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Study of Non-Negative Solution for a Tenth-Order Boundary Value Problems via Fixed Point Theory

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Abstract. In this paper, we investigate the existence of non-negative solutions for a boundary value problem associated with a tenth-order differential equation

$$-\beta^{(10)}(\zeta) = F\left(\zeta, \beta(\zeta), \beta'(\zeta), \beta''(\zeta), \beta'''(\zeta), \beta^{(4)}(\zeta), \beta^{(5)}(\zeta), \beta^{(6)}(\zeta), \beta^{(7)}(\zeta), \beta^{(8)}(\zeta), \beta^{(9)}(\zeta)\right),$$

$$\beta(0) = \beta'(0) = \beta''(0) = \beta'''(0) = \beta^{(4)}(0) = \beta^{(5)}(1),$$
$$\beta^{(6)}(1) = \beta^{(7)}(1) = \beta^{(8)}(1) = \beta^{(9)}(1) = 0,$$

where $\zeta \in [0,1]$ and $F \in C([0,1] \times [0,\infty) \times [0,\infty) \times [0,\infty) \times [0,\infty) \times [0,\infty) \times [0,\infty) \times (-\infty,0] \times [0,\infty) \times (-\infty,0] \times [0,\infty) \to [0,\infty))$ is a non-negative continuous function. We employ the nonlinear Leray-Schauder alternative and the Leray-Schauder fixed point theorem to prove the existence of at least one non-negative solution. The complete continuity of the integral operator is established rigorously via the Arzelà–Ascoli theorem, with explicit verification of uniform boundedness and equicontinuity. The non-decreasingness and convexity of the solution are derived from the sign structure of the Green function derivatives. Sharp L^∞–L^1 kernel estimates are obtained and their optimality is discussed. The theoretical results are validated through an explicit example with full numerical verification of all hypotheses.

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1. Introduction

Boundary value problems (BVPs) of high order have attracted sustained attention in applied mathematics, both from the perspective of theoretical analysis and the development of solution methods. Their importance stems from the essential role they play in the mathematical modeling of phenomena in applied sciences and modern technology. Higher-order BVPs arise naturally in the study of physical and chemical phenomena, including fluid dynamics, magnetohydrodynamic stability, long-wave modeling, and the bending of elastic beams under various loading conditions. A variety of analytical and numerical solution techniques have been developed to handle such problems [1–3]. Several researchers have established the existence and uniqueness of solutions to higher-order BVPs using fixed point theorems; detailed results and further references can be found in [4–8]. In [9], nonnegative solutions for nonlinear higher-order BVPs were derived. In [10], higher-order nonlinear ordinary differential equations with four-point nonlocal integral boundary conditions were studied. In [11], approximate solutions to higher-order BVPs were obtained via a modified decomposition scheme. A homotopic perturbation technique for higher-order BVPs was introduced in [12].

More recently, several authors have focused on constructing efficient numerical methods to approximate solutions to tenth-order BVPs [13–17]. In [18], a novel numerical scheme for tenth-order BVPs was proposed. Reproducing kernel functions for linear tenth-order BVPs were derived in [19]. Numerical techniques for tenth- and twelfth-order linear and nonlinear differential equations were introduced in [20]. A reliable algorithm for tenth-order BVPs was given in [21]. The weighted residual method was applied to tenth- and twelfth-order BVPs in [22]. Numerical methods for higher-order eigenvalue problems arising in thermal instability were proposed in [23]. A generalized Adomian decomposition method for tenth- and twelfth-order BVPs was presented in [24].

This study aims to investigate the presence of non-negative solutions for the boundary value problem

$$-\beta^{(10)}(\zeta) = \digamma\left(\zeta, \beta(\zeta), \beta'(\zeta), \beta''(\zeta), \beta'''(\zeta), \beta^{(4)}(\zeta), \beta^{(5)}(\zeta), \beta^{(6)}(\zeta), \beta^{(7)}(\zeta), \beta^{(8)}(\zeta), \beta^{(9)}(\zeta)\right), \tag{1}$$

$$\begin{aligned} &\beta(0) = \beta'(0) = \beta''(0) = \beta'''(0) = \beta^{(4)}(0) = \beta^{(5)}(1) = \\ &\qquad \beta^{(6)}(1) = \beta^{(7)}(1) = \beta^{(8)}(1) = \beta^{(9)}(1) = 0. \end{aligned} \tag{2}$$

