddot{I}} \right] \right\rangle \right) \middle| v \in U \right\},$$

where $\left[\underset{\cdot}{a}_{\ddot{I}}, \dot{b}_{\ddot{I}} \right]$ and $\left[\dot{c}_{\ddot{I}}, \underset{\wedge}{d}_{\ddot{I}} \right]$ are identified as MS and NMS functions, respectively which satisfy the condition $0 \leq \dot{b}_{\ddot{I}} + \underset{\wedge}{d}_{\ddot{I}} \leq 1$.

Definition 3 ([31]). *Let U demonstrate any US. A CIFS $\dot{C}$ introduced over U is formulated as*

$$\dot{C} = \left\{ \left(v, \left\langle T_{\dot{C}}(v) e^{I 2\pi \eta_{\dot{C}}(v)}, F_{\dot{C}}(v) e^{I 2\pi \sigma_{\dot{C}}(v)} \right\rangle \right) \middle| v \in U \right\},$$

where $T_{\dot{C}}(v) e^{I 2\pi \eta_{\dot{C}}(v)}$ and $F_{\dot{C}}(v) e^{I 2\pi \sigma_{\dot{C}}(v)}$ are identified as MS and NMS functions, respectively which satisfy the condition $0 \leq T_I(v) + F_I(v) \leq 1$ and $0 \leq 2\pi\eta_{\dot{C}}(v) + 2\pi\sigma_{\dot{C}}(v) \leq 1$.

Definition 4 ([46]). *Let U demonstrate any US. A CIVIFS $\mathcal{C}$ defined over U is formulated as*

$$\mathcal{C} = \left\{ \left(v, \left\langle \left[\underset{\cdot}{a}_{\mathcal{C}}, \dot{b}_{\mathcal{C}} \right] e^{i\left[2\pi\lambda_{\mathcal{C}}, 2\pi\sigma_{\mathcal{C}}\right]}, \left[\dot{c}_{\mathcal{C}}, \underset{\wedge}{d}_{\mathcal{C}} \right] e^{i\left[2\pi\eta_{\mathcal{C}}, 2\pi\gamma_{\mathcal{C}}\right]} \right\rangle \right) \middle| v \in U \right\},$$

where $\left[\underset{\cdot}{a}_{\mathcal{C}}, \dot{b}_{\mathcal{C}} \right] e^{i\left[2\pi\lambda_{\mathcal{C}}, 2\pi\sigma_{\mathcal{C}}\right]}$ and $\left[\dot{c}_{\mathcal{C}}, \underset{\wedge}{d}_{\mathcal{C}} \right] e^{i\left[2\pi\eta_{\mathcal{C}}, 2\pi\gamma_{\mathcal{C}}\right]}$ are identified as MS and NMS functions, respectively which satisfy the condition $0 \leq \dot{b}_{\mathcal{C}} + \underset{\wedge}{d}_{\mathcal{C}} \leq 1$ and $0 \leq 2\pi\sigma_{\mathcal{C}}(v) + 2\pi\gamma_{\mathcal{C}}(v) \leq 1$.

3. Complex Interval-Valued Intuitionistic Fuzzy Bonferroni Operational Laws

This section demonstrates innovative Bonferroni operation laws for CIVIFNs.

For simplification, assume that $\mathbb{C}_j = \left(\left[\underset{\cdot}{a}_j, \dot{b}_j \right] e^{i[2\pi\lambda_j, 2\pi\sigma_j]}, \left[\dot{c}_j, \underset{\wedge}{d}_j \right] e^{i[2\pi\eta_j, 2\pi\gamma_j]} \right)$ represent a CIVIFN, where $\underset{\cdot}{a}_j$, $\dot{b}_j$, λ_j, σ_j, $\dot{c}_j, \underset{\wedge}{d}_j$, $\eta_j, \gamma_j \in [0,1]$ with $\dot{b}_j + \underset{\wedge}{d}_j \leq 1$ and $2\pi\sigma_j + 2\pi\gamma_j \leq 2\pi$ or $\sigma_j + \gamma_j \leq 1$.

The newly defined innovative Bonferroni operational laws are demonstrated as follows:

Definition 5. *Let $\mathcal{C}_j = \begin{pmatrix} \left[\underset{\cdot}{a}_j, \dot{b}_j \right] e^{i[2\pi\lambda_j, 2\pi\sigma_j]}, \\ \left[\dot{c}_j, \underset{\wedge}{d}_j \right] e^{i[2\pi\eta_j, 2\pi\gamma_j]} \end{pmatrix}$ and $\mathcal{C}_k = \begin{pmatrix} \left[\underset{\cdot}{a}_k, \dot{b}_k \right] e^{i[2\pi\lambda_k, 2\pi\sigma_k]}, \\ \left[\dot{c}_k, \underset{\wedge}{d}_k \right] e^{i[2\pi\eta_k, 2\pi\gamma_k]} \end{pmatrix}$ describe two CIVIFNs that are defined on a US U. Let $\check{S}$, $\check{T} > 0$, then the proposed Bonferroni operational law is introduced as:*

$$\mathcal{C}_j \otimes \mathcal{C}_k =$$

$$\left(\left[\underset{\cdot}{a}_j, \dot{b}_j \right] e^{i[2\pi\lambda_j, 2\pi\sigma_j]}, \left[\dot{c}_j, \underset{\wedge}{d}_j \right] e^{i[2\pi\eta_j, 2\pi\gamma_j]} \right) \otimes \left(\left[\underset{\cdot}{a}_k, \dot{b}_k \right] e^{i[2\pi\lambda_k, 2\pi\sigma_k]}, \left[\dot{c}_k, \underset{\wedge}{d}_k \right] e^{i[2\pi\eta_k, 2\pi\gamma_k]} \right),$$

$\mathbb{C}_j \otimes \mathbb{C}_k =$

$$\left(\begin{array}{c} \left[\begin{array}{c} 1-\left(1-\left[\left(1-(1-\underset{\cdot}{a}_j)^{\check{S}}.(1-\underset{\cdot}{a}_k)^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1-(1-\underset{\cdot}{a}_k)^{\check{S}}.(1-\underset{\cdot}{a}_j)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ 1-\left(1-\left[\left(1-\left(1-\dot{b}_j\right)^{\check{S}}.\left(1-\dot{b}_k\right)^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1-\left(1-\dot{b}_k\right)^{\check{S}}.\left(1-\dot{b}_j\right)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{array}\right] \\ e^{i\left[\begin{array}{c} 2\pi\left(1-\left(1-\left[\left(1-(1-\lambda_j)^{\check{S}}.(1-\lambda_k)^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1-(1-\lambda_k)^{\check{S}}.(1-\lambda_j)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\right), \\ 2\pi\left(\left(1-\left(1-\left[\left(1-(1-\sigma_j)^{\check{S}}.(1-\sigma_k)^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1-(1-\sigma_k)^{\check{S}}.(1-\sigma_j)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\right)\right) \end{array}\right]}, \\ \left[\begin{array}{c} \left(1-\left[\left(1-\dot{c}_j{}^{\check{S}}\dot{c}_k{}^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1-\dot{c}_k{}^{\check{S}}.\dot{c}_j{}^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ \left(1-\left[\left(1-\underset{\hat{}}{d}_j{}^{\check{S}}\underset{\hat{}}{d}_k{}^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1-\underset{\hat{}}{d}_k{}^{\check{S}}.\underset{\hat{}}{d}_j{}^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{array}\right] e^{i\left[\begin{array}{c} 2\pi\left(1-\left[\left(1-\eta_j{}^{\check{S}}\eta_k{}^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1-\eta_k{}^{\check{S}}.\eta_j{}^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ 2\pi\left(1-\left[\left(1-\gamma_j{}^{\check{S}}\gamma_k{}^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1-\gamma_k{}^{\check{S}}.\gamma_j{}^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{array}\right]} \end{array}\right).$$

Example 1. *Let* $\mathbb{C}_j = \begin{pmatrix} [0.1, 0.5]e^{i[2\pi(0.2), 2\pi(0.6)]}, \\ [0.2, 0.4]e^{i[2\pi(0.1), 2\pi(0.3)]} \end{pmatrix}$ *and* $\mathbb{C}_j = \begin{pmatrix} [0.4, 0.7]e^{i[2\pi(0.1), 2\pi(0.4)]}, \\ [0.1, 0.3]e^{i[2\pi(0.2), 2\pi(0.6)]} \end{pmatrix}$ *be two CIVIFSs. For* $\check{S} = 2,\ \check{T} = 3,$ *then*

$\mathbb{C}_j \otimes \mathbb{C}_k =$

$$\left([0.1, 0.5]e^{i[2\pi(0.2), 2\pi(0.6)]}, [0.2, 0.4]e^{i[2\pi(0.1), 2\pi(0.3)]}\right) \otimes \left([0.4, 0.7]e^{i[2\pi(0.1), 2\pi(0.4)]}, [0.1, 0.3]e^{i[2\pi(0.2), 2\pi(0.6)]}\right),$$

$$\mathbb{C}_j \otimes \mathbb{C}_k = \left([0.25, 0.45]e^{i[2\pi(0.25), 2\pi(0.41)]}, [0.18, 0.49]e^{i[2\pi(0.21), 2\pi(0.45)]}\right).$$

Definition 6. *Let* $\mathbb{C}_j = \begin{pmatrix} \left[\underset{\cdot}{a}_j, \dot{b}_j\right] e^{i[2\pi\lambda_j, 2\pi\sigma_j]}, \\ \left[\dot{c}_j, \underset{\hat{}}{d}_j\right] e^{i[2\pi\eta_j, 2\pi\gamma_j]} \end{pmatrix}$ *and* $\mathbb{C}_k = \begin{pmatrix} \left[\underset{\cdot}{a}_k, \dot{b}_k\right] e^{i[2\pi\lambda_k, 2\pi\sigma_k]}, \\ \left[\dot{c}_k, \underset{\hat{}}{d}_k\right] e^{i[2\pi\eta_k, 2\pi\gamma_k]} \end{pmatrix}$ *describe two CIVIFNs that are defined on a US* U*. Let* $\check{S},\ \check{T} > 0$*, then the proposed weighted Bonferroni operational law is introduced as:*

$\mathfrak{w}_j \mathbb{C}_j \otimes \mathfrak{w}_k \mathbb{C}_k =$

$$\left(\begin{array}{c}
\left[\begin{array}{c}
1-\left(1-\left[\left(1-(1-\underset{\cdot}{a}_j{}^{\mathfrak{w}_j})^{\check{S}}.(1-\underset{\cdot}{a}_k{}^{\mathfrak{w}_k})^{\check{T}}\right)^{\frac{1}{2(2-1)}}\times\left(1-(1-\underset{\cdot}{a}_k{}^{\mathfrak{w}_k})^{\check{S}}.(1-\underset{\cdot}{a}_j{}^{\mathfrak{w}_j})^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\
1-\left(1-\left[\left(1-\left(1-\dot{b}_j{}^{\mathfrak{w}_j}\right)^{\check{S}}.\left(1-\dot{b}_k{}^{\mathfrak{w}_k}\right)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\times\left(1-\left(1-\dot{b}_k{}^{\mathfrak{w}_k}\right)^{\check{S}}.\left(1-\dot{b}_j{}^{\mathfrak{w}_j}\right)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}
\end{array}\right] \\
e^{i\left[\begin{array}{c}
2\pi\left(1-\left(1-\left[\left(1-(1-\lambda_j{}^{\mathfrak{w}_j})^{\check{S}}.(1-\lambda_k{}^{\mathfrak{w}_k})^{\check{T}}\right)^{\frac{1}{2(2-1)}}\times\left(1-(1-\lambda_k{}^{\mathfrak{w}_k})^{\check{S}}.(1-\lambda_j{}^{\mathfrak{w}_j})^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\right), \\
2\pi\left(\left(1-\left(1-\left[\left(1-(1-\sigma_j{}^{\mathfrak{w}_j})^{\check{S}}.(1-\sigma_j{}^{\mathfrak{w}_j})^{\check{T}}\right)^{\frac{1}{2(2-1)}}\times\left(1-(1-\sigma_j{}^{\mathfrak{w}_j})^{\check{S}}.(1-\sigma_j{}^{\mathfrak{w}_j})^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\right)\right)
\end{array}\right]}, \\
\left[\begin{array}{c}
\left(1-\left[\begin{array}{c}\left(1-(1-(1-\dot{c}_j)^{\mathfrak{w}_j})^{\check{S}}.(1-(1-\dot{c}_k)^{\mathfrak{w}_k})^{\check{T}}\right)^{\frac{1}{2(2-1)}}\times \\ \left(1-(1-(1-\dot{c}_k)^{\mathfrak{w}_k})^{\check{S}}.(1-(1-\dot{c}_j)^{\mathfrak{w}_j})^{\check{T}}\right)^{\frac{1}{2(2-1)}}\end{array}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\
\left(1-\left[\begin{array}{c}\left(1-\left(1-\left(1-\underset{\hat{}}{d}_j\right)^{\mathfrak{w}_j}\right)^{\check{S}}.\left(1-\left(1-\underset{\hat{}}{d}_k\right)^{\mathfrak{w}_k}\right)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\times \\ \left(1-\left(1-\left(1-\underset{\hat{}}{d}_k\right)^{\mathfrak{w}_k}\right)^{\check{S}}.\left(1-\left(1-\underset{\hat{}}{d}_j\right)^{\mathfrak{w}_j}\right)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\end{array}\right]\right)^{\frac{1}{\check{S}+\check{T}}}
\end{array}\right] \\
e^{i\left[\begin{array}{c}
\left(1-\left[\begin{array}{c}\left(1-(1-(1-\eta_j)^{\mathfrak{w}_j})^{\check{S}}.(1-(1-\eta_k)^{\mathfrak{w}_k})^{\check{T}}\right)^{\frac{1}{2(2-1)}}\times \\ \left(1-(1-(1-\eta_k)^{\mathfrak{w}_k})^{\check{S}}.(1-(1-\eta_j)^{\mathfrak{w}_j})^{\check{T}}\right)^{\frac{1}{2(2-1)}}\end{array}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\
\left(1-\left[\begin{array}{c}\left(1-(1-(1-\gamma_j)^{\mathfrak{w}_j})^{\check{S}}.(1-(1-\gamma_k)^{\mathfrak{w}_k})^{\check{T}}\right)^{\frac{1}{2(2-1)}}\times \\ \left(1-(1-(1-\gamma_k)^{\mathfrak{w}_k})^{\check{S}}.(1-(1-\gamma_j)^{\mathfrak{w}_j})^{\check{T}}\right)^{\frac{1}{2(2-1)}}\end{array}\right]\right)^{\frac{1}{\check{S}+\check{T}}}
\end{array}\right]}
\end{array}\right)$$

Theorem 1. *Let* $\mathbb{C}_j = \left(\begin{array}{c}\left[\underset{\cdot}{a}_j, \dot{b}_j\right] e^{i[2\pi\lambda_j, 2\pi\sigma_j]}, \\ \left[\dot{c}_j, \underset{\hat{}}{d}_j\right] e^{i[2\pi\eta_j, 2\pi\gamma_j]}\end{array}\right)$ *and* $\mathbb{C}_k = \left(\begin{array}{c}\left[\underset{\cdot}{a}_k, \dot{b}_k\right] e^{i[2\pi\lambda_k, 2\pi\sigma_k]}, \\ \left[\dot{c}_k, \underset{\hat{}}{d}_k\right] e^{i[2\pi\eta_k, 2\pi\gamma_k]}\end{array}\right)$ *describe two complex interval-valued intuitionistic fuzzy numbers that are defined on a universal set* U*. Then,*

i) $\mathbb{C}_j \otimes \mathbb{C}_k$ *is a CIVIFN.*

ii) $\mathfrak{w}_j \mathbb{C}_j \otimes \mathfrak{w}_k \mathbb{C}_k$ *is a CIVIFN.*

The proof for Theorem 1 is provided in Appendix A.

Theorem 2. *Let* $\mathbb{C}_j = \left(\begin{array}{c}\left[\underset{\cdot}{a}_j, \dot{b}_j\right] e^{i[2\pi\lambda_j, 2\pi\sigma_j]}, \\ \left[\dot{c}_j, \underset{\hat{}}{d}_j\right] e^{i[2\pi\eta_j, 2\pi\gamma_j]}\end{array}\right)$ *and* $\mathbb{C}_k = \left(\begin{array}{c}\left[\underset{\cdot}{a}_k, \dot{b}_k\right] e^{i[2\pi\lambda_k, 2\pi\sigma_k]}, \\ \left[\dot{c}_k, \underset{\hat{}}{d}_k\right] e^{i[2\pi\eta_k, 2\pi\gamma_k]}\end{array}\right)$ *describe two complex interval-valued intuitionistic fuzzy numbers that are defined on a universal set* U*. Then,*

i) $\mathbb{C}_j \otimes \mathbb{C}_k = \mathbb{C}_k \otimes \mathbb{C}_j$.

ii) $\mathfrak{w}_j \mathbb{C}_j \otimes \mathfrak{w}_k \mathbb{C}_k = \mathfrak{w}_k \mathbb{C}_k \otimes \mathfrak{w}_j \mathbb{C}_j$.

The proof for Theorem 2 is provided in Appendix B.

Theorem 3. *Let* $\mathbb{C}_j = \left(\begin{array}{c}\left[\underset{\cdot}{a}_j, \dot{b}_j\right] e^{i[2\pi\lambda_j, 2\pi\sigma_j]}, \\ \left[\dot{c}_j, \underset{\hat{}}{d}_j\right] e^{i[2\pi\eta_j, 2\pi\gamma_j]}\end{array}\right)$ *and* $\mathbb{C}_k = \left(\begin{array}{c}\left[\underset{\cdot}{a}_k, \dot{b}_k\right] e^{i[2\pi\lambda_k, 2\pi\sigma_k]}, \\ \left[\dot{c}_k, \underset{\hat{}}{d}_k\right] e^{i[2\pi\eta_k, 2\pi\gamma_k]}\end{array}\right)$

describe two complex interval-valued intuitionistic fuzzy numbers that are defined on a universal set U. Then,

i) $(\mathcal{C}_j \otimes \mathcal{C}_k) \otimes \mathcal{C}_l = \mathcal{C}_j \otimes (\mathcal{C}_k \otimes \mathcal{C}_l)$.

ii) $(\mathfrak{w}_j \mathcal{C}_j \otimes \mathfrak{w}_k \mathcal{C}_k) \otimes \mathfrak{w}_l \mathcal{C}_l = \mathfrak{w}_j \mathcal{C}_j \otimes (\mathfrak{w}_k \mathcal{C}_k \otimes \mathfrak{w}_l \mathcal{C}_l)$.

Proof: The proof is obvious.

4. Proposed Complex Interval-Valued Intuitionistic Fuzzy Geometric Bonferroni Aggregation Operators (CIVIFGBAOs)

This section demonstrates some novel geometric Bonferroni AOs using CIVIFNs. The necessity to effectively manage the interdependencies between criteria in uncertain D-making environments motivates the development of Bonferroni functions in the CIVIFS framework. Traditional AOs, such as geometric or arithmetic means, assume that all attributes are independent, which is rarely true in real-world situations. The Bonferroni function addresses this restriction by enabling interactions between pairs of inputs via its parameters, leading to more comprehensive modeling of mutual influence among criteria. Its parameterized structure facilitates decision-makers to modify the level of interaction sensitivity, which is useful for adapting the model to specific problem situations. When incorporated with the CIVIFS model, which already offers complex uncertainty via interval-valued membership and non-membership degrees, as well as phase information, the Bonferroni function provides another layer of precision and flexibility.

Some of the desirable properties of the proposed CIVIFGBAOs have been discussed. In the proposed notion, C_j defines a family of all CIVIFNs, which is,

$$V_j = \begin{pmatrix} [a_j, b_j]\, e^{i[2\pi\lambda_j, 2\pi\sigma_j]}, \\ [c_j, d_j]\, e^{i[2\pi\eta_j, 2\pi\gamma_j]} \end{pmatrix}$$

where $a_j, b_j, \lambda_j, \sigma_j, c_j, d_j, \eta_j, \gamma_j \in [0,1]$ and w_j indicate the corresponding positive weighted vector of C_j with $\sum_{j=1}^{n} w_j$.