where $\zeta \in [0,1]$ and $\digamma \in C([0,1] \times [0,\infty) \times [0,\infty) \times [0,\infty) \times [0,\infty) \times [0,\infty) \times [0,\infty) \times (-\infty,0] \times [0,\infty) \times (-\infty,0] \times [0,\infty) \to [0,\infty))$ is a non-negative continuous function. The paper is organized as follows. Section 2 recalls the definitions and fixed point theorem used throughout. Section 3 establishes the main existence results via the nonlinear Leray-Schauder alternative and the Leray-Schauder fixed point theorem. An illustrative example with explicit numerical verification is provided in Section 4. Conclusions are collected in Section 5.

2. Preliminaries

In this section, we will define the Leray-Schauder fixed point theorem and present its fundamental theorem to find a non-negative solution to the problem described in equations (1) and (2).

Definition 1. *Assume Γ is a Banach space that includes a cone Δ. The application $\Pi : \Delta \to [0, \infty)$ is a concave operator if it is continuous and satisfies*

$$\Pi(\varrho\delta + (1 - \varrho)\upsilon) \geq \varrho\ \Pi(\delta) + (1 - \varrho)\Pi(\upsilon) \tag{3}$$

for all $\delta, \upsilon \in \Delta$ and $\varrho \in [0, 1]$.

Likewise, we can state that the operator $\Theta : \Delta \to [0, \infty)$ is a positive continuous convex operator on Δ if it is continuous and satisfies the following condition:

$$\Theta(\varrho\delta + (1 - \varrho)\upsilon) \leq \varrho\Theta(\delta) + (1 - \varrho)\Theta(\upsilon) \tag{4}$$

for all $\delta, \upsilon \in \Delta$ and $\varrho \in [0, 1]$.

Theorem 1. *([25, 26])*

Let Γ be a Banach space and $\Lambda \subset \Gamma$ be a bounded open subset with $0 \in \Lambda$. Suppose $\Xi : \overline{\Lambda} \to \Gamma$ is a completely continuous operator. Then, either

1. *There exists $\beta \in \partial\Lambda$ and $\mu > 1$ such that $\Xi(\beta) = \mu\beta$, or*
2. *There exists $\beta^* \in \overline{\Lambda}$ such that β^* is a fixed point of Ξ, i.e., $\Xi(\beta^*) = \beta^*$.*

3. Main Results

Within this section, we outline the necessary conditions for the function β to utilize the nonlinear Leray-Schauder framework and the Leray-Schauder fixed point theorem. These conditions are crucial for demonstrating the presence of at least one non-negative solution to the problem defined by equations (1) and (2).

Lemma 1. *Let Γ be the Banach space*

$$\Gamma = \Big\{\beta \in C^9([0,1]) : \beta(0) = \beta^{'}(0) = \beta^{''}(0) = \beta^{'''}(0) = \beta^{(4)}(0) = \beta^{(5)}(1) = \beta^{(6)}(1) = \beta^{(7)}(1) = \beta^{(8)}(1) = 0\Big\}, \tag{5}$$

equipped by the maximum norm

$$\|\beta\| = \max\left\{|\beta|_0, |\beta^{'}|_0, |\beta^{''}|_0, |\beta^{'''}|_0, |\beta^{(4)}|_0, |\beta^{(5)}|_0, |\beta^{(6)}|_0, |\beta^{(7)}|_0, |\beta^{(8)}|_0\right\},$$

where $|\beta|_0 = \max_{0\leq\zeta\leq1} |\beta(\zeta)|$. So for each $\beta \in \Gamma$, we have

$$\|\beta\| = |\beta^{(9)}|_0 \text{ and } |\beta|_0 \leq \frac{1}{5184}\|\beta\|,\ |\beta^{'}|_0 \leq \frac{1}{1152}\|\beta\|,\ |\beta^{''}|_0 \leq \frac{1}{336}\|\beta\|,\ |\beta^{'''}|_0 \leq \frac{1}{144}\|\beta\|,$$

$$|\beta^{(4)}|_0 \leq \frac{1}{120}\|\beta\|,\ |\beta^{(5)}|_0 \leq \frac{1}{24}\|\beta\|,\ |\beta^{(6)}|_0 \leq \frac{1}{6}\|\beta\|,\ |\beta^{(7)}|_0 \leq \frac{1}{2}\|\beta\|,\ |\beta^{(8)}|_0 \leq \|\beta\|.$$

Proof. Consider the homogeneous boundary value problem of the ninth-order

$$\beta^{(9)}(\zeta) = 0, \quad 0 < \zeta < 1,$$

with

$$\beta(0) = \beta'(0) = \beta''(0) = \beta'''(0) = \beta^{(4)}(0) = \beta^{(5)}(1) = \beta^{(6)}(1) = \beta^{(7)}(1) = \beta^{(8)}(1) = 0.$$

Then its Green function $\hbar(\zeta, \eta)$ is given by

$$\hbar(\zeta, \eta) = \frac{1}{8!} \begin{cases} \eta^8 - 8\eta^7\zeta + 28\eta^6\zeta^2 - 56\eta^5\zeta^3 + 70\eta^4\zeta^4, & 0 \le \eta \le \zeta \le 1, \\ -\zeta^8 + 8\eta\zeta^7 - 28\eta^2\zeta^6 + 56\eta^3\zeta^5, & 0 \le \zeta \le \eta \le 1. \end{cases} \tag{6}$$

From equation (6), it follows that

$$\hbar(\zeta, \eta) \ge 0, \ \frac{\partial \hbar(\zeta, \eta)}{\partial \zeta} \ge 0, \ \frac{\partial^2 \hbar(\zeta, \eta)}{\partial \zeta^2} \ge 0, \ \frac{\partial^3 \hbar(\zeta, \eta)}{\partial \zeta^3} \ge 0, \ \frac{\partial^4 \hbar(\zeta, \eta)}{\partial \zeta^4} \ge 0,$$

$$\frac{\partial^5 \hbar(\zeta, \eta)}{\partial \zeta^5} \ge 0, \ \frac{\partial^6 \hbar(\zeta, \eta)}{\partial \zeta^6} \le 0, \ \frac{\partial^7 \hbar(\zeta, \eta)}{\partial \zeta^7} \ge 0, \ \frac{\partial^8 \hbar(\zeta, \eta)}{\partial \zeta^8} \le 0, \tag{7}$$

Computing the integrals of the kernel and its derivatives, one finds

$$\int_0^1 |\hbar(\zeta, \eta)| d\eta = \int_0^1 \hbar(\zeta, \eta) d\eta = \frac{1}{8!}(\frac{1}{9}\zeta^9 - \zeta^8 + 4\zeta^7 - \frac{28}{3}\zeta^6 + 14\zeta^5),$$

$$\int_0^1 \left| \frac{\partial \hbar(\zeta, \eta)}{\partial \zeta} \right| d\eta = \int_0^1 \frac{\partial \hbar(\zeta, \eta)}{\partial \zeta} d\eta = \frac{1}{8!}(\zeta^8 - 8\zeta^7 + 28\zeta^6 - 56\zeta^5 + 70\zeta^4),$$

$$\int_0^1 \left| \frac{\partial^2 \hbar(\zeta, \eta)}{\partial \zeta^2} \right| d\eta = \int_0^1 \frac{\partial^2 \hbar(\zeta, \eta)}{\partial \zeta^2} d\eta = \frac{1}{8!}(8\zeta^7 - 56\zeta^6 + 168\zeta^5 - 280\zeta^4 + 280\zeta^3),$$

$$\int_0^1 \left| \frac{\partial^3 \hbar(\zeta, \eta)}{\partial \zeta^3} \right| d\eta = \int_0^1 \frac{\partial^3 \hbar(\zeta, \eta)}{\partial \zeta^3} d\eta = \frac{1}{8!}(56\zeta^6 - 336\zeta^5 + 840\zeta^4 - 1120\zeta^3 + 840\zeta^2),$$

$$\int_0^1 \left| \frac{\partial^4 \hbar(\zeta, \eta)}{\partial \zeta^4} \right| d\eta = \int_0^1 \frac{\partial^4 \hbar(\zeta, \eta)}{\partial \zeta^4} d\eta = \frac{1}{8!}(336\zeta^5 - 1680\zeta^4 + 3360\zeta^3 - 3360\zeta^2 + 1680\zeta),$$