The innovative CIVIFGBAOs are introduced based on the proposed operational laws, which are given as:

Definition 7. *Let* $V_j = \begin{pmatrix} [a_j, b_j]\, e^{i[2\pi\lambda_j, 2\pi\sigma_j]}, \\ [c_j, d_j]\, e^{i[2\pi\eta_j, 2\pi\gamma_j]} \end{pmatrix}$ *be a family of n CIVIFNs. Then, CIVIFGBAO is a map CIVIFGBAO:* ${V_j}^n \to V_j$ *defined by*

$$CIVIFGBAO(V_1, V_2, V_3, \ldots, V_n) = \otimes_{j,k=1}^{n} (V_j \otimes V_k); \quad j \neq k.$$

$CIVIFGBAO(V_1, V_2, V_3, \ldots, V_n) =$

$$\left(\begin{array}{r} \left[\begin{array}{c} 1-\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-(1-a_j)^p\cdot(1-a_k)^q\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{p+q}}, \\ 1-\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-(1-b_j)^p\cdot(1-b_k)^q\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{p+q}} \end{array}\right] e^{i\left[\begin{array}{c} 2\pi\left(1-\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-(1-\lambda_j)^p\cdot(1-\lambda_k)^q\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{p+q}}\right), \\ 2\pi\left(1-\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-(1-\sigma_j)^p\cdot(1-\sigma_k)^q\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{p+q}}\right) \end{array}\right]}, \\ \left[\begin{array}{c} \left(1-\left[\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left(1-{c_j}^p\cdot{c_k}^q\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{p+q}}, \\ \left(1-\left[\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left(1-{d_j}^p\cdot{d_k}^q\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{p+q}} \end{array}\right] e^{i\left[\begin{array}{c} 2\pi\left(1-\left[\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left(1-{\eta_j}^p\cdot{\eta_k}^q\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{p+q}}, \\ 2\pi\left(1-\left[\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left(1-{\gamma_j}^p\cdot{\gamma_k}^q\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{p+q}}, \end{array}\right]} \end{array}\right).$$

Definition 8. *Let* $V_j = \begin{pmatrix} [a_j, b_j]\, e^{i[2\pi\lambda_j, 2\pi\sigma_j]}, \\ [c_j, d_j]\, e^{i[2\pi\eta_j, 2\pi\gamma_j]} \end{pmatrix}$ *be a family of* n *CIVIFNs. Then, CIVIFWGBAO is a map CIVIFWGBAO:* ${V_j}^n \to V_j$ *defined by*

$$CIVIFWGBAO(V_1, V_2, V_3, \ldots, V_n) = \otimes_{j,k=1}^{n} w_j V_j \otimes w_k V_k); \quad j \neq k.$$

$CIVIFWGBAO(V_1, V_2, V_3, \ldots, V_n) =$

$$\left(\begin{array}{r} \left[\begin{array}{c} 1-\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-(1-{a_j}^{w_j})^p\cdot(1-{a_k}^{w_k})^q\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{p+q}}, \\ 1-\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-(1-{b_j}^{w_j})^p\cdot(1-{b_k}^{w_k})^q\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{p+q}} \end{array}\right] e^{i\left[\begin{array}{c} 2\pi\left(1-\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-(1-{\lambda_j}^{w_j})^p\cdot(1-{\lambda_k}^{w_k})^q\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{p+q}}\right), \\ 2\pi\left[2\pi\left(1-\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-(1-{\sigma_j}^{w_j})^p\cdot(1-{\sigma_k}^{w_k})^q\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{p+q}}\right)\right] \end{array}\right]}, \\ \left[\begin{array}{c} \left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-(1-(1-c_j)^{w_j})^p\cdot(1-(1-c_k)^{w_k})^q\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{p+q}}, \\ \left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-(1-(1-d_j)^{w_j})^p\cdot(1-(1-d_k)^{w_k})^q\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{p+q}} \end{array}\right] e^{i\left[\begin{array}{c} 2\pi\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-(1-(1-\eta_j)^{w_j})^p\cdot(1-(1-\eta_k)^{w_k})^q\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{p+q}}, \\ 2\pi\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-(1-(1-\gamma_j)^{w_j})^p\cdot(1-(1-\gamma_k)^{w_k})^q\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{p+q}} \end{array}\right]} \end{array}\right),$$

where w_j *represent the* j*-th weight with* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{n} w_j$.

Theorem 4. *Let* $V_j = \begin{pmatrix} [a_j, b_j]\, e^{i[2\pi\lambda_j, 2\pi\sigma_j]}, \\ [c_j, d_j]\, e^{i[2\pi\eta_j, 2\pi\gamma_j]} \end{pmatrix}$ *represent a family of* n *complex interval-valued intuitionistic fuzzy numbers. Then, for* $p, q > 0$, *the aggregated value of* C_j *by using CIVIFGBAO is a complex interval-valued intuitionistic fuzzy number.*

The proof for Theorem 4 is provided in Appendix C.

Theorem 5. *Let* $V_j = \begin{pmatrix} [a_j, b_j]\, e^{i[2\pi\lambda_j, 2\pi\sigma_j]}, \\ [c_j, d_j]\, e^{i[2\pi\eta_j, 2\pi\gamma_j]} \end{pmatrix}$ *represent a family of* n *complex interval-valued intuitionistic fuzzy numbers. Then, for* $p, q > 0$, *the aggregated value of* C_j *by using CIVIFGBAO is a CIVIFN.*

Proof: *The proof of the given statement is straightforward and follows the same approach as Theorem 4.*

Theorem 6 (Idempotency)**.** *If all $V_j; j = 1, 2, 3, ..., n$, are equal, that is, $V_j = V$ for all j, then*

$$CIVIFGBAO(V_1, V_2, V_3, \ldots, V_n) = V.$$

The proof for Theorem 6 is provided in Appendix D.

Theorem 7 (Monotonicity)**.** *Let* $V_j = \begin{pmatrix} [a_j, b_j]\, e^{i[2\pi\lambda_j, 2\pi\sigma_j]}, \\ [c_j, d_j]\, e^{i[2\pi\eta_j, 2\pi\gamma_j]} \end{pmatrix}$ *and* $V_j' = \begin{pmatrix} \left[a_j', b_j'\right] e^{i\left[2\pi\lambda_j', 2\pi\sigma_j'\right]}, \\ \left[c_j', d_j'\right] e^{i\left[2\pi\eta_j', 2\pi\gamma_j'\right]} \end{pmatrix}$ *be collection of complex interval-valued intuitionistic fuzzy numbers such that* $a_j \le a_j',\ b_j \le b_j',\ \lambda_j \le \lambda_j',\ \sigma_j \le \sigma_j',\ c_j \ge c_j',\ d_j \ge d_j',\ \eta_j \ge \eta_j',\ \gamma_j \ge \gamma_j'$ *for all j, then*

$$CIVIFGBAO(V_1, V_2, V_3, \ldots, V_n) \le CIVIFGBAO\left(V_1', V_2', V_3', \ldots, V_n'\right).$$

The proof for Theorem 7 is provided in Appendix E.

Theorem 8 (Monotonicity)**.** *Let* $V_j = \begin{pmatrix} [a_j, b_j]\, e^{i[2\pi\lambda_j, 2\pi\sigma_j]}, \\ [c_j, d_j]\, e^{i[2\pi\eta_j, 2\pi\gamma_j]} \end{pmatrix}$ *and* $V_j' = \begin{pmatrix} \left[a_j', b_j'\right] e^{i\left[2\pi\lambda_j', 2\pi\sigma_j'\right]}, \\ \left[c_j', d_j'\right] e^{i\left[2\pi\eta_j', 2\pi\gamma_j'\right]} \end{pmatrix}$ *be collection of complex interval-valued intuitionistic fuzzy numbers such that* $a_j \le a_j',\ b_j \le b_j',\ \lambda_j \le \lambda_j',\ \sigma_j \le \sigma_j',\ c_j \ge c_j',\ d_j \ge d_j',\ \eta_j \ge \eta_j',\ \gamma_j \ge \gamma_j'$ *for all j, then*

$$CIVIFWGBAO(V_1, V_2, V_3, \ldots, V_n) \le CIVIFWGBAO\left(V_1', V_2', V_3', \ldots, V_n'\right).$$

Proof: *The proof of the given statement is straightforward and follows the same approach as Theorem 7.*

Theorem 9. *If all $C_j; j = 1, 2, 3, \ldots, n$ be a family of n complex interval-valued intuitionistic fuzzy numbers. Then,*

i) $CIVIFGBAO(V_1, V_2, V_3, \ldots, V_n) = CIVIFGBAO(V_1, V_2, V_3, \ldots, V_n)$.

ii) $CIVIFWGBAO(V_1, V_2, V_3, \ldots, V_n) = CIVIFWGBAO(V_1, V_2, V_3, \ldots, V_n)$.

where $V_1, V_2, V_3, \ldots, V_n$ is any permutation of $V_1, V_2, V_3, \ldots, V_n$.

Proof: *The proof of the given statement is straightforward and follows the same approach as Theorem 2.*

Theorem 10. *If all $C_j; j = 1, 2, 3, \ldots, n$ be a family of n complex interval-valued intuitionistic fuzzy numbers. Then,*

i) $\overline{V} \le CIVIFGBAO(V_1, V_2, V_3, \ldots, V_n) \le V^+$,

ii) $\overline{V} \le CIVIFWGBAO(V_1, V_2, V_3, \ldots, V_n) \le V^+$,

where

$$\overline{V} = \begin{pmatrix} [\min_j a_j, \min_j b_j]\, e^{i[2\pi \min_j \lambda_j, 2\pi \min_j \sigma_j]}, \\ [\max_j c_j, \max_j d_j]\, e^{i[2\pi \max_j \eta_j, 2\pi \max_j \gamma_j]} \end{pmatrix},$$

and

$$V^+ = \begin{pmatrix} [\max_j a_j, \max_j b_j]\, e^{i[2\pi \min_j \lambda_j, 2\pi \min_j \sigma_j]}, \\ [\min_j c_j, \min_j d_j]\, e^{i[2\pi \min_j \eta_j, 2\pi \min_j \gamma_j]} \end{pmatrix}.$$

Proof: *The proof of the given statement is straightforward and follows the Monotonicity Property.*

5. Complex Interval-Valued Intuitionistic Fuzzy Distance Measures

This section presents some novel DMs for CIVIFSs. Here, V_j denotes a CIVIFS and $V(U)$ represents a family of CIVIFSs and w_j be the corresponding weighted vector of C_j with $\sum_{j=1}^{n} w_j$.

Definition 9. *Let $V(U)$ denote a collection of CIVIFSs and d be a mapping defined on $V(U)$ such that $d : V(U) \times V(U) \rightarrow [0,1]$. Then, d is termed a DM of CIVIFSs if the given conditions are fulfilled.*

i) $0 \leq d(V_j, V_k) \leq 1$,

ii) $d(V_j, V_k) = 0$, *if and only if* $V_j = V_k$,

iii) $d(V_j, V_k) = d(V_k, V_j)$,

iv) $d(V_j, V_l) \leq d(V_j, V_k) + d(V_k, V_l)$.

The function proposed for DM of CIVIFSs can be defined as:

$$d(V_j, V_k) =$$

$$\left[\frac{\begin{array}{c} \sup_{\upsilon_i \in U} |a_j(\upsilon_i) - a_k(\upsilon_i)| + \sup_{\upsilon_i \in U} |b_j(\upsilon_i) - b_k(\upsilon_i)| + \frac{1}{2\pi} \sup_{\upsilon_i \in U} |2\pi\lambda_j(\upsilon_i) - 2\pi\lambda_k(\upsilon_i)| + \frac{1}{2\pi} \sup_{\upsilon_i \in U} |2\pi\sigma_j(\upsilon_i) - 2\pi\sigma_k(\upsilon_i)| + \\ \sup_{\upsilon_i \in U} |c_j(\upsilon_i) - c_k(\upsilon_i)| + \sup_{\upsilon_i \in U} |d_j(\upsilon_i) - d_k(\upsilon_i)| + \frac{1}{2\pi} \sup_{\upsilon_i \in U} |2\pi\eta_j(\upsilon_i) - 2\pi\eta_k(\upsilon_i)| + \frac{1}{2\pi} \sup_{\upsilon_i \in U} |2\pi\gamma_j(\upsilon_i) - 2\pi\gamma_k(\upsilon_i)| \end{array}}{16} + \frac{\max\left(\begin{array}{c} \sup_{\upsilon_i \in U} |a_j(\upsilon_i) - a_k(\upsilon_i)|, \sup_{\upsilon_i \in U} |b_j(\upsilon_i) - b_k(\upsilon_i)|, \frac{1}{2\pi} \sup_{\upsilon_i \in U} |2\pi\lambda_j(\upsilon_i) - 2\pi\lambda_k(\upsilon_i)|, \frac{1}{2\pi} \sup_{\upsilon_i \in U} |2\pi\sigma_j(\upsilon_i) - 2\pi\sigma_k(\upsilon_i)|, \\ \sup_{\upsilon_i \in U} |c_j(\upsilon_i) - c_k(\upsilon_i)|, \sup_{\upsilon_i \in U} |d_j(\upsilon_i) - d_k(\upsilon_i)|, \frac{1}{2\pi} \sup_{\upsilon_i \in U} |2\pi\eta_j(\upsilon_i) - \eta_k(\upsilon_i)|, \frac{1}{2\pi} \sup_{\upsilon_i \in U} |2\pi\gamma_j(\upsilon_i) - 2\pi\gamma_k(\upsilon_i)| \end{array} \right)}{2} \right].$$

Theorem 11. *The function d given in Definition 9 demonstrates a DM of CIVIFSs.*

The proof for Theorem 11 is provided in Appendix F.

Definition 10. *Let $V(U)$ denote a collection of CIVIFSs and d be a mapping defined on $V(U)$ such that $d_w : V(U) \times V(U) \rightarrow [0,1]$. Then, d is termed a DM of CIVIFSs if the given conditions are fulfilled:*

i) $0 \leq d_w(V_j, V_k) \leq 1$,

ii) $d_w(V_j, V_k) = 0$, *if and only if* $V_j = V_k$,

iii) $d_w(V_j, V_k) = d_w(V_k, V_j)$,

iv) $d_w(V_j, V_l) \leq d_w(V_j, V_k) + d_w(V_k, V_l)$.

The function proposed for DM of CIVIFSs can be defined as:

$$d_w(V_j, V_k) =$$

$$\left[\frac{\begin{array}{c} w_j \cdot \sup_{\upsilon_i \in U} |a_j(\upsilon_i) - a_k(\upsilon_i)| + w_j \cdot \sup_{\upsilon_i \in U} |b_j(\upsilon_i) - b_k(\upsilon_i)| + \frac{w_j}{2\pi} \sup_{\upsilon_i \in U} |2\pi\lambda_j(\upsilon_i) - 2\pi\lambda_k(\upsilon_i)| + \frac{w_j}{2\pi} \sup_{\upsilon_i \in U} |2\pi\sigma_j(\upsilon_i) - 2\pi\sigma_k(\upsilon_i)| + \\ w_j \sup_{\upsilon_i \in U} |c_j(\upsilon_i) - c_k(\upsilon_i)| + w_j \sup_{\upsilon_i \in U} |d_j(\upsilon_i) - d_k(\upsilon_i)| + \frac{w_j}{2\pi} \sup_{\upsilon_i \in U} |2\pi\eta_j(\upsilon_i) - 2\pi\eta_k(\upsilon_i)| + \frac{w_j}{2\pi} \sup_{\upsilon_i \in U} |2\pi\gamma_j(\upsilon_i) - 2\pi\gamma_k(\upsilon_i)| \end{array}}{16} + \frac{\max\left(\begin{array}{c} w_j \sup_{\upsilon_i \in U} |a_j(\upsilon_i) - a_k(\upsilon_i)|, w_j \sup_{\upsilon_i \in U} |b_j(\upsilon_i) - b_k(\upsilon_i)|, \frac{w_j}{2\pi} \sup_{\upsilon_i \in U} |2\pi\lambda_j(\upsilon_i) - 2\pi\lambda_k(\upsilon_i)|, \frac{w_j}{2\pi} \sup_{\upsilon_i \in U} |2\pi\sigma_j(\upsilon_i) - 2\pi\sigma_k(\upsilon_i)|, \\ w_j \sup_{\upsilon_i \in U} |c_j(\upsilon_i) - c_k(\upsilon_i)|, w_j \sup_{\upsilon_i \in U} |d_j(\upsilon_i) - d_k(\upsilon_i)|, \frac{w_j}{2\pi} \sup_{\upsilon_i \in U} |2\pi\eta_j(\upsilon_i) - \eta_k(\upsilon_i)|, \frac{w_j}{2\pi} \sup_{\upsilon_i \in U} |2\pi\gamma_j(\upsilon_i) - 2\pi\gamma_k(\upsilon_i)| \end{array} \right)}{2} \right].$$

Theorem 12. *The function d_w given in Definition 10 demonstrates a weighted DM of CIVIFSs.*

Proof: *The proof of the given statement is straightforward and follows the same approach as Theorem 11.*

6. Decision-Making Applications for Proposed Aggregation Operators and Distance Measures

MCDM is a systematic and structured approach for comparing and evaluating a finite set of alternatives when making decisions that involve multiple, often conflicting attributes or criteria. In real-world applications, decision-makers need to consider several quantitative and qualitative factors when choosing a cybersecurity tool, supplier, investment strategy, or location. Usually, the D-making process starts with developing a decision matrix, in which each column represents a criterion and each row denotes an alternative. Each alternative's score or performance corresponding to each criterion is expressed by the entries in this matrix. These values may be crisp numbers, fuzzy values, or interval numbers depending on the nature of the data. When criteria have been evaluated in different units, the next step is to normalize the matrix for the purpose of removing scale effects. Then, an aggregation procedure is utilized to combine the normalized ratings with weights. Some advanced MADM methods incorporate MDs to compute how close each alternative is to an ideal and anti-ideal solution. The final step is to calculate the closeness coefficient or similarity score for each alternative, which helps to identify the alternatives from most to least preferable. In complex D-making problems, MADM demonstrates an objective, clear, and reproducible method, providing significant support to analysts, policymakers, and industry leaders across different domains.

The D-making procedure is a significant research task among academics, mostly due to the fact that it is essential to many different real-world applications. Over the years, researchers have applied and developed different methodologies to increase the efficiency and quality of D-making techniques. These techniques include neural data networks, management system design, data mining techniques, and principles from medical decision theory, but they are not limited. Each of these techniques provides unique tools and perspectives that help to solve complex real-life problems where numerous attributes must be considered simultaneously. D-making has thus developed as an interdisciplinary field of study, capturing interest from distinct domains like healthcare, cybersecurity, artificial intelligence, engineering, and business.

This section aims to conduct a precise evaluation of the MADM procedure by using the newly defined Bonferroni AOs and DMs to present the practical reliability and effectiveness of the newly defined techniques. Bonferroni AOs are preferred in the proposed study on Dombi, Hamy, and Einstein operators because of their unique capability to model dependency over criteria, which other operators frequently neglect. While Dombi and Einstein AOs can be effective at capturing nonlinear relationships and boundness in aggregation process, they often assume independent input arguments and do not explicitly account for interactions between pairs of criteria. Although Hamy AOs are more adaptable than basic means, they aggregate in a more comprehensive but non-relational manner. Bonferroni AOs, on the other hand, enable pairwise interactions among input values to have an effect on the aggregated result, which plays a vital role in real-world MADM scenarios where attributes are rarely isolated. For example, environmental impact and cost may be inversely related, while safety and performance may be positively correlated. The Bonferroni AOs offer accessible control over the relationships between them, allow decision-makers to clearly represent complementarity or redundancy among criteria. Thus, Bonferroni AOs provide a richer, interaction-sensitive aggregation technique, resulting in more realistic, interpretable, and adaptive decision results than Hamy, Einstein, or Dombi operators.

A structured MADM algorithm is developed using the proposed AOs and DMs, incorporating them into the framework for D-making. The proposed algorithm is then applied and validated through a carefully constructed numerical example, which illustrates the consistency, applicability, and decision-support capabilities of the proposed approaches under real or simulated conditions. Finally, the superiority of newly defined approaches is assessed with a comparative analysis between existing and newly defined approaches.

To conduct a comprehensive assessment or evaluation of any given D-making problem, we proceed by identifying and choosing a collection of possible alternatives $\breve{A} = \{\breve{A}_1, \breve{A}_2, \ldots, \breve{A}_n\}$. The selected alternatives indicate the distinct options that can be examined. For each alternative, we choose a finite family of relevant criteria or attributes to guide our comparison. The set of criteria is denoted by $\dot{C} = \{\dot{C}_1, \dot{C}_2, \ldots, \dot{C}_m\}$. After establishing the alternatives and their related attributes, we proceed by developing a D-making matrix (DMM). To represent the uncertainty that can be found in available data or expert opinions, we assign CIVIFSs to each item of information in the DMM. In DMM, we will employ the CIVIFSs to each value of the $j-th$ alternative and the $k-th$ attribute, that is,

$$\breve{A}_j = \left(\left[\underset{\cdot}{\mathrm{a}}_{jk}, \dot{b}_{jk}\right] e^{i\left[2\pi\lambda_{jk}, 2\pi\sigma_{jk}\right]}, \left[\dot{c}_{jk}, \underset{\hat{}}{\mathrm{d}}_{jk}\right] e^{i\left[2\pi\eta_{jk}, 2\pi\gamma_{jk}\right]}\right),$$

where $\left[\underset{\cdot}{\mathrm{a}}_{jk}, \dot{b}_{jk}\right] e^{i\left[2\pi\lambda_{jk}, 2\pi\sigma_{jk}\right]}$ and $\left[\dot{c}_{jk}, \underset{\hat{}}{\mathrm{d}}_{jk}\right] e^{i\left[2\pi\eta_{jk}, 2\pi\gamma_{jk}\right]}$ represent truth grade and false grade, respectively, and $\dot{b}_{jk} + \underset{\hat{}}{\mathrm{d}}_{jk} \leq 1,\ \sigma_{jk} + \gamma_{jk} \leq 1$. The flowchart of the D-making method is explained using the following Figure 2.