$$\int_0^1 \left| \frac{\partial^5 \hbar(\zeta, \eta)}{\partial \zeta^5} \right| d\eta = \int_0^1 \frac{\partial^5 \hbar(\zeta, \eta)}{\partial \zeta^5} d\eta = \frac{1}{8!}(1680\zeta^4 - 6720\zeta^3 + 10080\zeta^2 - 6720\zeta + 1680),$$

$$\int_0^1 \left| \frac{\partial^6 \hbar(\zeta, \eta)}{\partial \zeta^6} \right| d\eta = -\int_0^1 \frac{\partial^6 \hbar(\zeta, \eta)}{\partial \zeta^6} d\eta = \frac{1}{8!}(-6720\zeta^3 + 20160\zeta^2 - 20160\zeta + 6720),$$

$$\int_0^1 \left| \frac{\partial^7 \hbar(\zeta, \eta)}{\partial \zeta^7} \right| d\eta = \int_0^1 \frac{\partial^7 \hbar(\zeta, \eta)}{\partial \zeta^7} d\eta = \frac{1}{8!}(20160\zeta^2 - 40320\zeta + 20160),$$

$$\int_0^1 \left| \frac{\partial^8 \hbar(\zeta, \eta)}{\partial \zeta^8} \right| d\eta = -\int_0^1 \frac{\partial^8 \hbar(\zeta, \eta)}{\partial \zeta^8} d\eta = \frac{1}{8!}(-40320\zeta + 40320). \tag{8}$$

By integration, we find

$$\max_{0\le\zeta\le1}\int_0^1|\hbar(\zeta,\eta)|d\eta=\frac{1}{5184},\ \max_{0\le\zeta\le1}\int_0^1\left|\frac{\partial\hbar(\zeta,\eta)}{\partial\zeta}\right|d\eta=\frac{1}{1152},\ \max_{0\le\zeta\le1}\int_0^1\left|\frac{\partial^2\hbar(\zeta,\eta)}{\partial\zeta^2}\right|d\eta=\frac{1}{336},$$

$$\max_{0\le\zeta\le1}\int_0^1\left|\frac{\partial^3\hbar(\zeta,\eta)}{\partial\zeta^3}\right|d\eta=\frac{1}{144},\ \max_{0\le\zeta\le1}\int_0^1\left|\frac{\partial^4\hbar(\zeta,\eta)}{\partial\zeta^4}\right|d\eta=\frac{1}{120},\ \max_{0\le\zeta\le1}\int_0^1\left|\frac{\partial^5\hbar(\zeta,\eta)}{\partial\zeta^5}\right|d\eta=\frac{1}{24},$$

$$\max_{0\le\zeta\le1}\int_0^1\left|\frac{\partial^6\hbar(\zeta,\eta)}{\partial\zeta^6}\right|d\eta=\frac{1}{6},\ \max_{0\le\zeta\le1}\int_0^1\left|\frac{\partial^7\hbar(\zeta,\eta)}{\partial\zeta^7}\right|d\eta=\frac{1}{2},\ \max_{0\le\zeta\le1}\int_0^1\left|\frac{\partial^8\hbar(\zeta,\eta)}{\partial\zeta^8}\right|d\eta=1.$$

We have $\beta\in\Gamma$ and $\|\beta\|=\varkappa$. So

$$\beta(\zeta)=\int_0^1\hbar(\zeta,\eta)[\beta^{(9)}(\eta)]d\eta,\ \beta'(\zeta)=\int_0^1\frac{\partial\hbar(\zeta,\eta)}{\partial\zeta}[\beta^{(9)}(\eta)]d\eta,$$

$$\beta''(\zeta)=\int_0^1\frac{\partial^2\hbar(\zeta,\eta)}{\partial\zeta^2}[\beta^{(9)}(\eta)]d\eta,\ \beta'''(\zeta)=\int_0^1\frac{\partial^3\hbar(\zeta,\eta)}{\partial\zeta^3}[\beta^{(9)}(\eta)]d\eta,$$

$$\beta^{(4)}(\zeta)=\int_0^1\frac{\partial^4\hbar(\zeta,\eta)}{\partial\zeta^4}[\beta^{(9)}(\eta)]d\eta,\ \beta^{(5)}(\zeta)=\int_0^1\frac{\partial^5\hbar(\zeta,\eta)}{\partial\zeta^5}[\beta^{(9)}(\eta)]d\eta,$$

$$\beta^{(6)}(\zeta)=\int_0^1\frac{\partial^6\hbar(\zeta,\eta)}{\partial\zeta^6}[\beta^{(9)}(\eta)]d\eta,\ \beta^{(7)}(\zeta)=\int_0^1\frac{\partial^7\hbar(\zeta,\eta)}{\partial\zeta^7}[\beta^{(9)}(\eta)]d\eta,$$

$$\beta^{(8)}(\zeta)=\int_0^1\frac{\partial^8\hbar(\zeta,\eta)}{\partial\zeta^8}[\beta^{(9)}(\eta)]d\eta.$$

Therefore

$$|\beta|_0\le\max_{0\le\zeta\le1}\int_0^1|\hbar(\zeta,\eta)||\beta^{(9)}(\eta)|d\eta\le|\beta^{(9)}|_0\max_{0\le\zeta\le1}\int_0^1|\hbar(\zeta,\eta)|d\eta=\frac{1}{5184}|\beta^{(9)}|_0,$$

$$|\beta'|_0\le\max_{0\le\zeta\le1}\int_0^1\left|\frac{\partial\hbar(\zeta,\eta)}{\partial\zeta}\right||\beta^{(9)}(\eta)|d\eta\le|\beta^{(9)}|_0\max_{0\le\zeta\le1}\int_0^1\left|\frac{\partial\hbar(\zeta,\eta)}{\partial\zeta}\right|d\eta=\frac{1}{1152}|\beta^{(9)}|_0,$$

$$|\beta''|_0\le\max_{0\le\zeta\le1}\int_0^1\left|\frac{\partial^2\hbar(\zeta,\eta)}{\partial\zeta^2}\right||\beta^{(9)}(\eta)|d\eta\le|\beta^{(9)}|_0\max_{0\le\zeta\le1}\int_0^1\left|\frac{\partial^2\hbar(\zeta,\eta)}{\partial\zeta^2}\right|d\eta=\frac{1}{336}|\beta^{(9)}|_0,$$

$$\begin{aligned}|\beta'''|_0&\le\max_{0\le\zeta\le1}\int_0^1\left|\frac{\partial^3\hbar(\zeta,\eta)}{\partial\zeta^3}\right||\beta^{(9)}(\eta)|d\eta\le|\beta^{(9)}|_0\max_{0\le\zeta\le1}\int_0^1\left|\frac{\partial^3\hbar(\zeta,\eta)}{\partial\zeta^3}\right|d\eta\\&=\frac{1}{144}|\beta^{(9)}|_0,\end{aligned}$$

$$\begin{aligned}|\beta^{(4)}|_0&\le\max_{0\le\zeta\le1}\int_0^1\left|\frac{\partial^4\hbar(\zeta,\eta)}{\partial\zeta^4}\right||\beta^{(9)}(\eta)|d\eta\le|\beta^{(9)}|_0\max_{0\le\zeta\le1}\int_0^1\left|\frac{\partial^4\hbar(\zeta,\eta)}{\partial\zeta^4}\right|d\eta\\&=\frac{1}{120}|\beta^{(9)}|_0,\end{aligned}$$

$$|\beta^{(5)}|_0 \le \max_{0\le\zeta\le 1}\int_0^1 \left|\frac{\partial^5 \hbar(\zeta,\eta)}{\partial\zeta^5}\right| |\beta^{(9)}(\eta)|d\eta \le |\beta^{(9)}|_0 \max_{0\le\zeta\le 1}\int_0^1 \left|\frac{\partial^5 \hbar(\zeta,\eta)}{\partial\zeta^5}\right| d\eta$$
$$= \frac{1}{24}|\beta^{(9)}|_0,$$
$$|\beta^{(6)}|_0 \le \max_{0\le\zeta\le 1}\int_0^1 \left|\frac{\partial^6 \hbar(\zeta,\eta)}{\partial\zeta^6}\right| |\beta^{(9)}(\eta)|d\eta \le |\beta^{(9)}|_0 \max_{0\le\zeta\le 1}\int_0^1 \left|\frac{\partial^6 \hbar(\zeta,\eta)}{\partial\zeta^6}\right| d\eta$$
$$= \frac{1}{6}|\beta^{(9)}|_0,$$
$$|\beta^{(7)}|_0 \le \max_{0\le\zeta\le 1}\int_0^1 \left|\frac{\partial^7 \hbar(\zeta,\eta)}{\partial\zeta^7}\right| |\beta^{(9)}(\eta)|d\eta \le |\beta^{(9)}|_0 \max_{0\le\zeta\le 1}\int_0^1 \left|\frac{\partial^7 \hbar(\zeta,\eta)}{\partial\zeta^7}\right| d\eta$$
$$= \frac{1}{2}|\beta^{(9)}|_0,$$
$$|\beta^{(8)}|_0 \le \max_{0\le\zeta\le 1}\int_0^1 \left|\frac{\partial^8 \hbar(\zeta,\eta)}{\partial\zeta^8}\right| |\beta^{(9)}(\eta)|d\eta \le |\beta^{(9)}|_0 \max_{0\le\zeta\le 1}\int_0^1 \left|\frac{\partial^8 \hbar(\zeta,\eta)}{\partial\zeta^8}\right| d\eta$$
$$= |\beta^{(9)}|_0.$$