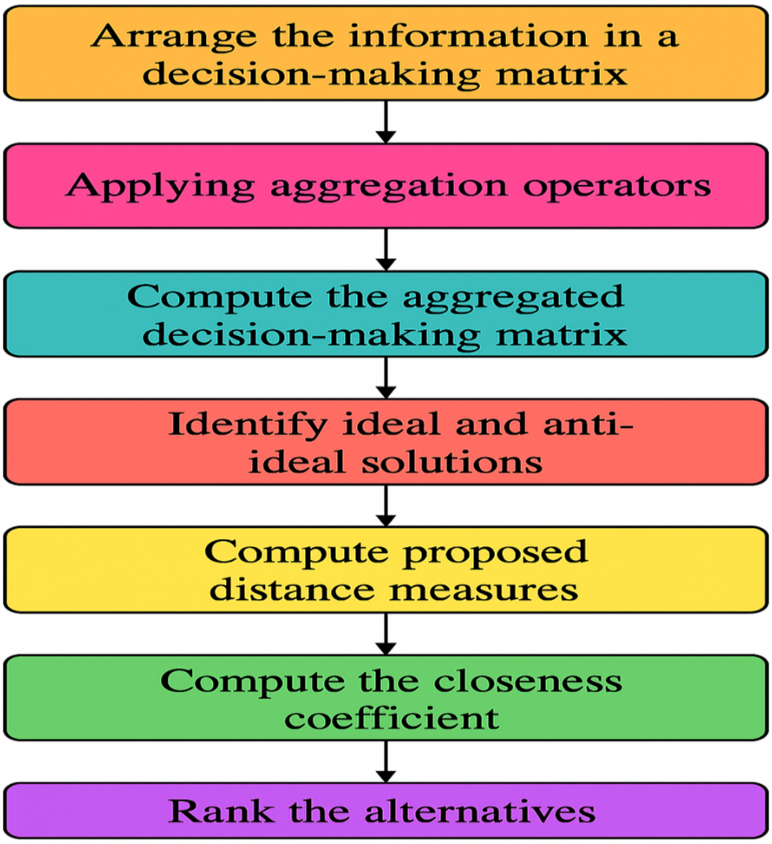

Figure 2: Flowchart of the proposed Algorithm

We aim to implement a D-making process that not only accounts for uncertainty but also offers a rational and transparent framework for ranking or selecting the best alternative. The detailed steps of the proposed method are discussed below:

Step 1: The CIVIF information provided by the decision experts is organized properly into a DMM. In the proposed DMM, each row indicates an alternative, and each column shows distinct evaluation criterion. The entries of the DMM are represented as CIVIFNs. After developing the DMM, the attributes are analyzed to categorize them as either beneficial or non-beneficial. The use of this classification is essential throughout the normalization and aggregation phases of the D-making process to enable a fair comparison of alternatives with different attribute alignments.
Step 2: To integrate the evaluations contributed by multiple experts for each alternative, we employ the proposed AOs to aggregate the individual opinions corresponding to each criterion within the framework of CIVIFS. For each alternative, the information providing by $i-th$ expert is expressed as CIVIFNs. This individual information is then combined based on the defined AOs, such as the CIVIFGBAO or CIVIFWGBAO, depending on the assumptions and characteristics of the D-making settings. The

aggregation procedure produces a single CIVIFS score for each alternative under each criterion, expressing the collective judgment of all experts while retaining the ambiguity and uncertainty in their assessments.

We use the following AOs to aggregate the experts information.

$$CIVIFGBAO\left(\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \ldots, \mathbb{C}_n\right)$$

$$= \left(\begin{array}{c} \begin{bmatrix} 1-\left(1-\prod^{n}{}_{\substack{j,k=1\\ j\neq k}} \left[\left(1-\left(1-\underset{\cdot}{a}_j\right)^{\check{S}} . \left(1-\underset{\cdot}{a}_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ 1-\left(1-\prod^{n}{}_{\substack{j,k=1\\ j\neq k}} \left[\left(1-\left(1-\dot{b}_j\right)^{\check{S}} . \left(1-\dot{b}_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{bmatrix} e^{i\begin{bmatrix} 2\pi\left(1-\left(1-\prod^{n}{}_{\substack{j,k=1\\ j\neq k}} \left[\left(1-\left(1-\lambda_j\right)^{\check{S}} . \left(1-\lambda_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\right), \\ 2\pi\left(1-\left(1-\prod^{n}{}_{\substack{j,k=1\\ j\neq k}} \left[\left(1-\left(1-\sigma_j\right)^{\check{S}} . \left(1-\sigma_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\right) \end{bmatrix}}, \\ \begin{bmatrix} \left(1-\left[\prod^{n}{}_{\substack{j,k=1\\ j\neq k}} \left(1-\dot{c}_j{}^{\check{S}}\dot{c}_k{}^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ \left(1-\left[\prod^{n}{}_{\substack{j,k=1\\ j\neq k}} \left(1-\underset{\wedge}{d}_j{}^{\check{S}}\underset{\wedge}{d}_k{}^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{bmatrix} e^{i\begin{bmatrix} 2\pi\left(1-\left[\prod^{n}{}_{\substack{j,k=1\\ j\neq k}} \left(1-\eta_j{}^{\check{S}}\eta_k{}^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ 2\pi\left(1-\left[\prod^{n}{}_{\substack{j,k=1\\ j\neq k}} \left(1-\gamma_j{}^{\check{S}}\gamma_k{}^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \end{bmatrix}} \end{array} \right),$$

and

$$CIVIFWGBAO\left(\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \ldots, \mathbb{C}_n\right)$$

$$= \left(\begin{array}{c} \begin{bmatrix} 1-\left(1-\prod^{n}{}_{\substack{J,K=1\\ j\neq k}} \left[\left(1-\left(1-\underset{\cdot}{a}_j{}^{\mathfrak{w}_j}\right)^{\check{S}} . \left(1-\underset{\cdot}{a}_k{}^{\mathfrak{w}_k}\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ 1-\left(1-\prod^{n}{}_{\substack{j,k=1\\ j\neq k}} \left[\left(1-\left(1-\dot{b}_j{}^{\mathfrak{w}_j}\right)^{\check{S}} . \left(1-\dot{b}_k{}^{\mathfrak{w}_k}\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{bmatrix} e^{i\begin{bmatrix} 2\pi\left(1-\left(1-\prod^{n}{}_{\substack{J,K=1\\ j\neq k}} \left[\left(1-\left(1-\lambda_j{}^{\mathfrak{w}_j}\right)^{\check{S}} . \left(1-\lambda_k{}^{\mathfrak{w}_k}\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\right), \\ 2\pi\left[2\pi\left(1-\left(1-\prod^{n}{}_{\substack{J,K=1\\ j\neq k}} \left[\left(1-\left(1-\sigma_j{}^{\mathfrak{w}_j}\right)^{\check{S}} . \left(1-\sigma_k{}^{\mathfrak{w}_k}\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\right)\right] \end{bmatrix}}, \\ \begin{bmatrix} \left(1-\prod^{n}{}_{\substack{j,k=1\\ j\neq k}} \left[\left(1-\left(1-\left(1-\dot{c}_j\right)^{\mathfrak{w}_j}\right)^{\check{S}} . \left(1-\left(1-\dot{c}_k\right)^{\mathfrak{w}_k}\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ \left(1-\prod^{n}{}_{\substack{j,k=1\\ j\neq k}} \left[\left(1-\left(1-\left(1-\underset{\wedge}{d}_j\right)^{\mathfrak{w}_j}\right)^{\check{S}} . \left(1-\left(1-\underset{\wedge}{d}_k\right)^{\mathfrak{w}_k}\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{bmatrix} e^{i\begin{bmatrix} 2\pi\left(1-\prod^{n}{}_{\substack{j,k=1\\ j\neq k}} \left[\left(1-\left(1-\left(1-\eta_j\right)^{\mathfrak{w}_j}\right)^{\check{S}} . \left(1-\left(1-\eta_k\right)^{\mathfrak{w}_k}\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ 2\pi\left(1-\prod^{n}{}_{\substack{j,k=1\\ j\neq k}} \left[\left(1-\left(1-\left(1-\gamma_j\right)^{\mathfrak{w}_j}\right)^{\check{S}} . \left(1-\left(1-\gamma_k\right)^{\mathfrak{w}_k}\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{bmatrix}} \end{array} \right).$$

Step 3: The Aggregated DMM is formed by aggregating each expert's individual evaluations for each alternative under each criterion using the proposed CIVIFGBAO or CIVIFWGBAO. Table 1 shows the collective evaluations for each alternative in terms of each attribute. Each element in this aggregated DMM represents a CIVIFS. The proposed aggregated DMM represented by Table 1.

Table 1: Expert information within the framework of CIVIFSs

Aggregated DMM	$\dot{C}_1$	$\dot{C}_2$	$\dot{C}_3$	.	.	.	$\dot{C}_m$
$\breve{A}_1$	$\breve{A}_1(\dot{C}_1)$	$\breve{A}_1(\dot{C}_2)$	$\breve{A}_1(\dot{C}_3)$	.	.	.	$\breve{A}_1(\dot{C}_m)$
$\breve{A}_2$	$\breve{A}_2(\dot{C}_1)$	$\breve{A}_2(\dot{C}_2)$	$\breve{A}_2(\dot{C}_3)$	.	.	.	$\breve{A}_2(\dot{C}_m)$
$\breve{A}_3$	$\breve{A}_3(\dot{C}_1)$	$\breve{A}_3(\dot{C}_2)$	$\breve{A}_3(\dot{C}_3)$	.	.	.	$\breve{A}_3(\dot{C}_m)$
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
$\breve{A}_n$	$\breve{A}_n(\dot{C}_1)$	$\breve{A}_n(\dot{C}_2)$	$\breve{A}_n(\dot{C}_3)$	.	.	.	$\breve{A}_n(\dot{C}_m)$

Step 3: Indicate the ideal and anti-ideal solution of an aggregated DMM. For this, first of all, we identify whether the criterion is beneficial, then higher/maximum values are preferred for truth grade and lower/minimum values for false grade. In case of non-beneficial, lower/minimum values are preferred for truth grade, and higher/maximum values are preferred for false grade.
Step 4: Following the identification of the ideal and anti-ideal solutions for each criterion using the aggregated DMM, the DMs are calculated for each alternative from both the ideal and anti-ideal solutions. The ideal solution corresponds to the best value for each alternative, whereas the **anti-ideal solution** indicates the least favorable value for each alternative.

The proposed DM to the ideal solution is:

$$d^{+}\left(\breve{A}_j, \breve{A}^{+}\right)$$

$$= \left[\frac{\begin{array}{c} \sup_{\mathfrak{A}_i \in U} \left|\underset{\cdot}{a}_j - \underset{\cdot}{a}^{+}\right| + \sup_{\mathfrak{A}_i \in U} \left|\dot{b}_j - \dot{b}^{+}\right| + \frac{1}{2\pi} \sup_{\mathfrak{A}_i \in U} \left|2\pi\lambda_j - 2\pi\lambda^{+}\right| + \frac{1}{2\pi} \sup_{\mathfrak{A}_i \in U} \left|2\pi\sigma_j - 2\pi\sigma^{+}\right| + \\ \sup_{\mathfrak{A}_i \in U} \left|\dot{c}_j - \dot{c}^{+}\right| + \sup_{\mathfrak{A}_i \in U} \left|\underset{\hat{}}{d}_j - \underset{\hat{}}{d}^{+}\right| + \frac{1}{2\pi} \sup_{\mathfrak{A}_i \in U} \left|2\pi\eta_j - 2\pi\eta^{+}\right| + \frac{1}{2\pi} \sup_{\mathfrak{A}_i \in U} \left|2\pi\gamma_j - 2\pi\gamma^{+}\right| \end{array}}{16} + \frac{\max\left(\begin{array}{c} \sup_{\mathfrak{A}_i \in U} \left|\underset{\cdot}{a}_j - \underset{\cdot}{a}^{+}\right|, \sup_{\mathfrak{A}_i \in U} \left|\dot{b}_j - \dot{b}^{+}\right|, \frac{1}{2\pi} \sup_{\mathfrak{A}_i \in U} \left|2\pi\lambda_j - 2\pi\lambda^{+}\right|, \frac{1}{2\pi} \sup_{\mathfrak{A}_i \in U} \left|2\pi\sigma_j - 2\pi\sigma^{+}\right|, \\ \sup_{\mathfrak{A}_i \in U} \left|\dot{c}_j - \dot{c}^{+}\right|, \sup_{\mathfrak{A}_i \in U} \left|\underset{\hat{}}{d}_j - \underset{\hat{}}{d}^{+}\right|, \frac{1}{2\pi} \sup_{\mathfrak{A}_i \in U} \left|2\pi\eta_j - 2\pi\eta^{+}\right|, \frac{1}{2\pi} \sup_{\mathfrak{A}_i \in U} \left|2\pi\gamma_j - 2\pi\gamma^{+}\right| \end{array} \right)}{2} \right].$$

The proposed DM to ideal solution is:

$d^{-}\left(\breve{A}_j, \breve{A}^{+}\right)$

$$= \left[\frac{\begin{array}{c} \sup_{\mathfrak{A}_i \in U} \left|\underset{.}{a}_j - \underset{.}{a}^{-}\right| + \sup_{\mathfrak{A}_i \in U} \left|\dot{b}_j - \dot{b}^{-}\right| + \frac{1}{2\pi} \sup_{\mathfrak{A}_i \in U} \left|2\pi\lambda_j - 2\pi\lambda^{-}\right| + \frac{1}{2\pi} \sup_{\mathfrak{A}_i \in U} \left|2\pi\sigma_j - 2\pi\sigma^{-}\right| + \\ \sup_{\mathfrak{A}_i \in U} \left|\dot{c}_j - \dot{c}^{-}\right| + \sup_{\mathfrak{A}_i \in U} \left|\underset{\hat{}}{d}_j - \underset{\hat{}}{d}^{-}\right| + \frac{1}{2\pi} \sup_{\mathfrak{A}_i \in U} \left|2\pi\eta_j - 2\pi\eta^{-}\right| + \frac{1}{2\pi} \sup_{\mathfrak{A}_i \in U} \left|2\pi\gamma_j - 2\pi\gamma^{-}\right| \end{array}}{16} + \frac{\max \left(\begin{array}{c} \sup_{\mathfrak{A}_i \in U} \left|\underset{.}{a}_j - \underset{.}{a}^{-}\right|, \sup_{\mathfrak{A}_i \in U} \left|\dot{b}_j - \dot{b}^{-}\right|, \frac{1}{2\pi} \sup_{\mathfrak{A}_i \in U} \left|2\pi\lambda_j - 2\pi\lambda^{-}\right|, \frac{1}{2\pi} \sup_{\mathfrak{A}_i \in U} \left|2\pi\sigma_j - 2\pi\sigma^{-}\right|, \\ \sup_{\mathfrak{A}_i \in U} \left|\dot{c}_j - \dot{c}^{-}\right|, \sup_{\mathfrak{A}_i \in U} \left|\underset{\hat{}}{d}_j - \underset{\hat{}}{d}^{-}\right|, \frac{1}{2\pi} \sup_{\mathfrak{A}_i \in U} \left|2\pi\eta_j - 2\pi\eta^{-}\right|, \frac{1}{2\pi} \sup_{\mathfrak{A}_i \in U} \left|2\pi\gamma_j - 2\pi\gamma^{-}\right| \end{array} \right)}{2} \right].$$

Step 6: After computing the proposed DMs of each alternative from the ideal solution and anti-ideal solution, we calculate the closeness coefficient (CC) for each alternative in order to determine its proximity to the ideal scenario. The CC measures how favorable one choice is in comparison to the others. The alternative with the **highest CC** is considered the most favorable based on the collective expert evaluations and the decision criteria. The closeness coefficient of the $j-th$ alternative is calculated by the following formula:

$$cc_j = \frac{d^{-}\left(C_j, C^{+}\right)}{d^{+}\left(C_j, C^{+}\right) + d^{-}\left(C_j, C^{+}\right)}.$$

Step 7: Rank the alternatives. Any alternative with the highest CC identifies the best alternative, and with the least CC indicates the worst alternative.

We will present a real-world application of cybersecurity to illustrate the implementation of the proposed model to make it easier to understand and assess its usefulness and superiority. Despite the crucial importance of cybersecurity in safeguarding digital infrastructures, there is still a visible gap in the integration of effective D-making techniques into cybersecurity systems. Most cybersecurity techniques concentrate on technical features like threat mitigation, intrusion detection, and encryption while ignoring formal D-making methods that enable for uncertainty, expert judgments, and multi-criteria assessment. TOPSIS, VIKOR, AHP, and BWM are some of the most used MCDM approaches. TOPSIS approach is useful for ranking alternatives; it struggles to deal with uncertainty. VIKOR emphasizes compromise solutions, but it demands precise weights, which are difficult to achieve in uncertain conditions.

This gap restricts the capacity to make informed, transparent, and context-aware security decisions, particularly in complex environments like cloud networks, industrial systems, and critical infrastructures. To improve risk assessment, response prioritization, and strategic security planning, it is critical to bridge the gap between cybersecurity and formal D-making theories such as MCDM. Our proposed study combines Bonferroni aggregation procedures with CIVIFSs, allowing it to represent two-dimensional uncertainty while maintaining robust during D-making.

In the below section, we will concentrate on cybersecurity D-making, using the proposed AOs and DMs.

7. Application to Cybersecurity Problem

The purpose of this section is to utilize the proposed AOs and DMs, which present an innovative and comprehensive framework for determining, evaluating, and selecting the best cybersecurity approaches and solutions for administrative protection. The proposed study focuses on institutions or enterprises that are more sensitive to rising cyber threats, including malware attacks, data breaches, phishing campaigns, and ransomware. The key goal of cybersecurity D-making is to secure regulatory compliance and operational continuity while preserving the integrity, confidentiality, and availability of digital assets. In the cybersecurity area, securing sensitive data, systems, and infrastructure from interruption access is important to maintaining company resilience and customer trust. Companies and institutions face more difficulties in determining threats, addressing crises, and establishing strong defenses in place as they expand their IT infrastructure and adopt more digital platforms. The ability of the company to reduce risk, avoid financial loss, and safeguard its brand in a very unstable threat landscape is essentially what contributes to making wise cybersecurity decisions so crucial. For an organization, a selection of security techniques, including firewalls, data encryption, intrusion detection systems, incident response capabilities, and endpoint protection, must be carefully assessed. Additionally, investigating cybersecurity includes access controls, evaluating third-party risk, employee training programs, and conformance to ISO/IEC or NIST 27001 standards. The company must require solutions that are both effective and adaptive while considering the trade-offs between performance and security investment in a fast-paced, more complicated cyberattack environment. An informative D-making procedure boosts an organization's technical stability as well as its strategic adaptability to novel threats.

Organizations must carefully weigh a number of attributes during the selection method in order to improve the accuracy and quality of cybersecurity defenses. These attributes include threat detection, user impact, cost-effectiveness, and response time. These attributes improve the cybersecurity structure's overall development and performance. Selecting the best cybersecurity approach in today's challenging digital world is important to corporate risk management and long-term operational performance. To develop a robust and safe cyber infrastructure, this complex decision requires establishing a balance between human factors, technology, regulation, and business objectives.

Analyzing the most significant attributes that contribute to the D-making procedure's overall performance is important because it involves multiple attributes that are interrelated. The proposed study is structured around four main attributes, each of which is significant in identifying the quality and clarity of the conclusion. The following four attributes represent the basic components that collectively define the structure for D-making:

1. Threat Detection "$\dot{C}_1$": In cybersecurity, threat detection is important because it enables organizations to detect and stop harmful behavior before it does extensive damage. To identify any unusual intrusions, or vulnerabilities, activity, it continuously monitors networks, systems, and data. Threat detection helps detect and

address possible security breaches before they have a chance to cause harm, which makes it highly beneficial in cybersecurity.

2. User Impact "$\dot{C}_2$": User impact is significant when it addresses the problems with cybersecurity because end users are frequently the first line of defense and the most vulnerable point in a security system. Well-informed, security-conscious, and alert users may significantly reduce the threat environment, but careless or inexperienced users can put systems at major risk. This means that the more users involved, the higher the chance of mistakes or risks.

3. Cost-effectiveness "$\dot{C}_3$": In cybersecurity, cost-effectiveness describes to putting in place security measures that provide the best protection with the least possible budget. Employee training, risk assessments, scalable technologies, and proactive monitoring are common types of cost-effective strategies that can be used to reduce possible losses while maintaining under spending limits. It is a non-beneficial criterion.

4. Response Time "$\dot{C}_4$": Response time in cybersecurity plays an important role in minimizing the harm caused by cyber threats. The risk of data loss, financial impact, and reputational damage reduces with the speed at which an organization detects and reacts to an attack. It is a beneficial criterion.

The information discussed above is said to be attributes or criteria that can be used to choose the most preferred alternative. In any case, the cybersecurity choice may differ based on the particular needs and concerns of an organization. The main objectives of the proposed study are to identify the best one among the following three alternatives.

1. Firewall X "$\breve{A}_1$": By providing strong protection against external threats and internal networks, Firewall X plays a significant role to cybersecurity. It monitors and analyzes incoming and outgoing traffic based on predefined security rules, helping to prevent unauthorized access, malware infiltration, and data breaches. Firewall X is a next-generation or advanced firewall that typically employs intrusion prevention, application-level filtering, and deep packet inspection functions.

2. Endpoint Security Y "$\breve{A}_2$": Endpoint security Y is essential to cybersecurity because it protects individual devices that connect to a network, including laptops, servers, smartphones, and desktops. Hackers often use these endpoints as channels of entry for malware, ransomware, or data theft. Endpoint Security Y uses features including device control, behavioral analysis, real-time threat detection, and antivirus protection to make sure every device is safe.

3. Zero Trust Z "$\breve{A}_3$": A key component of cybersecurity is Zero Trust Z, which enforces the principle of "never trust, always verify." It demands least-privilege access to all resources, rigorous identity verification, and ongoing authentication. This strategy lowers the chance of data breaches, stops attackers from moving laterally, and improves network resilience in general.

For the proposed D-making process, we consider a set of three cybersecurity tools $A = \{A_1,\ A_2,\ A_3\}$ where $A_1,\ A_2,$ and A_3 represent Firewall X, Endpoint Security Y, and Zero Trust Z, respectively. The set of criteria is denoted by $\dot{C} = \{\dot{C}_1, \dot{C}_2, \dot{C}_3, \dot{C}_4\}$ where $\dot{C}_1, \dot{C}_2, \dot{C}_3, \dot{C}_4$ identifies threat detection, user impact, cost effectiveness and response time respectively. Assume that $\mathfrak{w}$ be the weighted vector corresponding to four alternatives. Let $E = \{E_1, E_2, E_3\}$ be a set of expert team which play a key role in the proposed D-making method. Each expert assigns weights to the four criteria as follows:

Table 2: Positive weight vectors assigned by Experts to each criterion

Expert	$\dot{C}_1$	$\dot{C}_2$	$\dot{C}_3$	$\dot{C}_4$
E_1	0.3	0.2	0.3	0.2
E_2	0.25	0.25	0.2	0.3
E_3	0.2	0.25	0.35	0.2

Now, we will employ the proposed algorithm given in Section 6 to evaluate the best cybersecurity tool.