Then, $|\beta^{(9)}|_0 = \|\beta\| = \varkappa$, and thus the proof is complete.

Remark 1. *The constants* 5184, 1152, 336, 144, 120, 24, 6, 2, *and* 1 *appearing in Lemma 1 arise as the reciprocals of the quantities*

$$\max_{0\le\zeta\le 1}\int_0^1 \left|\frac{\partial^k \hbar(\zeta,\eta)}{\partial\zeta^k}\right| d\eta, \quad k = 0, 1, \ldots, 8,$$

computed explicitly from the Green function (6)*. Specifically, evaluating the polynomial antiderivatives at their maxima on* $[0,1]$ *gives*

$$\frac{1}{5184} = \frac{14}{8!}, \quad \frac{1}{1152} = \frac{35}{8!}, \quad \frac{1}{336} = \frac{120}{8!}, \quad \frac{1}{144} = \frac{280}{8!}, \quad \frac{1}{120} = \frac{336}{8!},$$
$$\frac{1}{24} = \frac{1680}{8!}, \quad \frac{1}{6} = \frac{6720}{8!}, \quad \frac{1}{2} = \frac{20160}{8!}, \quad 1 = \frac{40320}{8!}.$$

These are the tightest (smallest) upper bounds achievable from the L^∞*–*L^1 *kernel estimate, and they are* sharp *in the sense that equality is attained at the maximizing value of* ζ*. The constant* $\mho$ *defined in* (9) *is therefore an optimal spectral-radius type bound for the weighted growth of* $\digamma$*: the condition* $\mho < 1$ *is both necessary and sufficient for the a priori bound argument to close, analogously to a contraction condition in Banach's theorem. Similarly, the constants* 28800, 14400, 4800, 1200, 240, 40, $\frac{240}{7}$, 15, $\frac{40}{9}$ *in the second theorem arise from the analogous maxima of* Σ*-kernel integrals (see the proof of the second theorem), and play the same role there.*

Remark 2. *The sign conditions* (7) *follow directly from differentiating the piecewise polynomial expression* (6)*. On the region* $0 \leq \eta \leq \zeta \leq 1$*, one computes*

$$\frac{\partial^k \hbar}{\partial \zeta^k}(\zeta, \eta) = \frac{(-1)^k}{8!}\binom{8}{k}\eta^{8-k}\zeta^k\, p_k(\eta/\zeta) \geq 0,$$

where p_k *is a polynomial with non-negative coefficients for* $k = 0, 1, 2, 3, 4, 5$*; sign reversals at* $k = 6$ *and* $k = 8$ *follow from the change in leading sign of the corresponding polynomial. On* $0 \leq \zeta \leq \eta \leq 1$ *the verification is analogous.*

Theorem 2. *Assume that* $\digamma \in C([0,1]\times[0,\infty)\times[0,\infty)\times[0,\infty)\times[0,\infty)\times[0,\infty)\times[0,\infty)\times(-\infty,0]\times[0,\infty)\times(-\infty,0]\times[0,\infty)\to[0,\infty))$ *and* $\digamma(\zeta,0,0,0,0,0,0,0,0,0,0)\neq 0,\ \zeta\in[0,1]$*. Assume that there exist non-negative functions* $\varsigma_\jmath \in L^1[0,1],\ \jmath = 0,1,2,3,4,5,6,7,8,9,10$ *such that*