Step 1: In this step, the expert panel's evaluations are organized and arranged into DMMs for each expert $E_i; i = 1, 2, 3$. These DMMs describe the evaluations of the given set of alternatives $A = \{A_1,\ A_2,\ A_3\}$ against the set of criteria $\dot{C} = \left\{\dot{C}_1, \dot{C}_2, \dot{C}_3, \dot{C}_4\right\}$. Each expert individually assesses how well each alternative meets each criterion, and their numerical findings are tabulated. The rows of each matrix indicates the alternatives $A_i; i = 1, 2, 3,$ while the columns indicate the defined criteria $\dot{C}_j; j = 1, 2, 3, 4$.

Formally, for each $E_i; i = 1, 2, 3,$ the proposed DMM is established as follows.

Table 3: Information evaluated by Expert E_1

DMM	$\dot{C}_1$	$\dot{C}_2$	$\dot{C}_3$	$\dot{C}_4$
$\mathfrak{w}$	0.3	0.2	0.3	0.2
$\breve{A}_1$	$\begin{pmatrix}[0.2, 0.4]e^{i[2\pi(0.3), 2\pi(0.5)]}, \\ [0.3, 0.6]e^{i[2\pi(0.1), 2\pi(0.3)]}\end{pmatrix}$	$\begin{pmatrix}[0.4, 0.7]e^{i[2\pi(0.2), 2\pi(0.4)]}, \\ [0.1, 0.2]e^{i[2\pi(0.3), 2\pi(0.5)]}\end{pmatrix}$	$\begin{pmatrix}[0.1, 0.6]e^{i[2\pi(0.2), 2\pi(0.4)]}, \\ [0.1, 0.3]e^{i[2\pi(0.4), 2\pi(0.6)]}\end{pmatrix}$	$\begin{pmatrix}[0.3, 0.4]e^{i[2\pi(0.1), 2\pi(0.3)]}, \\ [0.4, 0.5]e^{i[2\pi(0.2), 2\pi(0.5)]}\end{pmatrix}$
$\breve{A}_2$	$\begin{pmatrix}[0.1, 0.6]e^{i[2\pi(0.2), 2\pi(0.4)]}, \\ [0.2, 0.3]e^{i[2\pi(0.3), 2\pi(0.5)]}\end{pmatrix}$	$\begin{pmatrix}[0.4, 0.6]e^{i[2\pi(0.2), 2\pi(0.4)]}, \\ [0.1, 0.2]e^{i[2\pi(0.3), 2\pi(0.6)]}\end{pmatrix}$	$\begin{pmatrix}[0.2, 0.5]e^{i[2\pi(0.4), 2\pi(0.6)]}, \\ [0.1, 0.5]e^{i[2\pi(0.2), 2\pi(0.4)]}\end{pmatrix}$	$\begin{pmatrix}[0.3, 0.4]e^{i[2\pi(0.2), 2\pi(0.5)]}, \\ [0.1, 0.2]e^{i[2\pi(0.3), 2\pi(0.4)]}\end{pmatrix}$
$\breve{A}_3$	$\begin{pmatrix}[0.1, 0.3]e^{i[2\pi(0.2), 2\pi(0.7)]}, \\ [0.2, 0.5]e^{i[2\pi(0.1), 2\pi(0.3)]}\end{pmatrix}$	$\begin{pmatrix}[0.5, 0.6]e^{i[2\pi(0.3), 2\pi(0.7)]}, \\ [0.2, 0.4]e^{i[2\pi(0.1), 2\pi(0.2)]}\end{pmatrix}$	$\begin{pmatrix}[0.1, 0.3]e^{i[2\pi(0.2), 2\pi(0.4)]}, \\ [0.2, 0.5]e^{i[2\pi(0.3), 2\pi(0.6)]}\end{pmatrix}$	$\begin{pmatrix}[0.2, 0.6]e^{i[2\pi(0.1), 2\pi(0.4)]}, \\ [0.2, 0.3]e^{i[2\pi(0.2), 2\pi(0.5)]}\end{pmatrix}$

Table 4: Information evaluated by Expert E_2

DMM	$\dot{C}_1$	$\dot{C}_2$	$\dot{C}_3$	$\dot{C}_4$
$\mathfrak{w}$	0.25	0.25	0.2	0.3
$\breve{A}_1$	$\left(\begin{matrix}[0.1,0.5]e^{i[2\pi(0.2),2\pi(0.6)]},\\ [0.2,0.5]e^{i[2\pi(0.2),2\pi(0.3)]}\end{matrix}\right)$	$\left(\begin{matrix}[0.3,0.7]e^{i[2\pi(0.1),2\pi(0.5)]},\\ [0.1,0.2]e^{i[2\pi(0.2),2\pi(0.5)]}\end{matrix}\right)$	$\left(\begin{matrix}[0.2,0.5]e^{i[2\pi(0.2),2\pi(0.5)]},\\ [0.2,0.4]e^{i[2\pi(0.3),2\pi(0.5)]}\end{matrix}\right)$	$\left(\begin{matrix}[0.4,0.5]e^{i[2\pi(0.1),2\pi(0.4)]},\\ [0.3,0.5]e^{i[2\pi(0.1),2\pi(0.4)]}\end{matrix}\right)$
$\breve{A}_2$	$\left(\begin{matrix}[0.2,0.6]e^{i[2\pi(0.3),2\pi(0.4)]},\\ [0.1,0.3]e^{i[2\pi(0.2),2\pi(0.5)]}\end{matrix}\right)$	$\left(\begin{matrix}[0.3,0.5]e^{i[2\pi(0.1),2\pi(0.5)]},\\ [0.2,0.4]e^{i[2\pi(0.2),2\pi(0.4)]}\end{matrix}\right)$	$\left(\begin{matrix}[0.1,0.4]e^{i[2\pi(0.3),2\pi(0.7)]},\\ [0.2,0.5]e^{i[2\pi(0.2),2\pi(0.3)]}\end{matrix}\right)$	$\left(\begin{matrix}[0.2,0.5]e^{i[2\pi(0.1),2\pi(0.7)]},\\ [0.2,0.4]e^{i[2\pi(0.1),2\pi(0.3)]}\end{matrix}\right)$
$\breve{A}_3$	$\left(\begin{matrix}[0.2,0.4]e^{i[2\pi(0.3),2\pi(0.7)]},\\ [0.1,0.5]e^{i[2\pi(0.2),2\pi(0.3)]}\end{matrix}\right)$	$\left(\begin{matrix}[0.4,0.6]e^{i[2\pi(0.2),2\pi(0.6)]},\\ [0.3,0.4]e^{i[2\pi(0.2),2\pi(0.4)]}\end{matrix}\right)$	$\left(\begin{matrix}[0.2,0.4]e^{i[2\pi(0.1),2\pi(0.3)]},\\ [0.1,0.4]e^{i[2\pi(0.2),2\pi(0.5)]}\end{matrix}\right)$	$\left(\begin{matrix}[0.1,0.4]e^{i[2\pi(0.2),2\pi(0.5)]},\\ [0.1,0.4]e^{i[2\pi(0.2),2\pi(0.4)]}\end{matrix}\right)$

Table 5: Information evaluated by Expert E_3

DMM	$\dot{C}_1$	$\dot{C}_2$	$\dot{C}_3$	$\dot{C}_4$
$\mathfrak{w}$	0.2	0.25	0.35	0.2
$\breve{A}_1$	$\left(\begin{matrix}[0.2,0.5]e^{i[2\pi(0.1),2\pi(0.6)]},\\ [0.3,0.5]e^{i[2\pi(0.3),2\pi(0.4)]}\end{matrix}\right)$	$\left(\begin{matrix}[0.3,0.6]e^{i[2\pi(0.1),2\pi(0.4)]},\\ [0.1,0.3]e^{i[2\pi(0.2),2\pi(0.5)]}\end{matrix}\right)$	$\left(\begin{matrix}[0.1,0.4]e^{i[2\pi(0.2),2\pi(0.4)]},\\ [0.2,0.5]e^{i[2\pi(0.2),2\pi(0.5)]}\end{matrix}\right)$	$\left(\begin{matrix}[0.3,0.5]e^{i[2\pi(0.2),2\pi(0.6)]},\\ [0.1,0.5]e^{i[2\pi(0.1),2\pi(0.3)]}\end{matrix}\right)$
$\breve{A}_2$	$\left(\begin{matrix}[0.2,0.7]e^{i[2\pi(0.3),2\pi(0.5)]},\\ [0.1,0.2]e^{i[2\pi(0.2),2\pi(0.4)]}\end{matrix}\right)$	$\left(\begin{matrix}[0.1,0.5]e^{i[2\pi(0.2),2\pi(0.5)]},\\ [0.2,0.5]e^{i[2\pi(0.2),2\pi(0.4)]}\end{matrix}\right)$	$\left(\begin{matrix}[0.1,0.6]e^{i[2\pi(0.3),2\pi(0.6)]},\\ [0.2,0.4]e^{i[2\pi(0.2),2\pi(0.4)]}\end{matrix}\right)$	$\left(\begin{matrix}[0.2,0.5]e^{i[2\pi(0.4),2\pi(0.7)]},\\ [0.2,0.4]e^{i[2\pi(0.1),2\pi(0.3)]}\end{matrix}\right)$
$\breve{A}_3$	$\left(\begin{matrix}[0.1,0.5]e^{i[2\pi(0.2),2\pi(0.6)]},\\ [0.1,0.4]e^{i[2\pi(0.2),2\pi(0.3)]}\end{matrix}\right)$	$\left(\begin{matrix}[0.4,0.6]e^{i[2\pi(0.3),2\pi(0.6)]},\\ [0.2,0.4]e^{i[2\pi(0.1),2\pi(0.4)]}\end{matrix}\right)$	$\left(\begin{matrix}[0.3,0.5]e^{i[2\pi(0.1),2\pi(0.4)]},\\ [0.2,0.4]e^{i[2\pi(0.2),2\pi(0.5)]}\end{matrix}\right)$	$\left(\begin{matrix}[0.2,0.6]e^{i[2\pi(0.1),2\pi(0.4)]},\\ [0.1,0.3]e^{i[2\pi(0.2),2\pi(0.3)]}\end{matrix}\right)$

Step 2&3: After developing individual DMMs from each expert's assessment in **Step 1**, the next crucial step is the combination of these expert evaluations into a single aggregated DMM. This is essential since multiple experts may offer different opinions, and we require a unified view for future decision analysis. Using the proposed AOs, we can aggregate the evaluations of each criterion for each alternative from many experts. Now, we employ the proposed AOs to aggregate the values $\breve{A}^{E_i}(\dot{C}); i = 1, 2, 3$, of attribute $\dot{C}$ in every alternative. The aggregated values are arranged in Tables 6 & 7. (For the sake of generality, we assume $\breve{S}$, $\breve{T}$=1).

Table 6: Aggregated DMM using CIVIFGBAOs

Aggregated DMM	$\dot{C}_1$	$\dot{C}_2$	$\dot{C}_3$	$\dot{C}_4$
$\breve{A}_1$	$\begin{pmatrix}[0.17, 0.47]e^{i[2\pi(0.2),2\pi(0.56)]},\\ [0.26, 0.53]e^{i[2\pi(0.19),2\pi(0.33)]}\end{pmatrix}$	$\begin{pmatrix}[0.33, 0.69]e^{i[2\pi(0.13),2\pi(0.43)]},\\ [0.1, 0.23]e^{i[2\pi(0.23),2\pi(0.5)]}\end{pmatrix}$	$\begin{pmatrix}[0.13, 0.5]e^{i[2\pi(0.2),2\pi(0.43)]},\\ [0.16, 0.4]e^{i[2\pi(0.29),2\pi(0.53)]}\end{pmatrix}$	$\begin{pmatrix}[0.33, 0.47]e^{i[2\pi(0.2),2\pi(0.6)]},\\ [0.25, 0.5]e^{i[2\pi(0.1),2\pi(0.3)]}\end{pmatrix}$
$\breve{A}_2$	$\begin{pmatrix}[0.17, 0.63]e^{i[2\pi(0.27),2\pi(0.43)]},\\ [0.13, 0.26]e^{i[2\pi(0.23),2\pi(0.47)]}\end{pmatrix}$	$\begin{pmatrix}[0.27, 0.53]e^{i[2\pi(0.17),2\pi(0.43)]},\\ [0.14, 0.36]e^{i[2\pi(0.23),2\pi(0.46)]}\end{pmatrix}$	$\begin{pmatrix}[0.3, 0.5]e^{i[2\pi(0.33),2\pi(0.63)]},\\ [0.16, 0.47]e^{i[2\pi(0.2),2\pi(0.37)]}\end{pmatrix}$	$\begin{pmatrix}[0.2, 0.47]e^{i[2\pi(0.23),2\pi(0.64)]},\\ [0.16, 0.33]e^{i[2\pi(0.15),2\pi(0.33)]}\end{pmatrix}$
$\breve{A}_3$	$\begin{pmatrix}[0.13, 0.4]e^{i[2\pi(0.23),2\pi(0.67)]},\\ [0.13, 0.47]e^{i[2\pi(0.16),2\pi(0.3)]}\end{pmatrix}$	$\begin{pmatrix}[0.37, 0.6]e^{i[2\pi(0.27),2\pi(0.63)]},\\ [0.23, 0.4]e^{i[2\pi(0.13),2\pi(0.33)]}\end{pmatrix}$	$\begin{pmatrix}[0.2, 0.4]e^{i[2\pi(0.13),2\pi(0.37)]},\\ [0.16, 0.43]e^{i[2\pi(0.23),2\pi(0.53)]}\end{pmatrix}$	$\begin{pmatrix}[0.17, 0.54]e^{i[2\pi(0.13),2\pi(0.43)]},\\ [0.13, 0.33]e^{i[2\pi(0.2),2\pi(0.39)]}\end{pmatrix}$

Table 7: Aggregated DMM using CIVIFWGBAOs

Aggregated DMM	$\dot{C}_1$	$\dot{C}_2$	$\dot{C}_3$	$\dot{C}_4$
$\mathfrak{w}$	0.3	0.2	0.3	0.2
$\breve{A}_1$	$\begin{pmatrix}[0.64, 0.83]e^{i[2\pi(0.67),2\pi(0.86)]},\\ [0.07, 0.17]e^{i[2\pi(0.05),2\pi(0.09)]}\end{pmatrix}$	$\begin{pmatrix}[0.77, 0.90]e^{i[2\pi(0.62),2\pi(0.82)]},\\ [0.02, 0.06]e^{i[2\pi(0.06),2\pi(0.14)]}\end{pmatrix}$	$\begin{pmatrix}[0.55, 0.81]e^{i[2\pi(0.62),2\pi(0.78)]},\\ [0.05, 0.13]e^{i[2\pi(0.09),2\pi(0.19)]}\end{pmatrix}$	$\begin{pmatrix}[0.78, 0.83]e^{i[2\pi(0.63),2\pi(0.83)]},\\ [0.08, 0.17]e^{i[2\pi(0.04),2\pi(0.13)]}\end{pmatrix}$
$\breve{A}_2$	$\begin{pmatrix}[0.64, 0.89]e^{i[2\pi(0.68),2\pi(0.80)]},\\ [0.03, 0.07]e^{i[2\pi(0.06),2\pi(0.14)]}\end{pmatrix}$	$\begin{pmatrix}[0.72, 0.86]e^{i[2\pi(0.65),2\pi(0.83)]},\\ [0.04, 0.1]e^{i[2\pi(0.06),2\pi(0.13)]}\end{pmatrix}$	$\begin{pmatrix}[0.54, 0.81]e^{i[2\pi(0.72),2\pi(0.87)]},\\ [0.05, 0.15]e^{i[2\pi(0.06),2\pi(0.12)]}\end{pmatrix}$	$\begin{pmatrix}[0.72, 0.84]e^{i[2\pi(0.70),2\pi(0.90)]},\\ [0.05, 0.1]e^{i[2\pi(0.05),2\pi(0.10)]}\end{pmatrix}$
$\breve{A}_3$	$\begin{pmatrix}[0.58, 0.78]e^{i[2\pi(0.68),2\pi(0.89)]},\\ [0.03, 0.14]e^{i[2\pi(0.04),2\pi(0.08)]}\end{pmatrix}$	$\begin{pmatrix}[0.82, 0.88]e^{i[2\pi(0.73),2\pi(0.89)]},\\ [0.06, 0.11]e^{i[2\pi(0.03),2\pi(0.09)]}\end{pmatrix}$	$\begin{pmatrix}[0.62, 0.76]e^{i[2\pi(0.54),2\pi(0.74)]},\\ [0.05, 0.15]e^{i[2\pi(0.07),2\pi(0.19)]}\end{pmatrix}$	$\begin{pmatrix}[0.66, 0.86]e^{i[2\pi(0.64),2\pi(0.83)]},\\ [0.04, 0.10]e^{i[2\pi(0.06),2\pi(0.13)]}\end{pmatrix}$

The MS Excel codes to compute the membership and non-membership functions are provided in Appendices G–I.

Step 4: Now, we have to identify the ideal and anti-ideal solutions of an aggregated DMM. For this, we indicate the maximum values and minimum values for truth grade and false grade. First, we discuss the performance aspects of the proposed criteria using Table 8.

Table 8: Performance aspects of the proposed criterion

Symbol	Criterion	Type
$\dot{C}_1$	Threat Detection	Beneficial
$\dot{C}_2$	User Impact (Higher= risk)	Non-beneficial
$\dot{C}_3$	Cost (lower = better)	Non-beneficial
$\dot{C}_4$	Response Time	Beneficial

Now, we construct Tables 9 & 10 to identify ideal and anti-ideal values:

Table 9: Ideal and anti-ideal values using Table 6

Criterion	Type	Ideal Value	Anti-ideal value
$\dot{C}_1$	Beneficial	$\underset{\cdot}{a}_j = 0.17,\ \dot{b}_j = 0.63,\ \lambda_j = 0.27,\ \sigma_j = 0.67,$ $\dot{c}_j = 0.13,\ \underset{\wedge}{d}_j = 0.26,\ \eta_j = 0.16, \gamma_j = 0.3$	$\underset{\cdot}{a}_j = 0.13,\ \dot{b}_j = 0.4,\ \lambda_j = 0.2,\ \sigma_j = 0.43,$ $\dot{c}_j = 0.26,\ \underset{\wedge}{d}_j = 0.53,\ \eta_j = 0.23, \gamma_j = 0.47$
$\dot{C}_2$	Non-beneficial	$\underset{\cdot}{a}_j = 0.27,\ \dot{b}_j = 0.53,\ \lambda_j = 0.13,\ \sigma_j = 0.43,$ $\dot{c}_j = 0.23,\ \underset{\wedge}{d}_j = 0.4,\ \eta_j = 0.23, \gamma_j = 0.5$	$\underset{\cdot}{a}_j = 0.37,\ \dot{b}_j = 0.69,\ \lambda_j = 0.27,\ \sigma_j = 0.63,$ $\dot{c}_j = 0.1,\ \underset{\wedge}{d}_j = 0.23,\ \eta_j = 0.13, \gamma_j = 0.33$
$\dot{C}_3$	Non-beneficial	$\underset{\cdot}{a}_j = 0.13,\ \dot{b}_j = 0.4,\ \lambda_j = 0.13,\ \sigma_j = 0.37,$ $\dot{c}_j = 0.16,\ \underset{\wedge}{d}_j = 0.47,\ \eta_j = 0.29, \gamma_j = 0.53$	$\underset{\cdot}{a}_j = 0.2,\ \dot{b}_j = 0.5,\ \lambda_j = 0.33,\ \sigma_j = 0.63,$ $\dot{c}_j = 0.16,\ \underset{\wedge}{d}_j = 0.4,\ \eta_j = 0.29, \gamma_j = 0.53$
$\dot{C}_4$	Beneficial	$\underset{\cdot}{a}_j = 0.33,\ \dot{b}_j = 0.54,\ \lambda_j = 0.23,\ \sigma_j = 0.64,$ $\dot{c}_j = 0.13,\ \underset{\wedge}{d}_j = 0.33,\ \eta_j = 0.1, \gamma_j = 0.3$	$\underset{\cdot}{a}_j = 0.17,\ \dot{b}_j = 0.47,\ \lambda_j = 0.23,\ \sigma_j = 0.43,$ $\dot{c}_j = 0.25,\ \underset{\wedge}{d}_j = 0.5,\ \eta_j = 0.2, \gamma_j = 0.39$

Table 10: Ideal and anti-ideal values using Table 7

Criterion	Type	Ideal Value	Anti-ideal value
$\dot{C}_1$	Beneficial	$\underset{\cdot}{a}_j = 0.64,\ \dot{b}_j = 0.89,\ \lambda_j = 0.68,\ \sigma_j = 0.89,$ $\dot{c}_j = 0.03,\ \underset{\wedge}{d}_j = 0.07,\ \eta_j = 0.04, \gamma_j = 0.08$	$\underset{\cdot}{a}_j = 0.58,\ \dot{b}_j = 0.78,\ \lambda_j = 0.67,\ \sigma_j = 0.80,$ $\dot{c}_j = 0.07,\ \underset{\wedge}{d}_j = 0.17,\ \eta_j = 0.06, \gamma_j = 0.14$
$\dot{C}_2$	Non-beneficial	$\underset{\cdot}{a}_j = 0.72,\ \dot{b}_j = 0.86,\ \lambda_j = 0.62,\ \sigma_j = 0.82,$ $\dot{c}_j = 0.06,\ \underset{\wedge}{d}_j = 0.11,\ \eta_j = 0.06, \gamma_j = 0.14$	$\underset{\cdot}{a}_j = 0.82,\ \dot{b}_j = 0.90,\ \lambda_j = 0.73,\ \sigma_j = 0.89,$ $\dot{c}_j = 0.02,\ \underset{\wedge}{d}_j = 0.06,\ \eta_j = 0.03, \gamma_j = 0.09$
$\dot{C}_3$	Non-beneficial	$\underset{\cdot}{a}_j = 0.54,\ \dot{b}_j = 0.76,\ \lambda_j = 0.54,\ \sigma_j = 0.74,$ $\dot{c}_j = 0.05,\ \underset{\wedge}{d}_j = 0.15,\ \eta_j = 0.09, \gamma_j = 0.19$	$\underset{\cdot}{a}_j = 0.62,\ \dot{b}_j = 0.81,\ \lambda_j = 0.72,\ \sigma_j = 0.87,$ $\dot{c}_j = 0.05,\ \underset{\wedge}{d}_j = 0.13,\ \eta_j = 0.06, \gamma_j = 0.12$
$\dot{C}_4$	Beneficial	$\underset{\cdot}{a}_j = 0.78,\ \dot{b}_j = 0.86,\ \lambda_j = 0.70,\ \sigma_j = 0.90,$ $\dot{c}_j = 0.04,\ \underset{\wedge}{d}_j = 0.1,\ \eta_j = 0.04, \gamma_j = 0.10$	$\underset{\cdot}{a}_j = 0.66,\ \dot{b}_j = 0.83,\ \lambda_j = 0.63,\ \sigma_j = 0.83,$ $\dot{c}_j = 0.08,\ \underset{\wedge}{d}_j = 0.17,\ \eta_j = 0.06, \gamma_j = 0.13$

Step 5: Tables 11 and 12 provide a complete illustration of the proposed DMs, which are employed to identify each alternative's proximity to the ideal and anti-ideal solutions. These tables present the calculated distances based on the established AOs and assessment criteria, allowing for the determination of the most desired alternative (closest to the ideal solution) and the least preferred alternative (closest to the anti-ideal solution). Table 11 summarizes the DMs between each alternative and the ideal solution, although Table 12 describes the DMs from the anti-ideal solution, allowing for a comparative comparison to facilitate in effective D-making.

Table 11: Ideal and anti-ideal solutions using Table 8

Ideal Solutions	Anti-ideal Solutions
$d^+\left(\breve{A}_1, \breve{A}^+\right) = 0.19$	$d^+\left(\breve{A}_1, \breve{A}^-\right) = 0.15$
$d^+\left(\breve{A}_2, \breve{A}^+\right) = 0.2$	$d^+\left(\breve{A}_2, \breve{A}^-\right) = 0.21$
$d^+\left(\breve{A}_3, \breve{A}^+\right) = 0.19$	$d^+\left(\breve{A}_3, \breve{A}^-\right) = 0.2$

Table 12: Ideal and anti-ideal solutions using Table 8

Ideal Solutions	Anti-ideal Solutions
$d^+\left(\breve{A}_1, \breve{A}^+\right) = 0.08$	$d^+\left(\breve{A}_1, \breve{A}^-\right) = 0.08$
$d^+\left(\breve{A}_2, \breve{A}^+\right) = 0.12$	$d^+\left(\breve{A}_2, \breve{A}^-\right) = 0.09$
$d^+\left(\breve{A}_3, \breve{A}^+\right) = 0.09$	$d^+\left(\breve{A}_3, \breve{A}^-\right) = 0.12$

Step 6: Table 13 presents closeness coefficient $cc_j; j = 1, 2, 3$, for each alternative. These coefficients are computed using the relative DMs of each alternative from the ideal and anti-ideal solutions. A higher value of closeness coefficient shows that the relevant alternative is closer to the ideal solution and farther from the anti-ideal solution, identifying a better choice. Table 13 thus offers renewable energy source rankings under two scenarios, which will contribute in the final D-making process.

Table 13: Closeness coefficient using CIVIFGBAO and CIVIFWGBAO

Alternative	Scenario 1 Rank	Scenario 2 Rank
Firewall X	0.44	0.5
Endpoint Security Y	0.512	0.43
Zero Trust Z	0.513	0.57

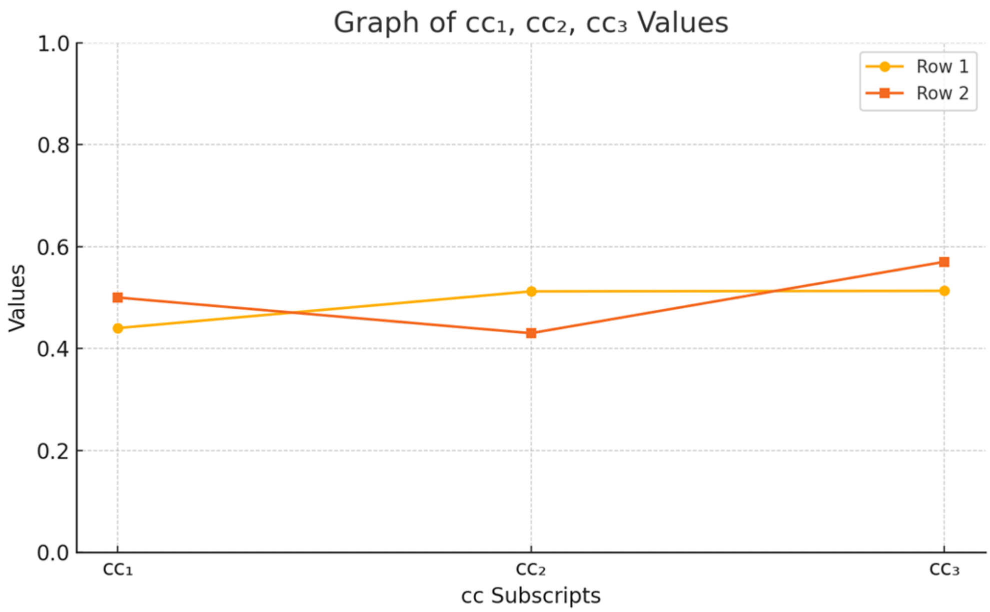

Figure 3: Graphical representation of ranking results using line graph

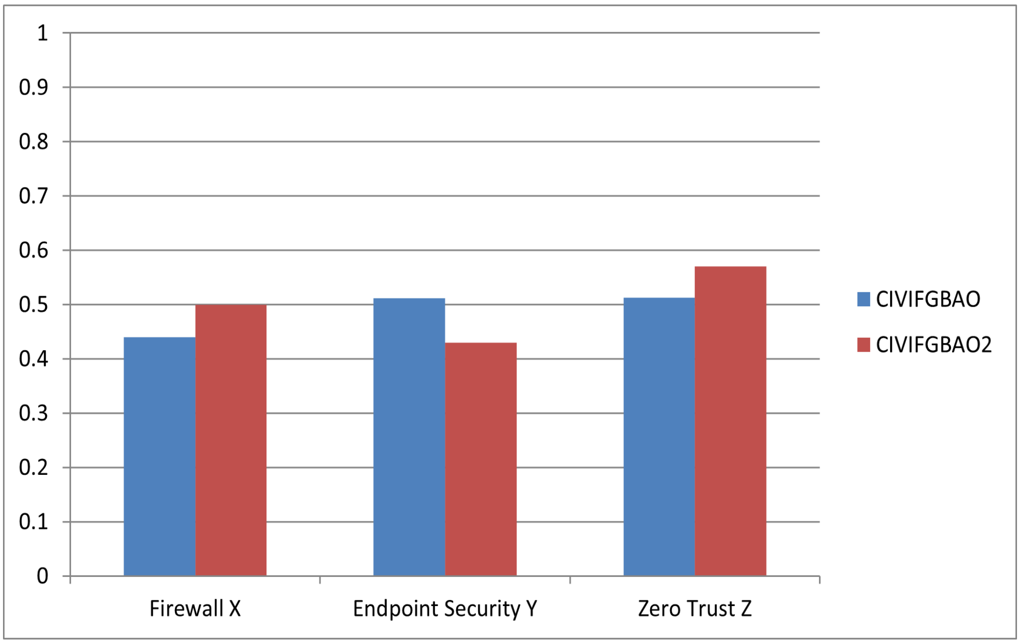

Figure 4: Graphical representation of ranking results using bar graph

Step 7: The ranking of the proposed alternatives, without considering the predefined weight vectors, is obtained as follows: $\breve{A}_3 \succ \breve{A}_2 \succ \breve{A}_1$, identifying that $\breve{A}_3$ is the most desirable choice under equal importance of attributes. Although, after giving specific weight vectors to represent the varying importance of each criteria in the D-making process, the ranking becomes: $\breve{A}_3 \succ \breve{A}_1 \succ \breve{A}_2$. This variation emphasizes the importance of attribute weights on the final selection while also demonstrating the proposed approach's flexibility and adaptation to both weighted and unweight settings.

Therefor the alternative Zero Trust Z "$\breve{A}_3$" is better choice according to proposed method.

Comparison Analysis

This section compares the ranking values produced using the existing technique with the proposed ranking outcomes in Table 11. The Bonferroni geometric AOs, Einstein geometric AOs, Elliptic AOs, Gaussian AOs, Prioritized AOs, have been widely utilized in many fuzzy contexts to improve D-making procedures by capturing the interactions of criteria. A comparison analysis is a comprehensive and powerful method for identifying the validity and supremacy of the newly defined AOs. The following data has been organized for comparative analysis. Bi et al. [12] proposed averaging operators using CFSs, and Bi et al. [[39]] introduced geometric operators in the complex fuzzy environment. Moreover, Zhou and He defined geometric Bonferroni operators for IFSs [59], Park et al. [[51]] developed weighted geometric Bonferroni operators using IFSs, and Unver [[25]] presented innovative Gaussian operators for IFSs. Additionally, Liu and Li proposed power Bonferroni operators using IVIFSs [[52]], and Zhang [[53]] developed geometric Bonferroni operators for IVIFSs. Recently, Rahman [[54]] introduced Einstein geometric operators for CIFSs, and Liu et al. [[55]] demonstrated prioritized operators using CIFSs. Consequently, Table 14 presents a comparison study using the information shown in Tables 3-5.

Table 14: Comparative Analysis of the proposed methods and existing methods

Methods	Score Values	Ranking Values
Bi [[19]]	0.0, 0.0, 0.0	Not possible
Bi [[39]]	0.0, 0.0, 0.0	Not possible
Zhou [[56]]	0.0, 0.0, 0.0	Not possible
Park [[51]]	0.0, 0.0, 0.0	Not possible
Unver [[25]]	0.0, 0.0, 0.0	Not possible
Liu [[52]]	0.0, 0.0, 0.0	Not possible
Zhang [[53]]	0.0, 0.0, 0.0	Not possible
Rahman [[54]]	0.0, 0.0, 0.0	Not possible
Liu [[55]]	0.0, 0.0, 0.0	Not possible
Proposed CIVIFGBAO	0.44, 0.512, 0.513	$\breve{A}_3 \succ \breve{A}_2 \succ \breve{A}_1$
Proposed CIVIFWGBAO	0.5, 0.43, 0.57	$\breve{A}_3 \succ \breve{A}_1 \succ \breve{A}_2$

Figure 5 illustrates a graphical representation of the proposed methods and existing methods.

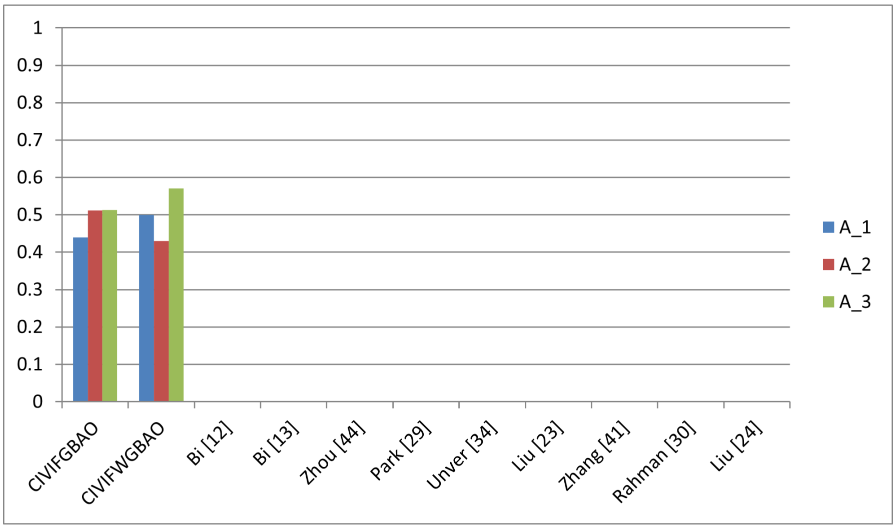

Figure 5: Graphical illustration of the proposed and existing methods

The alternative Zero Trust Z "$\breve{A}_3$" is the best choice according to proposed methods such as CIVIFGBAO and CIVIFWGBAO. However, existing methods cannot be used to tackle this problem since the proposed operators are developed with a new idea, the CIVIFSs, which is a more advanced and complete model than traditional methods. In the context of AOs, the progression from FSs to IVIFSs, CIFSs, and then to CIVIFSs presents a progressive improvement in the capacity to represent and control uncertainty. Table 15 presents a discussion of the characteristic-wise comparison between the proposed approaches and existing methodologies.

For handling interval-valued periodic and multidimensional data, the proposed methods provide the most extensive uncertainty modeling by integrating interval analysis, intuitionistic hesitation, and complex-valued logic. Moreover, the limitations and ambiguities inherent in existing techniques are discussed in the following.

(1). Bi et al. [[19]] proposed averaging operators, and Bi et al. [[39]] introduced geometric operators using CFSs to solve D-making problems. In the basic structure of CFS, the truth grade is expressed as a complex number, but this model is not sufficient to adequately capture the falsity function as well as the interval numbers. These approaches, therefore, cannot be used to assess the information in Tables 3-5, since they are only specific illustrations of the more general proposed framework.

(2). Zhou and He demonstrated geometric Bonferroni operators using IFSs [[56]], Park et al. [[51]] developed weighted geometric Bonferroni operators for IFSs, and Unver [[25]] defined novel Gaussian operators under IF information. In the framework of IFS, the truth grade and false grade are real-valued, but this model is not able to handle two-dimensional as well as interval numbers. Therefore, the IFS model cannot be used to evaluate the information given in Tables 3-5, since they are only a particular case of the proposed approaches.

(3). Liu and Li gave the idea of power Bonferroni operators based on IVIFSs [[52]], and Zhang [[53]] proposed geometric Bonferroni operators using IVIFSs. IVIFS model is

Table 15: Characteristic-wise comparison

Methods	Truth	Falsity	Two-dimensional	Interval support	Interval support with complex numbers
Bi [[19]]	Yes	No	Yes	NO	No
Bi [[39]]	Yes	No	Yes	No	No
Zhou [[56]]	Yes	Yes	No	No	No
Park [[51]]	Yes	Yes	No	No	No
Unver [[25]]	Yes	Yes	No	No	No
Liu [[52]]	Yes	Yes	No	Yes	No
Zhang [[53]]	Yes	Yes	No	Yes	No
Rahman [[54]]	Yes	Yes	Yes	No	No
Liu [[55]]	Yes	Yes	Yes	No	No
Proposed CIVIFGBAO	Yes	Yes	Yes	Yes	Yes
Proposed CIVIFWGBAO	Yes	Yes	Yes	Yes	Yes

an extension of IFS model where the truth grade and false grade are interval numbers, but this model cannot be able to present two-dimensional phenomena. Therefore, the IVIFS model cannot be used to evaluate the information given in Tables 3-5, since they are only the particular case of the proposed approaches.

(4). Rahman [[54]] proposed innovative Einstein geometric operators based on CIFSs, and Liu et al. [[55]] developed prioritized operators using CIFSs. CIFS model is an extension of IFS model where the truth grade and false grade are complex-valued, but this model cannot be able to present interval numbers. Therefore, the CIFS model cannot be used to evaluate the information given in Tables 3-5, since they are only the particular case of the proposed approaches.

The above discussion leads to the conclusion that the newly defined approaches are more advanced and complete models than the established models.

Key Advantages

The Bonferroni AOs addresses the restriction of existing approaches by enabling interactions between pairs of inputs via its parameters, leading to more comprehensive modeling of mutual influence among criteria. For example, environmental impact and cost may be inversely related, while safety and performance may be positively correlated. The Bonferroni AOs offer accessible control over the relationships between them, allow decision-makers to clearly represent complementarity or redundancy among criteria. Its parameterized structure facilitates decision-makers to modify the level of interaction sensitivity, which is useful for adapting the model to specific problem situations. When incorporated with the CIVIFS model, which already offers complex uncertainty via interval-valued membership and non-membership degrees, as well as phase information, the Bonferroni function provides another layer of precision and flexibility.

Demerits or Limitations

The Bonferroni operator within the framework of CIVIFSs improved real-life modeling by capturing interrelationships between criteria that are generally neglected by traditional aggregation approaches. The proposed operators generalized the Bonferroni operators in the environment of CIFSs, CFSs, IFSs, and FSs. In the proposed AOs,

$$CIVIFGBAO\left(\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \ldots, \mathbb{C}_n\right)$$

$$= \left(\begin{array}{c} \left[\begin{array}{c} 1-\left(1-\prod^n_{\substack{j,k=1\\ j\neq k}}\left[\left(1-\left(1-\underset{.}{a}_j\right)^{\check{S}}.\left(1-\underset{.}{a}_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ 1-\left(1-\prod^n_{\substack{j,k=1\\ j\neq k}}\left[\left(1-\left(1-\dot{b}_j\right)^{\check{S}}.\left(1-\dot{b}_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}}+\check{T}} \end{array} \right] e^{i\left[\begin{array}{c} 2\pi\left(1-\left(1-\prod^n_{\substack{j,k=1\\ j\neq k}}\left[\left(1-\left(1-\lambda_j\right)^{\check{S}}.\left(1-\lambda_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\right), \\ 2\pi\left(1-\left(1-\prod^n_{\substack{j,k=1\\ j\neq k}}\left[\left(1-\left(1-\sigma_j\right)^{\check{S}}.\left(1-\sigma_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\right) \end{array} \right]}, \\ \left[\begin{array}{c} \left(1-\left[\prod^n_{\substack{j,k=1\\ j\neq k}}\left(1-\dot{c}_j{}^{\check{S}}.\dot{c}_k{}^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ \left(1-\left[\prod^n_{\substack{j,k=1\\ j\neq k}}\left(1-\underset{\hat{}}{d}_j{}^{\check{S}}.\underset{\hat{}}{d}_k{}^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{array} \right] e^{i\left[\begin{array}{c} 2\pi\left(1-\left[\prod^n_{\substack{j,k=1\\ j\neq k}}\left(1-\eta_j{}^{\check{S}}.\eta_k{}^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ 2\pi\left(1-\left[\prod^n_{\substack{j,k=1\\ j\neq k}}\left(1-\gamma_j{}^{\check{S}}.\gamma_k{}^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \end{array} \right]} \end{array} \right),$$

and

$CIVIFWGBAO\left(\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \ldots, \mathbb{C}_n\right)$

$$
= \left(\begin{array}{c}
\left[\begin{array}{c} 1-\left(1-\prod^{n}_{\substack{J,K=1\\ j\neq k}}\left[\left(1-\left(1-\underset{\cdot}{a}_j{}^{\mathfrak{w}_j}\right)^{\check{S}}.\left(1-\underset{\cdot}{a}_k{}^{\mathfrak{w}_k}\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ 1-\left(1-\prod^{n}_{\substack{j,k=1\\ j\neq k}}\left[\left(1-\left(1-\dot{b}_j{}^{\mathfrak{w}_j}\right)^{\check{S}}.\left(1-\dot{b}_k{}^{\mathfrak{w}_k}\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{array} \right] e^{i\left[\begin{array}{c} 2\pi\left(1-\left(1-\prod^{n}_{\substack{J,K=1\\ j\neq k}}\left[\left(1-\left(1-\lambda_j{}^{\mathfrak{w}_j}\right)^{\check{S}}.\left(1-\lambda_k{}^{\mathfrak{w}_k}\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\right), \\ \left[2\pi\left(1-\left(1-\prod^{n}_{\substack{J,K=1\\ j\neq k}}\left[\left(1-\left(1-\sigma_j{}^{\mathfrak{w}_j}\right)^{\check{S}}.\left(1-\sigma_k{}^{\mathfrak{w}_k}\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\right)\right] \end{array} \right]}, \\
\left[\begin{array}{c} \left(1-\prod^{n}_{\substack{j,k=1\\ j\neq k}}\left[\left(1-\left(1-\left(1-\dot{c}_j\right)^{\mathfrak{w}_j}\right)^{\check{S}}.\left(1-\left(1-\dot{c}_k\right)^{\mathfrak{w}_k}\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ \left(1-\prod^{n}_{\substack{j,k=1\\ j\neq k}}\left[\left(1-\left(1-\left(1-\underset{\hat{}}{d}_j\right)^{\mathfrak{w}_j}\right)^{\check{S}}.\left(1-\left(1-\underset{\hat{}}{d}_k\right)^{\mathfrak{w}_k}\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{array} \right] e^{i\left[\begin{array}{c} 2\pi\left(1-\prod^{n}_{\substack{j,k=1\\ j\neq k}}\left[\left(1-\left(1-\left(1-\eta_j\right)^{\mathfrak{w}_j}\right)^{\check{S}}.\left(1-\left(1-\eta_k\right)^{\mathfrak{w}_k}\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ 2\pi\left(1-\prod^{n}_{\substack{j,k=1\\ j\neq k}}\left[\left(1-\left(1-\left(1-\gamma_j\right)^{\mathfrak{w}_j}\right)^{\check{S}}.\left(1-\left(1-\gamma_k\right)^{\mathfrak{w}_k}\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{array} \right]}
\end{array} \right),
$$

the following conditions are satisfied.

$$
1-\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-\left(1-\dot{b}_j\right)^{\check{S}}.\left(1-\dot{b}_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}+\left(1-\left[\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left(1-\underset{\hat{}}{d}_j{}^{\check{S}}.\underset{\hat{}}{d}_k{}^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\leq 1,
$$

$$
2\pi\left(1-\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-\left(1-\sigma_j\right)^{\check{S}}.\left(1-\sigma_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\right)+2\pi\left(1-\left[\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left(1-\gamma_j{}^{\check{S}}.\gamma_k{}^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\leq 2\pi,
$$

$$
1-\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-\left(1-\dot{b}_j{}^{\mathfrak{w}_j}\right)^{\check{S}}.\left(1-\dot{b}_k{}^{\mathfrak{w}_k}\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}+\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-\left(1-\left(1-\underset{\hat{}}{d}_j\right)^{\mathfrak{w}_j}\right)^{\check{S}}.\left(1-\left(1-\underset{\hat{}}{d}_k\right)^{\mathfrak{w}_k}\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\leq 1,
$$

and

$$
2\pi\left(1-\left(1-\prod_{\substack{J,K=1\\ j\neq k}}^{n}\left[\left(1-\left(1-\sigma_j{}^{\mathfrak{w}_j}\right)^{\check{S}}.\left(1-\sigma_k{}^{\mathfrak{w}_k}\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\right)+2\pi\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-\left(1-\left(1-\gamma_j\right)^{\mathfrak{w}_j}\right)^{\check{S}}.\left(1-\left(1-\gamma_k\right)^{\mathfrak{w}_k}\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\leq 2\pi.
$$

But if the information are given in the form of $\mathbb{C}_j = \left(\begin{array}{c} \left[\underset{\cdot}{\mathrm{a}}_j, \dot{b}_j \right] e^{i[2\pi\lambda_j, 2\pi\sigma_j]}, \\ \left[\dot{c}_j, \underset{\hat{}}{\mathrm{d}}_j \right] e^{i[2\pi\eta_j, 2\pi\gamma_j]} \end{array} \right)$ where $\dot{b}_j + \underset{\hat{}}{\mathrm{d}}_j > 1$ and $2\pi\sigma_j + 2\pi\gamma_j > 2\pi$. Then, the given conditions are failed. Therefore, the proposed model can be generalized to complex interval-valued Pythagorean fuzzy sets or complex interval-valued q-rung orthopair fuzzy sets.

8. Conclusion

Selecting the best cybersecurity tools to protect networks, data, and systems against changing cyber threats is a part of the D-making procedure. In order to improve D-making in complex circumstances, this study presented an extensive investigation of advanced AOs and DMs based on CIVIFS theory. The newly defined methods have been methodically utilized to develop an innovative D-making algorithm made especially for the selection of cybersecurity tools. The algorithm's reliability and effectiveness show how the proposed CIVIF AOs and DMs can be applied practically in real-world cybersecurity problems. Through a real-world case study involving three alternatives and four criteria, the proposed approach ranked $\breve{A}_3$ highest with a score function of 0.513 and 0.57 using CIVIFGBAOs and CIVIFWGBAOs. Comparative analysis revealed that our proposed method outperformed established MCDM techniques.

8.1. Future Direction

In the future, we will investigate applying the proposed geometric Bonferroni AOs, weighted geometric Bonferroni AOs, and newly defined DMs to some other domains, such as complex interval-valued Pythagorean fuzzy sets, interval-valued complex neutrosophic sets, interval-valued complex neutrosophic soft sets, and linguistic CIVIFSs, to enhance the quality of the newly defined approaches. The proposed approach could be utilized in diverse real-world D-making problems, including electric vehicle site selection and solid waste management. The proposed methodology could be further extended to handle more complex D-making situations such as MADM using q-rung orthopair FSs, including practical applications like electric vehicle charging station site selection. It can also be employed to assess municipal solid waste management systems utilizing a confidence level-based D-making framework in a q-rung orthopair image fuzzy environment [[57]]. Moreover, we will integrate the proposed approaches with artificial intelligence techniques such as neural networks [[58]], machine learning [[59]], and expert systems to improve the ability of artificial intelligence systems [[60]].

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Conflict of Interest

The authors declare that they have no conflict of interests.

Authorship Contributions

All authors contributed equally.

Data Availability

Our manuscript has no associated data.

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Appendix A

Let $\mathbb{C}_j = \begin{pmatrix} \left[\underset{\cdot}{a}_j, \dot{b}_j\right] e^{i[2\pi\lambda_j, 2\pi\sigma_j]}, \\ \left[\dot{c}_j, \underset{\wedge}{d}_j\right] e^{i[2\pi\eta_j, 2\pi\gamma_j]} \end{pmatrix}$ and $\mathbb{C}_k = \begin{pmatrix} \left[\underset{\cdot}{a}_k, \dot{b}_k\right] e^{i[2\pi\lambda_k, 2\pi\sigma_k]}, \\ \left[\dot{c}_k, \underset{\wedge}{d}_k\right] e^{i[2\pi\eta_k, 2\pi\gamma_k]} \end{pmatrix}$ describe two CIVIFNs that are defined on a US U. Then, by definition $\dot{b}_j + \underset{\wedge}{d}_j \leq 1$ and $2\pi\sigma_j + 2\pi\gamma_j \leq 2\pi$ or $\sigma_j + \gamma_j \leq 1$ with $\underset{\cdot}{a}_j$, $\dot{b}_j$, λ_j, σ_j, $\dot{c}_j, \underset{\wedge}{d}_j$, $\eta_j, \gamma_j \in [0,1]$. By Definition of proposed Bonferroni operational law, we have

$$\mathbb{C}_j \otimes \mathbb{C}_k$$

$$= \left(\begin{array}{c} \begin{bmatrix} 1 - \left(1 - \left[\left(1 - (1-\underset{\cdot}{a}_j)^{\check{S}} . (1-\underset{\cdot}{a}_k)^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1 - (1-\underset{\cdot}{a}_k)^{\check{S}} . (1-\underset{\cdot}{a}_j)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ 1 - \left(1 - \left[\left(1 - \left(1-\dot{b}_j\right)^{\check{S}} . \left(1-\dot{b}_k\right)^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1 - \left(1-\dot{b}_k\right)^{\check{S}} . \left(1-\dot{b}_j\right)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{bmatrix} \\ e^{i\begin{bmatrix} 2\pi\left(1 - \left(1 - \left[\left(1 - (1-\lambda_j)^{\check{S}} . (1-\lambda_k)^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1 - (1-\lambda_k)^{\check{S}} . (1-\lambda_j)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\right), \\ 2\pi\left(\left(1 - \left(1 - \left[\left(1 - (1-\sigma_j)^{\check{S}} . (1-\sigma_k)^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1 - (1-\sigma_k)^{\check{S}} . (1-\sigma_j)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\right)\right) \end{bmatrix}}, \\ \begin{bmatrix} \left(1 - \left[\left(1 - \dot{c}_j{}^{\check{S}} . \dot{c}_k{}^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1 - \dot{c}_k{}^{\check{S}} . \dot{c}_j{}^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ \left(1 - \left[\left(1 - \underset{\wedge}{d}_j{}^{\check{S}} . \underset{\wedge}{d}_k{}^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1 - \underset{\wedge}{d}_k{}^{\check{S}} . \underset{\wedge}{d}_j{}^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{bmatrix} e^{i\begin{bmatrix} 2\pi\left(1 - \left[\left(1 - \eta_j{}^{\check{S}} . \eta_k{}^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1 - \eta_k{}^{\check{T}} . \eta_j{}^{\check{S}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ 2\pi\left(1 - \left[\left(1 - \gamma_j{}^{\check{S}} . \gamma_k{}^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1 - \gamma_k{}^{\check{S}} . \gamma_j{}^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{bmatrix}} \end{array} \right)$$

We need to prove that

$$1 - \left(1 - \left[\left(1 - \left(1-\dot{b}_j\right)^{\check{S}} . \left(1-\dot{b}_k\right)^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1 - \left(1-\dot{b}_k\right)^{\check{S}} . \left(1-\dot{b}_j\right)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}$$
$$+ \left(1 - \left[\left(1 - \dot{c}_j{}^{\check{S}} . \dot{c}_k{}^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1 - \dot{c}_k{}^{\check{S}} . \dot{c}_j{}^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \leq 1,$$

and

$$\left(1 - \left(1 - \left[\left(1 - (1-\sigma_j)^{\check{S}} . (1-\sigma_k)^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1 - (1-\sigma_k)^{\check{S}} . (1-\sigma_j)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\right)$$
$$+ \left(1 - \left[\left(1 - \gamma_j{}^{\check{S}} . \gamma_k{}^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1 - \gamma_k{}^{\check{S}} . \gamma_j{}^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \leq 1.$$

Since,

$$1-\left(1-\left[\left(1-\left(1-\dot{b}_j\right)^{\check{S}}.\left(1-\dot{b}_k\right)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\times\left(1-\left(1-\dot{b}_k\right)^{\check{S}}.\left(1-\dot{b}_j\right)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}$$

$$+\left(1-\left[\left(1-\dot{c}_j{}^{\check{S}}.\dot{c}_k{}^{\check{T}}\right)^{\frac{1}{2(2-1)}}\times\left(1-\dot{c}_k{}^{\check{S}}.\dot{c}_j{}^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}$$

$$\leq 1-\left(1-\left[\left(1-\left(1-\dot{b}_j\right)^{\check{S}}.\left(1-\dot{b}_k\right)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\times\left(1-\left(1-\dot{b}_k\right)^{\check{S}}.\left(1-\dot{b}_j\right)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}$$

$$+\left(1-\left[\left(1-\left(1-\dot{b}_j\right)^{\check{S}}.\left(1-\dot{b}_k\right)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\times\left(1-\left(1-\dot{b}_k\right)^{\check{S}}.\left(1-\dot{b}_j\right)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}=1.$$

Also,

$$\left(1-\left(1-\left[\left(1-(1-\sigma_j)^{\check{S}}.(1-\sigma_k)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\times\left(1-(1-\sigma_k)^{\check{S}}.(1-\sigma_j)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\right)$$

$$+\left(1-\left[\left(1-\gamma_j{}^{\check{S}}.\gamma_k{}^{\check{T}}\right)^{\frac{1}{2(2-1)}}\times\left(1-\gamma_k{}^{\check{S}}.\gamma_j{}^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}$$

$$\leq 1-\left(1-\left[\left(1-(1-\sigma_j)^{\check{S}}.(1-\sigma_k)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\times\left(1-(1-\sigma_k)^{\check{S}}.(1-\sigma_j)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}$$

$$+\left(1-\left[\left(1-(1-\sigma_j)^{\check{S}}.(1-\sigma_k)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\times\left(1-(1-\sigma_k)^{\check{S}}.(1-\sigma_j)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}=1.$$

Thus, $\mathbb{C}_j\otimes\mathbb{C}_k$ is a CIVIFN.

Similarly, we can prove that $\mathfrak{w}_j\mathbb{C}_j\otimes\mathfrak{w}_k\mathbb{C}_k$ is a CIVIFN.

Appendix B

Let $\mathbb{C}_j=\left(\begin{array}{c}\left[\underset{\cdot}{a}_j,\dot{b}_j\right]e^{i[2\pi\lambda_j,2\pi\sigma_j]},\\ \left[\dot{c}_j,\underset{\frown}{d}_j\right]e^{i[2\pi\eta_j,2\pi\gamma_j]}\end{array}\right)$ and $\mathbb{C}_k=\left(\begin{array}{c}\left[\underset{\cdot}{a}_k,\dot{b}_k\right]e^{i[2\pi\lambda_k,2\pi\sigma_k]},\\ \left]\dot{c}_k,\underset{\frown}{d}_k\right]e^{i[2\pi\eta_k,2\pi\gamma_k]}\end{array}\right)$ be two CIVIFNs defined on a US U. Then, by Definition of $\mathbb{C}_j\otimes\mathbb{C}_k$, we have

$\mathbb{C}_j \otimes \mathbb{C}_k$

$$= \left(\begin{bmatrix} 1-\left(1-\left[\left(1-(1-\underset{\cdot}{a}_j)^{\check{S}}.(1-\underset{\cdot}{a}_k)^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1-(1-\underset{\cdot}{a}_k)^{\check{S}}.(1-\underset{\cdot}{a}_j)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ 1-\left(1-\left[\left(1-\left(1-\dot{b}_j\right)^{\check{S}}.\left(1-\dot{b}_k\right)^{\check{T}}\right)^{\frac{1}{(2-1)}} \times \left(1-\left(1-\dot{b}_k\right)^{\check{S}}.\left(1-\dot{b}_j\right)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{bmatrix} e^{i\begin{bmatrix} 2\pi\left(1-\left(1-\left[\left(1-(1-\lambda_j)^{\check{S}}.(1-\lambda_k)^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1-(1-\lambda_k)^{\check{S}}.(1-\lambda_j)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\right), \\ 2\pi\left(\left(1-\left(1-\left[\left(1-(1-\sigma_j)^{\check{S}}.(1-\sigma_k)^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1-(1-\sigma_k)^{\check{S}}.(1-\sigma_j)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\right)\right) \end{bmatrix}}, \\ \begin{bmatrix} \left(1-\left[\left(1-\dot{c}_j{}^{\check{S}}.\dot{c}_k{}^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1-\dot{c}_k{}^{\check{S}}.\dot{c}_j{}^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ \left(1-\left[\left(1-\underset{\wedge}{d}_j{}^{\check{S}}.\underset{\wedge}{d}_k{}^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1-\underset{\wedge}{d}_k{}^{\check{S}}.\underset{\wedge}{d}_j{}^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{bmatrix} e^{i\begin{bmatrix} 2\pi\left(1-\left[\left(1-\eta_j{}^{\check{S}}.\eta_k{}^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1-\eta_k{}^{\check{T}}.\eta_j{}^{\check{S}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ 2\pi\left(1-\left[\left(1-\gamma_j{}^{\check{S}}.\gamma_k{}^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1-\gamma_k{}^{\check{S}}.\gamma_j{}^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{bmatrix}} \right)$$

$\mathbb{C}_j \otimes \mathbb{C}_k$

$$= \left(\begin{bmatrix} 1-\left(1-\left[\left(1-(1-\underset{\cdot}{a}_k)^{\check{T}}.(1-\underset{\cdot}{a}_j)^{\check{S}}\right)^{\frac{1}{2(2-1)}} \times \left(1-(1-\underset{\cdot}{a}_j)^{\check{T}}.(1-\underset{\cdot}{a}_k)^{\check{S}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ 1-\left(1-\left[\left(1-\left(1-\dot{b}_k\right)^{\check{T}}.\left(1-\dot{b}_j\right)^{\check{S}}\right)^{\frac{1}{2(2-1)}} \times \left(1-\left(1-\dot{b}_j\right)^{\check{T}}.\left(1-\dot{b}_k\right)^{\check{S}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{bmatrix} e^{i\begin{bmatrix} 2\pi\left(1-\left(1-\left[\left(1-(1-\lambda_k)^{\check{T}}.(1-\lambda_j)^{\check{S}}\right)^{\frac{1}{2(2-1)}} \times \left(1-(1-\lambda_j)^{\check{T}}.(1-\lambda_k)^{\check{S}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\right), \\ 2\pi\left(\left(1-\left(1-\left[\left(1-(1-\sigma_k)^{\check{T}}.(1-\sigma_j)^{\check{S}}\right)^{\frac{1}{2(2-1)}} \times \left(1-(1-\sigma_j)^{\check{T}}.(1-\sigma_k)^{\check{S}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\right)\right) \end{bmatrix}}, \\ \begin{bmatrix} \left(1-\left[\left(1-\dot{c}_k{}^{\check{T}}.\dot{c}_j{}^{\check{S}}\right)^{\frac{1}{2(2-1)}} \times \left(1-\dot{c}_j{}^{\check{T}}.\dot{c}_k{}^{\check{S}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ \left(1-\left[\left(1-\underset{\wedge}{d}_k{}^{\check{T}}.\underset{\wedge}{d}_j{}^{\check{S}}\right)^{\frac{1}{2(2-1)}} \times \left(1-\underset{\wedge}{d}_j{}^{\check{T}}.\underset{\wedge}{d}_k{}^{\check{S}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{bmatrix} e^{i\begin{bmatrix} 2\pi\left(1-\left[\left(1-\eta_k{}^{\check{T}}.\eta_j{}^{\check{S}}\right)^{\frac{1}{2(2-1)}} \times \left(1-\eta_j{}^{\check{T}}.\eta_k{}^{\check{S}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ 2\pi\left(1-\left[\left(1-\gamma_k{}^{\check{T}}.\gamma_j{}^{\check{S}}\right)^{\frac{1}{2(2-1)}} \times \left(1-\gamma_j{}^{\check{T}}.\gamma_k{}^{\check{S}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{bmatrix}} \right)$$

$\mathbb{C}_j \otimes \mathbb{C}_k = \mathbb{C}_k \otimes \mathbb{C}_j$.

Similarly, we can demonstrate (ii).

Appendix C

We apply the principle of mathematical induction to demonstrate this theorem. We first verify the basic condition, namely, that the given result is true for $n = 2$. By the Bonferroni operational law, defined in Definition (5), we have

$$\mathbb{C}_j \otimes \mathbb{C}_k = \left(\left[\underset{\cdot}{a}_j, \dot{b}_j\right] e^{i[2\pi\lambda_j, 2\pi\sigma_j]}, \left[\dot{c}_j, \underset{\wedge}{d}_j\right] e^{i[2\pi\eta_j, 2\pi\gamma_j]}\right) \otimes \left(\left[\underset{\cdot}{a}_k, \dot{b}_k\right] e^{i[2\pi\lambda_k, 2\pi\sigma_k]}, \left[\dot{c}_k, \underset{\wedge}{d}_k\right] e^{i[2\pi\eta_k, 2\pi\gamma_k]}\right),$$

$\mathbb{C}_j \otimes \mathbb{C}_k$

$$= \left(\begin{array}{c} \left[\begin{array}{c} 1-\left(1-\left[\left(1-(1-a_j)^{\check{S}}.(1-a_k)^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1-(1-a_k)^{\check{S}}.(1-a_j)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ 1-\left(1-\left[\left(1-\left(1-\dot{b}_j\right)^{\check{S}}.\left(1-\dot{b}_k\right)^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1-\left(1-\dot{b}_k\right)^{\check{S}}.\left(1-\dot{b}_j\right)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{array} \right] \\ e^{i\left[\begin{array}{c} 2\pi\left(1-\left(1-\left[\left(1-(1-\lambda_j)^{\check{S}}.(1-\lambda_k)^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1-(1-\lambda_k)^{\check{S}}.(1-\lambda_j)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\right), \\ 2\pi\left(\left(1-\left(1-\left[\left(1-(1-\sigma_j)^{\check{S}}.(1-\sigma_k)^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1-(1-\sigma_k)^{\check{S}}.(1-\sigma_j)^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\right)\right) \end{array} \right]}, \\ \left[\begin{array}{c} \left(1-\left[\left(1-\dot{c}_j{}^{\check{S}}.\dot{c}_k{}^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1-\dot{c}_k{}^{\check{S}}.\dot{c}_j{}^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ \left(1-\left[\left(1-d_j{}^{\check{S}}.d_k{}^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1-d_k{}^{\check{S}}.d_j{}^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{array} \right] e^{i\left[\begin{array}{c} 2\pi\left(1-\left[\left(1-\eta_j{}^{\check{S}}.\eta_k{}^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1-\eta_k{}^{\check{S}}.\eta_j{}^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ 2\pi\left(1-\left[\left(1-\gamma_j{}^{\check{S}}.\gamma_k{}^{\check{T}}\right)^{\frac{1}{2(2-1)}} \times \left(1-\gamma_k{}^{\check{S}}.\gamma_j{}^{\check{T}}\right)^{\frac{1}{2(2-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{array} \right]} \end{array} \right).$$

which is hold for $n = 2$.

Assume that the given statement is hold for $n = m$.

We have to prove that the given statement is true for$n = m + 1$. For this, we have

$CIVIFWGBAO\,(\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \ldots, \mathbb{C}_{m+1}) = CIVIFWGBAO\,(\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \ldots, \mathbb{C}_m) \otimes C_{m+1}$,

$CIVIFWGBAO\,(\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \ldots, \mathbb{C}_{m+1})$

$$= \left(\begin{array}{c} \left[\begin{array}{c} 1-\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{m}\left[\left(1-(1-a_j)^{\check{S}}.(1-a_k)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ 1-\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{m}\left[\left(1-\left(1-\dot{b}_j\right)^{\check{S}}.\left(1-\dot{b}_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{array} \right] e^{i\left[\begin{array}{c} 2\pi\left(1-\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{m}\left[\left(1-(1-\lambda_j)^{\check{S}}.(1-\lambda_k)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\right), \\ 2\pi\left(1-\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{m}\left[\left(1-(1-\sigma_j)^{\check{S}}.(1-\sigma_k)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\right) \end{array} \right]}, \\ \left[\begin{array}{c} \left(1-\left[\prod_{\substack{j,k=1\\ j\neq k}}^{m}\left(1-\dot{c}_j{}^{\check{S}}.\dot{c}_k{}^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ \left(1-\left[\prod_{\substack{j,k=1\\ j\neq k}}^{m}\left(1-d_j{}^{\check{S}}.d_k{}^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{array} \right] e^{i\left[\begin{array}{c} 2\pi\left(1-\left[\prod_{\substack{j,k=1\\ j\neq k}}^{m}\left(1-\eta_j{}^{\check{S}}.\eta_k{}^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ 2\pi\left(1-\left[\prod_{\substack{j,k=1\\ j\neq k}}^{m}\left(1-\gamma_j{}^{\check{S}}.\gamma_k{}^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \end{array} \right]} \end{array} \right)$$

$$\otimes \left(\begin{array}{c} \left[a_{m+1}, \dot{b}_{m+1}\right] e^{i[2\pi\lambda_{m+1}, 2\pi\sigma_{m+1}]}, \\ \left[\dot{c}_{m+1}, d_{m+1}\right] e^{i[2\pi\eta_{m+1}, 2\pi\gamma_{m+1}]} \end{array} \right),$$

$$CIVIFWGBAO\left(\mathbb{C}_1,\mathbb{C}_2,\mathbb{C}_3,\ldots,\mathbb{C}_{m+1}\right)$$

$$= \left(\begin{array}{c} \left[\begin{array}{c} 1-\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{m+1}\left[\left(1-\left(1-\underset{.}{a}_j\right)^{\check{S}}\cdot\left(1-\underset{.}{a}_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ 1-\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{m+1}\left[\left(1-\left(1-\dot{b}_j\right)^{\check{S}}\cdot\left(1-\dot{b}_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{array} \right] e^{i\left[\begin{array}{c} 2\pi\left(1-\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{m+1}\left[\left(1-\left(1-\lambda_j\right)^{\check{S}}\cdot\left(1-\lambda_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\right), \\ 2\pi\left(1-\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{m+1}\left[\left(1-\left(1-\sigma_j\right)^{\check{S}}\cdot\left(1-\sigma_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\right) \end{array} \right]}, \\ \left[\begin{array}{c} \left(1-\left[\prod_{\substack{j,k=1\\ j\neq k}}^{m+1}\left(1-\dot{c}_j^{\check{S}}.\dot{c}_k^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ \left(1-\left[\prod_{\substack{j,k=1\\ j\neq k}}^{m+1}\left(1-\underset{.}{d}_j^{\check{S}}.\underset{.}{d}_k^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{array} \right] e^{i\left[\begin{array}{c} 2\pi\left(1-\left[\prod_{\substack{j,k=1\\ j\neq k}}^{m+1}\left(1-\eta_j^{\check{S}}.\eta_k^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ 2\pi\left(1-\left[\prod_{\substack{j,k=1\\ j\neq k}}^{m+1}\left(1-\gamma_j^{\check{S}}.\gamma_k^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \end{array} \right]} \end{array} \right).$$

Thus, the given statement is true for $n = m + 1$.

Hence by Principle of Mathematical induction the aggregated value of $\mathbb{C}_j; j = 1, 2, \ldots, n$ by using CIVIFGBAO is a CIVIFN.

Appendix D

If all $\mathbb{C}_j; j = 1, 2, 3, ..., n$, are equal, that is, $\mathbb{C}_j = \mathbb{C}$ for all$j$, then by Definition of CIVIFGBAO, we have

$$CIVIFGBAO\left(\mathbb{C}_1,\mathbb{C}_2,\mathbb{C}_3,\ldots,\mathbb{C}_n\right)$$

$$= \left(\begin{array}{c} \left[\begin{array}{c} 1-\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-\left(1-\underset{.}{a}_j\right)^{\check{S}}\cdot\left(1-\underset{.}{a}_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ 1-\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-\left(1-\dot{b}_j\right)^{\check{S}}\cdot\left(1-\dot{b}_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{array} \right] e^{i\left[\begin{array}{c} 2\pi\left(1-\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-\left(1-\lambda_j\right)^{\check{S}}\cdot\left(1-\lambda_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\right), \\ 2\pi\left(1-\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-\left(1-\sigma_j\right)^{\check{S}}\cdot\left(1-\sigma_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\right) \end{array} \right]}, \\ \left[\begin{array}{c} \left(1-\left[\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left(1-\dot{c}_j^{\check{S}}.\dot{c}_k^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ \left(1-\left[\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left(1-\underset{.}{d}_j^{\check{S}}.\underset{.}{d}_k^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{array} \right] e^{i\left[\begin{array}{c} 2\pi\left(1-\left[\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left(1-\eta_j^{\check{S}}.\eta_k^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ 2\pi\left(1-\left[\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left(1-\gamma_j^{\check{S}}.\gamma_k^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \end{array} \right]} \end{array} \right)$$

$CIVIFGBAO(\mathbb{C},\mathbb{C},\mathbb{C},\ldots,\mathbb{C})$

$$= \left(\begin{array}{c} \left[\begin{array}{c} 1-\left(1-\prod^{n}_{\substack{j,k=1\\ j\neq k}} \left[\left(1-(1-\underset{.}{a})^{\check{S}+\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ 1-\left(1-\prod^{n}_{\substack{j,k=1\\ j\neq k}} \left[\left(1-(1-\dot{b})^{\check{S}+\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{array} \right] e^{i\left[\begin{array}{c} 2\pi\left(1-\left(1-\prod^{n}_{\substack{j,k=1\\ j\neq k}} \left[\left(1-(1-\lambda)^{\check{S}++\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\right), \\ 2\pi\left(1-\left(1-\prod^{n}_{\substack{j,k=1\\ j\neq k}} \left[\left(1-(1-\sigma)^{\check{S}+\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\right) \end{array} \right]}, \\ \left[\begin{array}{c} \left(1-\left[\prod^{n}_{\substack{j,k=1\\ j\neq k}} \left(1-\dot{c}^{\check{S}+\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ \left(1-\left[\prod^{n}_{\substack{j,k=1\\ j\neq k}} \left(1-\underset{\hat{}}{d}^{\check{S}+\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{array} \right] e^{i\left[\begin{array}{c} \left(1-\left[\prod^{n}_{\substack{j,k=1\\ j\neq k}} \left(1-\eta_j^{\check{S}+\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ \left(1-\left[\prod^{n}_{\substack{j,k=1\\ j\neq k}} \left(1-\gamma^{\check{S}+\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{array} \right]} \end{array} \right),$$

$$CIVIFGBAO(\mathbb{C},\mathbb{C},\mathbb{C},\ldots,\mathbb{C}) = \left(\begin{array}{c} \left[\begin{array}{c} 1-\left((1-\underset{.}{a})^{\check{S}+\check{T}}\right)^{\frac{1}{\check{S}+\check{T}}}, \\ 1-\left((1-\dot{b})^{\check{S}+\check{T}}\right)^{\frac{1}{\check{S}+\check{T}}} \end{array} \right] e^{i\left[\begin{array}{c} 1-\left((1-\lambda)^{\check{S}+\check{T}}\right)^{\frac{1}{\check{S}+\check{T}}}, \\ 1-\left((1-\sigma)^{\check{S}+\check{T}}\right)^{\frac{1}{\check{S}+\check{T}}} \end{array} \right]}, \\ \left[\begin{array}{c} \left(1-\left[1-\dot{c}^{\check{S}+\check{T}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ \left(1-\left[1-\underset{\hat{}}{d}^{\check{S}+\check{T}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{array} \right] e^{i\left[\begin{array}{c} \left(1-\left[1-\eta^{\check{S}+\check{T}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}, \\ \left(1-\left[1-\gamma^{\check{S}+\check{T}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \end{array} \right]} \end{array} \right),$$

$$CIVIFGBAO(\mathbb{C},\mathbb{C},\mathbb{C},\ldots,\mathbb{C}) = \left([\underset{.}{a},\dot{b}]e^{i[2\pi\lambda,2\pi\sigma]},[\dot{c},\underset{\hat{}}{d}]e^{i[2\pi\eta,2\pi\gamma]}\right),$$

$$CIVIFGBAO(\mathbb{C},\mathbb{C},\mathbb{C},\ldots,\mathbb{C}) = \mathbb{C}.$$

Appendix E

Since $\underset{.}{a}_j \leq \underset{.}{a}'_j,\ \dot{b}_j \leq \dot{b}'_j,\ \lambda_j \leq \lambda'_j,\ \sigma_j \leq \sigma'_j,\ \dot{c}_j \geq \dot{c}'_j,\ \underset{.}{d}_j \geq \underset{.}{d}'_j,\ \eta_j \geq \eta'_j,\ \gamma_j \geq \gamma'_j$ for all j, then

$$\begin{gathered}
\left(1-\d{a}_j\right)^{\check{S}}.\left(1-\d{a}_k\right)^{\check{T}} \geq \left(1-\d{a}'_j\right)^{\check{S}}.\left(1-\d{a}'_k\right)^{\check{T}},\\
\left(1-\dot{b}_j\right)^{\check{S}}.\left(1-\dot{b}_k\right)^{\check{T}} \geq \left(1-\dot{b}'_j\right)^{\check{S}}.\left(1-\dot{b}'_k\right)^{\check{T}},\\
\left(1-\lambda_j\right)^{\check{S}}.\left(1-\lambda_k\right)^{\check{T}} \geq \left(1-\lambda'_j\right)^{\check{S}}.\left(1-\lambda'_k\right)^{\check{T}},\\
\left(1-\sigma_j\right)^{\check{S}}.\left(1-\sigma_k\right)^{\check{T}} \geq \left(1-\sigma'_j\right)^{\check{S}}.\left(1-\sigma'_k\right)^{\check{T}},
\end{gathered}$$

and

$$\begin{gathered}
\dot{c}^{\check{S}}.\dot{c}^{\check{T}} \geq \dot{c}'^{\check{S}}.\dot{c}'^{\check{T}},\\
\d{d}^{\check{S}}.\d{d}^{\check{T}} \geq \d{d}'^{\check{S}}.\d{d}'^{\check{T}},\\
\eta^{\check{S}}.\eta^{\check{T}} \geq \eta'^{\check{S}}.\eta'^{\check{T}},\\
\gamma^{\check{S}}.\gamma^{\check{T}} \geq \gamma'^{\check{S}}.\gamma'^{\check{T}}.
\end{gathered}$$

This implies that

$$\begin{gathered}
1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-\left(1-\d{a}_j\right)^{\check{S}}.\left(1-\d{a}_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right] \geq 1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-\left(1-\d{a}'_j\right)^{\check{S}}.\left(1-\d{a}'_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right],\\
1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-\left(1-\dot{b}_j\right)^{\check{S}}.\left(1-\dot{b}_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right] \geq 1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-\left(1-\dot{b}'_j\right)^{\check{S}}.\left(1-\dot{b}'_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right],\\
1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-\left(1-\lambda_j\right)^{\check{S}}.\left(1-\lambda_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right] \geq 1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-\left(1-\lambda'_j\right)^{\check{S}}.\left(1-\lambda'_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right],\\
1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-\left(1-\sigma_j\right)^{\check{S}}.\left(1-\sigma_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right] \geq 1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-\left(1-\sigma'_j\right)^{\check{S}}.\left(1-\sigma'_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right],
\end{gathered}$$

and

$$\begin{gathered}
\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left(1-\dot{c}^{\check{S}+\check{T}}\right)^{\frac{1}{n(n-1)}} \leq \prod_{\substack{j,k=1\\ j\neq k}}^{n}\left(1-\dot{c}'^{\check{S}+\check{T}}\right)^{\frac{1}{n(n-1)}},\\
\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left(1-\d{d}^{\check{S}+\check{T}}\right)^{\frac{1}{n(n-1)}} \leq \prod_{\substack{j,k=1\\ j\neq k}}^{n}\left(1-\d{d}'^{\check{S}+\check{T}}\right)^{\frac{1}{n(n-1)}},
\end{gathered}$$

$$\prod_{\substack{j,k=1\\j\neq k}}^{n}\left(1-\eta^{\check{S}+\check{T}}\right)^{\frac{1}{n(n-1)}} \leq \prod_{\substack{j,k=1\\j\neq k}}^{n}\left(1-\eta'^{\check{S}+\check{T}}\right)^{\frac{1}{n(n-1)}},$$

$$\prod_{\substack{j,k=1\\j\neq k}}^{n}\left(1-\gamma^{\check{S}+\check{T}}\right)^{\frac{1}{n(n-1)}} \leq \prod_{\substack{j,k=1\\j\neq k}}^{n}\left(1-\gamma'^{\check{S}+\check{T}}\right)^{\frac{1}{n(n-1)}}.$$

Therefore,

$$1-\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-\left(1-ạ_j\right)^{\check{S}}\cdot\left(1-ạ_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}$$

$$\leq 1-\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-\left(1-ạ'_j\right)^{\check{S}}\cdot\left(1-ạ'_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}},$$

$$1-\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-\left(1-\dot{b}_j\right)^{\check{S}}\cdot\left(1-\dot{b}_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}$$

$$\leq 1-\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-\left(1-\dot{b}'_j\right)^{\check{S}}\cdot\left(1-\dot{b}'_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}},$$

$$1-\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-\left(1-\lambda_j\right)^{\check{S}}\cdot\left(1-\lambda_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}$$

$$\leq 1-\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-\left(1-\lambda'_j\right)^{\check{S}}\cdot\left(1-\lambda'_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}},$$

$$1-\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-\left(1-\sigma_j\right)^{\check{S}}\cdot\left(1-\sigma_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}$$

$$\leq 1-\left(1-\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left[\left(1-\left(1-\sigma'_j\right)^{\check{S}}\cdot\left(1-\sigma'_k\right)^{\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}.$$

and

$$\left(1-\left[\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left(1-\dot{c}^{\check{S}+\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}\geq\left(1-\left[\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left(1-\dot{c}'^{\check{S}+\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}},$$

$$\left(1-\left[\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left(1-\underset{\cdot}{d}^{\check{S}+\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \geq \left(1-\left[\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left(1-\underset{\cdot}{d}'^{\check{S}+\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}},$$

$$\left(1-\left[\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left(1-\eta^{\check{S}+\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \geq \left(1-\left[\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left(1-\eta'^{\check{S}+\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}},$$

$$\left(1-\left[\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left(1-\gamma^{\check{S}+\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}} \geq \left(1-\left[\prod_{\substack{j,k=1\\ j\neq k}}^{n}\left(1-\gamma'^{\check{S}+\check{T}}\right)^{\frac{1}{n(n-1)}}\right]\right)^{\frac{1}{\check{S}+\check{T}}}.$$

Thus, from the above inequalities, we conclude that

$$CIVIFGBAO\left(\mathbb{C}_1,\mathbb{C}_2,\mathbb{C}_3,\ldots,\mathbb{C}_n\right) \leq CIVIFGBAO\left(\mathbb{C}'_1,\mathbb{C}'_2,\mathbb{C}'_3,\ldots,\mathbb{C}_n\right).$$

Appendix F

Let $\mathbb{C}_j = \begin{pmatrix} \left[\underset{\cdot}{a}_j, \dot{b}_j\right] e^{i[2\pi\lambda_j, 2\pi\sigma_j]}, \\ \left[\dot{c}_j, \underset{\wedge}{d}_j\right] e^{i[2\pi\eta_j, 2\pi\gamma_j]} \end{pmatrix}$ and $\mathbb{C}_k = \begin{pmatrix} \left[\underset{\cdot}{a}_k, \dot{b}_k\right] e^{i[2\pi\lambda_k, 2\pi\sigma_k]}, \\ \left[\dot{c}_k, \underset{\wedge}{d}_k\right] e^{i[2\pi\eta_k, 2\pi\gamma_k]} \end{pmatrix}$ describe two CIVIFNs defined on a US U. Then, the function d defined between two CIVIFSs C_j and C_k is given by

$$d\left(\mathbb{C}_j,\mathbb{C}_k\right)$$

$$= \left[\frac{\begin{array}{c} \sup_{\upsilon_i\in U}\left|\underset{\cdot}{a}_j(\upsilon_i)-\underset{\cdot}{a}_j(\upsilon_i)\right| + \sup_{\upsilon_i\in U}\left|\dot{b}_j(\upsilon_i)-\dot{b}_j(\upsilon_i)\right| + \frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|2\pi\lambda_j(\upsilon_i)-2\pi\lambda_j(\upsilon_i)\right| + \frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|2\pi\sigma_j(\upsilon_i)-2\pi\sigma_j(\upsilon_i)\right| + \\ \sup_{\upsilon_i\in U}\left|\dot{c}_j(\upsilon_i)-\dot{c}_j(\upsilon_i)\right| + \sup_{\upsilon_i\in U}\left|\underset{\wedge}{d}_j(\upsilon_i)-\underset{\wedge}{d}_j(\upsilon_i)\right| + \frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|2\pi\eta_j(\upsilon_i)-2\pi\eta_j(\upsilon_i)\right| + \frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|2\pi\gamma_j(\upsilon_i)-2\pi\gamma_j(\upsilon_i)\right| \end{array}}{16} + \frac{\max\begin{pmatrix} \sup_{\upsilon_i\in U}\left|\underset{\cdot}{a}_j(\upsilon_i)-\underset{\cdot}{a}_j(\upsilon_i)\right|, \sup_{\upsilon_i\in U}\left|\dot{b}_j(\upsilon_i)-\dot{b}_j(\upsilon_i)\right|, \frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|2\pi\lambda_j(\upsilon_i)-2\pi\lambda_j(\upsilon_i)\right|, \frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|2\pi\sigma_j(\upsilon_i)-2\pi\sigma_j(\upsilon_i)\right|, \\ \sup_{\upsilon_i\in U}\left|\dot{c}_j(\upsilon_i)-\dot{c}_j(\upsilon_i)\right|, \sup_{\upsilon_i\in U}\left|\underset{\wedge}{d}_j(\upsilon_i)-\underset{\wedge}{d}_j(\upsilon_i)\right|, \frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|2\pi\eta_j(\upsilon_i)-\eta_j(\upsilon_i)\right|, \frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|2\pi\gamma_j(\upsilon_i)-2\pi\gamma_j(\upsilon_i)\right| \end{pmatrix}}{2}\right].$$

Clearly, by Definition $d\left(\mathbb{C}_j,\mathbb{C}_k\right)\geq 0$. To prove that $d\left(\mathbb{C}_j,\mathbb{C}_k\right)\leq 1$, we have

$d\left(\mathbb{C}_j, \mathbb{C}_k\right)$

$$= \left[\frac{\begin{array}{c} \sup_{\upsilon_{\mathrm{i}} \in U} \left|\underset{\cdot}{\mathrm{a}}_j\left(\upsilon_{\mathrm{i}}\right) - \underset{\cdot}{\mathrm{a}}_j\left(\upsilon_{\mathrm{i}}\right)\right| + \sup_{\upsilon_{\mathrm{i}} \in U} \left|\dot{b}_j\left(\upsilon_{\mathrm{i}}\right) - \dot{b}_j\left(\upsilon_{\mathrm{i}}\right)\right| + \frac{1}{2\pi} \sup_{\upsilon_{\mathrm{i}} \in U} \left|2\pi\lambda_j\left(\upsilon_{\mathrm{i}}\right) - 2\pi\lambda_j\left(\upsilon_{\mathrm{i}}\right)\right| + \frac{1}{2\pi} \sup_{\upsilon_{\mathrm{i}} \in U} \left|2\pi\sigma_j\left(\upsilon_{\mathrm{i}}\right) - 2\pi\sigma_j\left(\upsilon_{\mathrm{i}}\right)\right| + \\ \sup_{\upsilon_{\mathrm{i}} \in U} \left|\dot{c}_j\left(\upsilon_{\mathrm{i}}\right) - \dot{c}_j\left(\upsilon_{\mathrm{i}}\right)\right| + \sup_{\upsilon_{\mathrm{i}} \in U} \left|\underset{\wedge}{\mathrm{d}}_j\left(\upsilon_{\mathrm{i}}\right) - \underset{\wedge}{\mathrm{d}}_j\left(\upsilon_{\mathrm{i}}\right)\right| + \frac{1}{2\pi} \sup_{\upsilon_{\mathrm{i}} \in U} \left|2\pi\eta_j\left(\upsilon_{\mathrm{i}}\right) - 2\pi\eta_j\left(\upsilon_{\mathrm{i}}\right)\right| + \frac{1}{2\pi} \sup_{\upsilon_{\mathrm{i}} \in U} \left|2\pi\gamma_j\left(\upsilon_{\mathrm{i}}\right) - 2\pi\gamma_j\left(\upsilon_{\mathrm{i}}\right)\right| \end{array}}{16} + \frac{\max \left(\begin{array}{c} \sup_{\upsilon_{\mathrm{i}} \in U} \left|\underset{\cdot}{\mathrm{a}}_j\left(\upsilon_{\mathrm{i}}\right) - \underset{\cdot}{\mathrm{a}}_j\left(\upsilon_{\mathrm{i}}\right)\right|, \sup_{\upsilon_{\mathrm{i}} \in U} \left|\dot{b}_j\left(\upsilon_{\mathrm{i}}\right) - \dot{b}_j\left(\upsilon_{\mathrm{i}}\right)\right|, \frac{1}{2\pi} \sup_{\upsilon_{\mathrm{i}} \in U} \left|2\pi\lambda_j\left(\upsilon_{\mathrm{i}}\right) - 2\pi\lambda_j\left(\upsilon_{\mathrm{i}}\right)\right|, \frac{1}{2\pi} \sup_{\upsilon_{\mathrm{i}} \in U} \left|2\pi\sigma_j\left(\upsilon_{\mathrm{i}}\right) - 2\pi\sigma_j\left(\upsilon_{\mathrm{i}}\right)\right|, \\ \sup_{\upsilon_{\mathrm{i}} \in U} \left|\dot{c}_j\left(\upsilon_{\mathrm{i}}\right) - \dot{c}_j\left(\upsilon_{\mathrm{i}}\right)\right|, \sup_{\upsilon_{\mathrm{i}} \in U} \left|\underset{\wedge}{\mathrm{d}}_j\left(\upsilon_{\mathrm{i}}\right) - \underset{\wedge}{\mathrm{d}}_j\left(\upsilon_{\mathrm{i}}\right)\right|, \frac{1}{2\pi} \sup_{\upsilon_{\mathrm{i}} \in U} \left|2\pi\eta_j\left(\upsilon_{\mathrm{i}}\right) - \eta_j\left(\upsilon_{\mathrm{i}}\right)\right|, \frac{1}{2\pi} \sup_{\upsilon_{\mathrm{i}} \in U} \left|2\pi\gamma_j\left(\upsilon_{\mathrm{i}}\right) - 2\pi\gamma_j\left(\upsilon_{\mathrm{i}}\right)\right| \end{array} \right)}{2} \right],$$

$$d\left(\mathbb{C}_j, \mathbb{C}_k\right) \leq \left[\frac{\begin{array}{c} 1 + 1 + \frac{1}{2\pi}(2\pi) + \frac{1}{2\pi}(2\pi) + \\ 1 + 1 + \frac{1}{2\pi}(2\pi) + \frac{1}{2\pi}(2\pi) \end{array}}{16} + \frac{\max \left(\begin{array}{c} 1, 1, \frac{1}{2\pi}(2\pi), \frac{1}{2\pi}(2\pi), \\ 1, 1, \frac{1}{2\pi}(2\pi), \frac{1}{2\pi}(2\pi) \end{array} \right)}{2} \right],$$

$$d\left(\mathbb{C}_j, \mathbb{C}_k\right) \leq \left[\frac{8}{16} + \frac{1}{2}\right] = 1.$$

Therefore, $0 \leq d\left(\mathbb{C}_j, \mathbb{C}_k\right) \leq 1$.

Conditions (ii) and (iii) are straightforward.

iv).

$d\left(\mathbb{C}_j, \mathbb{C}_l\right)$

$$= \left[\frac{\begin{array}{c} \sup_{\upsilon_{\mathrm{i}} \in U} \left|\underset{\cdot}{\mathrm{a}}_j\left(\upsilon_{\mathrm{i}}\right) - \underset{\cdot}{\mathrm{a}}_l\left(\upsilon_{\mathrm{i}}\right)\right| + \sup_{\upsilon_{\mathrm{i}} \in U} \left|\dot{b}_j\left(\upsilon_{\mathrm{i}}\right) - \dot{b}_l\left(\upsilon_{\mathrm{i}}\right)\right| + \frac{1}{2\pi} \sup_{\upsilon_{\mathrm{i}} \in U} \left|2\pi_j\left(\upsilon_{\mathrm{i}}\right) - 2\pi_l\left(\upsilon_{\mathrm{i}}\right)\right| + \frac{1}{2\pi} \sup_{\upsilon_{\mathrm{i}} \in U} \left|2\pi\sigma_j\left(\upsilon_{\mathrm{i}}\right) - 2\pi\sigma_l\left(\upsilon_{\mathrm{i}}\right)\right| + \\ \sup_{\upsilon_{\mathrm{i}} \in U} \left|\dot{c}_j\left(\upsilon_{\mathrm{i}}\right) - \dot{c}_l\left(\upsilon_{\mathrm{i}}\right)\right| + \sup_{\upsilon_{\mathrm{i}} \in U} \left|\underset{\wedge}{\mathrm{d}}_j\left(\upsilon_{\mathrm{i}}\right) - \underset{\wedge}{\mathrm{d}}_l\left(\upsilon_{\mathrm{i}}\right)\right| + \frac{1}{2\pi} \sup_{\upsilon_{\mathrm{i}} \in U} \left|2\pi\eta_j\left(\upsilon_{\mathrm{i}}\right) - 2\pi\eta_l\left(\upsilon_{\mathrm{i}}\right)\right| + \frac{1}{2\pi} \sup_{\upsilon_{\mathrm{i}} \in U} \left|2\pi\gamma_j\left(\upsilon_{\mathrm{i}}\right) - 2\pi\gamma_l\left(\upsilon_{\mathrm{i}}\right)\right| \end{array}}{16} + \frac{\max \left(\begin{array}{c} \sup_{\upsilon_{\mathrm{i}} \in U} \left|\underset{\cdot}{\mathrm{a}}_j\left(\upsilon_{\mathrm{i}}\right) - \underset{\cdot}{\mathrm{a}}_l\left(\upsilon_{\mathrm{i}}\right)\right|, \sup_{\upsilon_{\mathrm{i}} \in U} \left|\dot{b}_j\left(\upsilon_{\mathrm{i}}\right) - \dot{b}_l\left(\upsilon_{\mathrm{i}}\right)\right|, \frac{1}{2\pi} \sup_{\upsilon_{\mathrm{i}} \in U} \left|2\pi_j\left(\upsilon_{\mathrm{i}}\right) - 2\pi_l\left(\upsilon_{\mathrm{i}}\right)\right|, \frac{1}{2\pi} \sup_{\upsilon_{\mathrm{i}} \in U} \left|2\pi\sigma_j\left(\upsilon_{\mathrm{i}}\right) - 2\pi\sigma_l\left(\upsilon_{\mathrm{i}}\right)\right|, \\ \sup_{\upsilon_{\mathrm{i}} \in U} \left|\dot{c}_j\left(\upsilon_{\mathrm{i}}\right) - \dot{c}_l\left(\upsilon_{\mathrm{i}}\right)\right|, \sup_{\upsilon_{\mathrm{i}} \in U} \left|\underset{\wedge}{\mathrm{d}}_j\left(\upsilon_{\mathrm{i}}\right) - \underset{\wedge}{\mathrm{d}}_l\left(\upsilon_{\mathrm{i}}\right)\right|, \frac{1}{2\pi} \sup_{\upsilon_{\mathrm{i}} \in U} \left|2\pi\eta_j\left(\upsilon_{\mathrm{i}}\right) - 2\pi\eta_l\left(\upsilon_{\mathrm{i}}\right)\right|, \frac{1}{2\pi} \sup_{\upsilon_{\mathrm{i}} \in U} \left|2\pi\gamma_j\left(\upsilon_{\mathrm{i}}\right) - 2\pi\gamma_l\left(\upsilon_{\mathrm{i}}\right)\right| \end{array} \right)}{2} \right],$$

$d\left(\mathbb{C}_j, \mathbb{C}_l\right)$

$$= \left[\frac{\begin{array}{r} \sup_{\upsilon_{\mathrm{i}} \in U} \left|\underset{\cdot}{\mathrm{a}}_j\left(\upsilon_{\mathrm{i}}\right) - \underset{\cdot}{\mathrm{a}}_k\left(\upsilon_{\mathrm{i}}\right) + \underset{\cdot}{\mathrm{a}}_k\left(\upsilon_{\mathrm{i}}\right) - \underset{\cdot}{\mathrm{a}}_l\left(\upsilon_{\mathrm{i}}\right)\right| + \sup_{\upsilon_{\mathrm{i}} \in U} \left|\dot{b}_j\left(\upsilon_{\mathrm{i}}\right) - \dot{b}_k\left(\upsilon_{\mathrm{i}}\right) + \dot{b}_k\left(\upsilon_{\mathrm{i}}\right) - \dot{b}_l\left(\upsilon_{\mathrm{i}}\right)\right| + \\ \frac{1}{2\pi} \sup_{\upsilon_{\mathrm{i}} \in U} \left|2\pi_j\left(\upsilon_{\mathrm{i}}\right) - 2\pi_k\left(\upsilon_{\mathrm{i}}\right) + 2\pi_k\left(\upsilon_{\mathrm{i}}\right) - 2\pi\lambda_l\left(\upsilon_{\mathrm{i}}\right)\right| + \frac{1}{2\pi} \sup_{\upsilon_{\mathrm{i}} \in U} \left|2\pi\sigma_j\left(\upsilon_{\mathrm{i}}\right) - 2\pi\sigma_k\left(\upsilon_{\mathrm{i}}\right) + 2\pi\sigma_k\left(\upsilon_{\mathrm{i}}\right) - 2\pi\sigma_l\left(\upsilon_{\mathrm{i}}\right)\right| + \\ \sup_{\upsilon_{\mathrm{i}} \in U} \left|\dot{c}_j\left(\upsilon_{\mathrm{i}}\right) - \dot{c}_k\left(\upsilon_{\mathrm{i}}\right) + \dot{c}_k\left(\upsilon_{\mathrm{i}}\right) - \dot{c}_l\left(\upsilon_{\mathrm{i}}\right)\right| + \sup_{\upsilon_{\mathrm{i}} \in U} \left|\underset{\wedge}{\mathrm{d}}_j\left(\upsilon_{\mathrm{i}}\right) - \underset{\wedge}{\mathrm{d}}_k\left(\upsilon_{\mathrm{i}}\right) + \underset{\wedge}{\mathrm{d}}_k\left(\upsilon_{\mathrm{i}}\right) - \underset{\wedge}{\mathrm{d}}_l\left(\upsilon_{\mathrm{i}}\right)\right| + \\ \frac{1}{2\pi} \sup_{\upsilon_{\mathrm{i}} \in U} \left|2\pi\eta_j\left(\upsilon_{\mathrm{i}}\right) - 2\pi\eta_k\left(\upsilon_{\mathrm{i}}\right) + 2\pi\eta_k\left(\upsilon_{\mathrm{i}}\right) - 2\pi\eta_l\left(\upsilon_{\mathrm{i}}\right)\right| + \frac{1}{2\pi} \sup_{\upsilon_{\mathrm{i}} \in U} \left|2\pi\gamma_j\left(\upsilon_{\mathrm{i}}\right) - 2\pi\gamma_k\left(\upsilon_{\mathrm{i}}\right) + 2\pi\gamma_k\left(\upsilon_{\mathrm{i}}\right) - 2\pi\gamma_l\left(\upsilon_{\mathrm{i}}\right)\right| \end{array}}{16} + \frac{\max \left(\begin{array}{r} \sup_{\upsilon_{\mathrm{i}} \in U} \left|\underset{\cdot}{\mathrm{a}}_j\left(\upsilon_{\mathrm{i}}\right) - \underset{\cdot}{\mathrm{a}}_k\left(\upsilon_{\mathrm{i}}\right) + \underset{\cdot}{\mathrm{a}}_k\left(\upsilon_{\mathrm{i}}\right) - \underset{\cdot}{\mathrm{a}}_l\left(\upsilon_{\mathrm{i}}\right)\right|, \sup_{\upsilon_{\mathrm{i}} \in U} \left|\dot{b}_j\left(\upsilon_{\mathrm{i}}\right) - \dot{b}_k\left(\upsilon_{\mathrm{i}}\right) + \dot{b}_k\left(\upsilon_{\mathrm{i}}\right) - \dot{b}_l\left(\upsilon_{\mathrm{i}}\right)\right|, \\ \frac{1}{2\pi} \sup_{\upsilon_{\mathrm{i}} \in U} \left|2\pi\lambda_j\left(\upsilon_{\mathrm{i}}\right) - 2\pi\lambda_k\left(\upsilon_{\mathrm{i}}\right) + 2\pi\lambda_k\left(\upsilon_{\mathrm{i}}\right) - 2\pi\lambda_l\left(\upsilon_{\mathrm{i}}\right)\right|, \frac{1}{2\pi} \sup_{\upsilon_{\mathrm{i}} \in U} \left|2\pi\sigma_j\left(\upsilon_{\mathrm{i}}\right) - 2\pi\sigma_k\left(\upsilon_{\mathrm{i}}\right) + 2\pi\sigma_k\left(\upsilon_{\mathrm{i}}\right) - 2\pi\sigma_l\left(\upsilon_{\mathrm{i}}\right)\right|, \\ \sup_{\upsilon_{\mathrm{i}} \in U} \left|\dot{c}_j\left(\upsilon_{\mathrm{i}}\right) - \dot{c}_k\left(\upsilon_{\mathrm{i}}\right) + \dot{c}_k\left(\upsilon_{\mathrm{i}}\right) - \dot{c}_l\left(\upsilon_{\mathrm{i}}\right)\right|, \sup_{\upsilon_{\mathrm{i}} \in U} \left|\underset{\wedge}{\mathrm{d}}_j\left(\upsilon_{\mathrm{i}}\right) - \underset{\wedge}{\mathrm{d}}_k\left(\upsilon_{\mathrm{i}}\right) + \underset{\wedge}{\mathrm{d}}_k\left(\upsilon_{\mathrm{i}}\right) - \underset{\wedge}{\mathrm{d}}_l\left(\upsilon_{\mathrm{i}}\right)\right|, \\ \frac{1}{2\pi} \sup_{\upsilon_{\mathrm{i}} \in U} \left|2\pi\eta_j\left(\upsilon_{\mathrm{i}}\right) - 2\pi\eta_k\left(\upsilon_{\mathrm{i}}\right) + 2\pi\eta_k\left(\upsilon_{\mathrm{i}}\right) - 2\pi\eta_l\left(\upsilon_{\mathrm{i}}\right)\right|, \frac{1}{2\pi} \sup_{\upsilon_{\mathrm{i}} \in U} \left|2\pi\gamma_j\left(\upsilon_{\mathrm{i}}\right) - 2\pi\gamma_k\left(\upsilon_{\mathrm{i}}\right) + 2\pi\gamma_k\left(\upsilon_{\mathrm{i}}\right) - 2\pi\gamma_l\left(\upsilon_{\mathrm{i}}\right)\right| \end{array} \right)}{2} \right],$$

$$d\left(\mathbb{C}_j,\mathbb{C}_l\right)$$

$$\leq\left[\begin{array}{c}\dfrac{\begin{array}{c}\sup_{\upsilon_i\in U}\left|\underset{\cdot}{a}_j(\upsilon_i)-\underset{\cdot}{a}_k(\upsilon_i)\right|+\sup_{\upsilon_i\in U}\left|\underset{\cdot}{a}_k(\upsilon_i)-\underset{\cdot}{a}_l(\upsilon_i)\right|+\sup_{\upsilon_i\in U}\left|\dot{b}_j(\upsilon_i)-\dot{b}_k(\upsilon_i)\right|+\sup_{\upsilon_i\in U}\left|\dot{b}_k(\upsilon_i)-\dot{b}_l(\upsilon_i)\right|+\\ \frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|\lambda_j(\upsilon_i)-\lambda_k(\upsilon_i)\right|+\frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|\lambda_k(\upsilon_i)-\lambda_l(\upsilon_i)\right|+\frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|\sigma_j(\upsilon_i)-\sigma_k(\upsilon_i)\right|+\frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|\sigma_k(\upsilon_i)-\sigma_l(\upsilon_i)\right|+\\ \sup_{\upsilon_i\in U}\left|\dot{c}_j(\upsilon_i)-\dot{c}_k(\upsilon_i)\right|+\sup_{\upsilon_i\in U}\left|\dot{c}_k(\upsilon_i)-\dot{c}_l(\upsilon_i)\right|+\sup_{\upsilon_i\in U}\left|\underset{\wedge}{d}_j(\upsilon_i)-\underset{\wedge}{d}_k(\upsilon_i)\right|+\sup_{\upsilon_i\in U}\left|\underset{\wedge}{d}_k(\upsilon_i)-\underset{\wedge}{d}_l(\upsilon_i)\right|+\\ \frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|\eta_j(\upsilon_i)-\eta_k(\upsilon_i)\right|+\frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|\eta_k(\upsilon_i)-\eta_l(\upsilon_i)\right|+\frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|\gamma_j(\upsilon_i)-\gamma_k(\upsilon_i)\right|+\frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|\gamma_k(\upsilon_i)-\gamma_l(\upsilon_i)\right|\end{array}}{16}+\\ \dfrac{\max\left(\begin{array}{c}\sup_{\upsilon_i\in U}\left|\underset{\cdot}{a}_j(\upsilon_i)-\underset{\cdot}{a}_k(\upsilon_i)\right|+\sup_{\upsilon_i\in U}\left|\underset{\cdot}{a}_k(\upsilon_i)-\underset{\cdot}{a}_l(\upsilon_i)\right|,\sup_{\upsilon_i\in U}\left|\dot{b}_j(\upsilon_i)-\dot{b}_k(\upsilon_i)\right|+\sup_{\upsilon_i\in U}\left|\dot{b}_k(\upsilon_i)-\dot{b}_l(\upsilon_i)\right|,\\ \frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|\lambda_j(\upsilon_i)-\lambda_k(\upsilon_i)\right|+\frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|\lambda_k(\upsilon_i)-\lambda_l(\upsilon_i)\right|,\frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|\sigma_j(\upsilon_i)-\sigma_k(\upsilon_i)\right|+\frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|\sigma_k(\upsilon_i)-\sigma_l(\upsilon_i)\right|,\\ \sup_{\upsilon_i\in U}\left|\dot{c}_j(\upsilon_i)-\dot{c}_k(\upsilon_i)\right|+\sup_{\upsilon_i\in U}\left|\dot{c}_k(\upsilon_i)-\dot{c}_l(\upsilon_i)\right|,\sup_{\upsilon_i\in U}\left|\underset{\wedge}{d}_j(\upsilon_i)-\underset{\wedge}{d}_k(\upsilon_i)\right|+\sup_{\upsilon_i\in U}\left|\underset{\wedge}{d}_k(\upsilon_i)-\underset{\wedge}{d}_l(\upsilon_i)\right|,\\ \frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|\eta_j(\upsilon_i)-\eta_k(\upsilon_i)\right|+\frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|\eta_k(\upsilon_i)-\eta_l(\upsilon_i)\right|,\frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|\gamma_j(\upsilon_i)-\gamma_k(\upsilon_i)\right|+\frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|\gamma_k(\upsilon_i)-\gamma_l(\upsilon_i)\right|\end{array}\right)}{2}\end{array}\right],$$

$$d\left(\mathbb{C}_j,\mathbb{C}_l\right)$$

$$\leq\left[\begin{array}{c}\dfrac{\begin{array}{c}\sup_{\upsilon_i\in U}\left|\underset{\cdot}{a}_j(\upsilon_i)-\underset{\cdot}{a}_j(\upsilon_i)\right|+\sup_{\upsilon_i\in U}\left|\dot{b}_j(\upsilon_i)-\dot{b}_j(\upsilon_i)\right|+\frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|2\pi\lambda_j(\upsilon_i)-2\pi\lambda_j(\upsilon_i)\right|+\frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|2\pi\sigma_j(\upsilon_i)-2\pi\sigma_j(\upsilon_i)\right|+\\ \sup_{\upsilon_i\in U}\left|\dot{c}_j(\upsilon_i)-\dot{c}_j(\upsilon_i)\right|+\sup_{\upsilon_i\in U}\left|\underset{\wedge}{d}_j(\upsilon_i)-\underset{\wedge}{d}_j(\upsilon_i)\right|+\frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|2\pi\eta_j(\upsilon_i)-2\pi\eta_j(\upsilon_i)\right|+\frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|2\pi\gamma_j(\upsilon_i)-2\pi\gamma_j(\upsilon_i)\right|\end{array}}{16}+\\ \dfrac{\max\left(\begin{array}{c}\sup_{\upsilon_i\in U}\left|\underset{\cdot}{a}_j(\upsilon_i)-\underset{\cdot}{a}_j(\upsilon_i)\right|,\sup_{\upsilon_i\in U}\left|\dot{b}_j(\upsilon_i)-\dot{b}_j(\upsilon_i)\right|,\frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|2\pi\lambda_j(\upsilon_i)-2\pi\lambda_j(\upsilon_i)\right|,\frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|2\pi\sigma_j(\upsilon_i)-2\pi\sigma_j(\upsilon_i)\right|,\\ \sup_{\upsilon_i\in U}\left|\dot{c}_j(\upsilon_i)-\dot{c}_j(\upsilon_i)\right|,\sup_{\upsilon_i\in U}\left|\underset{\wedge}{d}_j(\upsilon_i)-\underset{\wedge}{d}_j(\upsilon_i)\right|,\frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|2\pi\eta_j(\upsilon_i)-\eta_j(\upsilon_i)\right|,\frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|2\pi\gamma_j(\upsilon_i)-2\pi\gamma_j(\upsilon_i)\right|\end{array}\right)}{2}\end{array}\right]$$

$$+\left[\begin{array}{c}\dfrac{\begin{array}{c}\sup_{\upsilon_i\in U}\left|\underset{\cdot}{a}_k(\upsilon_i)-\underset{\cdot}{a}_l(\upsilon_i)\right|+\sup_{\upsilon_i\in U}\left|\dot{b}_k(\upsilon_i)-\dot{b}_l(\upsilon_i)\right|+\frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|2\pi\lambda_k(\upsilon_i)-2\pi\lambda_l(\upsilon_i)\right|+\frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|2\pi\sigma_k(\upsilon_i)-2\pi\sigma_l(\upsilon_i)\right|+\\ \sup_{\upsilon_i\in U}\left|\dot{c}_k(\upsilon_i)-\dot{c}_l(\upsilon_i)\right|+\sup_{\upsilon_i\in U}\left|\underset{\wedge}{d}_k(\upsilon_i)-\underset{\wedge}{d}_l(\upsilon_i)\right|+\frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|2\pi\eta_k(\upsilon_i)-2\pi\eta_l(\upsilon_i)\right|+\frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|2\pi\gamma_k(\upsilon_i)-2\pi\gamma_l(\upsilon_i)\right|\end{array}}{16}+\\ \dfrac{\max\left(\begin{array}{c}\sup_{\upsilon_i\in U}\left|\underset{\cdot}{a}_k(\upsilon_i)-\underset{\cdot}{a}_l(\upsilon_i)\right|,\sup_{\upsilon_i\in U}\left|\dot{b}_k(\upsilon_i)-\dot{b}_l(\upsilon_i)\right|,\frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|2\pi\lambda_k(\upsilon_i)-2\pi\lambda_l(\upsilon_i)\right|,\frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|2\pi\sigma_k(\upsilon_i)-2\pi\sigma_l(\upsilon_i)\right|,\\ \sup_{\upsilon_i\in U}\left|\dot{c}_k(\upsilon_i)-\dot{c}_l(\upsilon_i)\right|,\sup_{\upsilon_i\in U}\left|\underset{\wedge}{d}_k(\upsilon_i)-\underset{\wedge}{d}_l(\upsilon_i)\right|,\frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|2\pi\eta_k(\upsilon_i)-\eta_l(\upsilon_i)\right|,\frac{1}{2\pi}\sup_{\upsilon_i\in U}\left|2\pi\gamma_k(\upsilon_i)-2\pi\gamma_l(\upsilon_i)\right|\end{array}\right)}{2}\end{array}\right].$$

$$d\left(\mathbb{C}_j,\mathbb{C}_l\right)\leq d\left(\mathbb{C}_j,\mathbb{C}_k\right)+d\left(\mathbb{C}_k,\mathbb{C}_l\right).$$