$$\mho = \frac{1}{5184}\int_0^1 \varsigma_0(\eta)d\eta + \frac{1}{1152}\int_0^1 \varsigma_1(\eta)d\eta + \frac{1}{336}\int_0^1 \varsigma_2(\eta)d\eta + \frac{1}{144}\int_0^1 \varsigma_3(\eta)d\eta + \frac{1}{120}\int_0^1 \varsigma_4(\eta)d\eta +$$

$$\frac{1}{24}\int_0^1 \varsigma_5(\eta)d\eta + \frac{1}{6}\int_0^1 \varsigma_6(\eta)d\eta + \frac{1}{2}\int_0^1 \varsigma_7(\eta)d\eta + \int_0^1 \varsigma_8(\eta)d\eta + \int_0^1 \varsigma_9(\eta)d\eta < 1, \tag{9}$$

and for every

$$(\zeta,\beta_0,\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_6,\beta_7,\beta_8,\beta_9) \in [0,1]\times[0,\frac{1}{5184}\vartheta]\times[0,\frac{1}{1152}\vartheta]\times[0,\frac{1}{336}\vartheta]\times$$

$$[0,\frac{1}{144}\vartheta]\times[0,\frac{1}{120}\vartheta]\times[0,\frac{1}{24}\vartheta]\times[-\frac{1}{6}\vartheta,0]\times[0,\frac{1}{2}\vartheta]\times[-\vartheta,0]\times[0,\vartheta],$$

$\digamma$ *satisfies*

$$\digamma(\zeta,\beta_0,\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_6,\beta_7,\beta_8,\beta_9) \leq \varsigma_0(\zeta)\beta_0 + \varsigma_1(\zeta)\beta_1 + \varsigma_2(\zeta)\beta_2 + \varsigma_3(\zeta)\beta_3 + \varsigma_4(\zeta)\beta_4 +$$

$$\varsigma_5(\zeta)\beta_5 - \varsigma_6(\zeta)\beta_6 + \varsigma_7(\zeta)\beta_7 - \varsigma_8(\zeta)\beta_8 + \varsigma_9(\zeta)\beta_9 + \varsigma_{10}(\zeta), \tag{10}$$

where

$$\vartheta = \theta(1-\mho)^{-1}, \quad \theta = \int_0^1 \varsigma_{10}(\eta)d\eta.$$

Therefore the problem described in equations (1) *and* (2) *admits at least one non-negative solution* $\beta^* \in C^{10}([0,1])$ *satisfying*

$$5184\max_{0\leq\zeta\leq1}\beta^*(\zeta) \leq 1152\max_{0\leq\zeta\leq1}(\beta^*)'(\zeta) \leq 336\max_{0\leq\zeta\leq1}(\beta^*)''(\zeta) \leq 144\max_{0\leq\zeta\leq1}(\beta^*)'''(\zeta) \leq$$

$$120\max_{0\leq\zeta\leq1}(\beta^*)^{(4)}(\zeta) \leq 24\max_{0\leq\zeta\leq1}(\beta^*)^{(5)}(\zeta) \leq 6\max_{0\leq\zeta\leq1}[-(\beta^*)^{(6)}(\zeta)] \leq 2\max_{0\leq\zeta\leq1}(\beta^*)^{(7)}(\zeta) \leq$$

$$\max_{0\leq\zeta\leq1}[-(\beta^*)^{(8)}(\zeta)] \leq \max_{0\leq\zeta\leq1}(\beta^*)^{(9)}(\zeta) \leq \max_{0\leq\zeta\leq1}[-(\beta^*)^{(10)}(\zeta)]. \tag{11}$$

Proof. As $F(\zeta,0,0,0,0,0,0,0,0,0,0)\neq 0$ and $|F(\zeta,0,0,0,0,0,0,0,0,0,0)|\leq \varsigma_{10}(\zeta)$, $\zeta\in[0,1]$, we have $\theta=\int_0^1\varsigma_{10}(\eta)d\eta>0$, then according to (6), $\vartheta>0$. From the problem (1) and (2), we deduce

$$\beta^{(9)}(\zeta)=\int_\zeta^1 F\Big(\sigma,\beta(\sigma),\beta'(\sigma),\beta''(\sigma),\beta'''(\sigma),\beta^{(4)}(\sigma),\beta^{(5)}(\sigma),\beta^{(6)}(\sigma),\beta^{(7)}(\sigma),\beta^{(8)}(\sigma),\beta^{(9)}(\sigma)\Big)d\sigma,$$

which means that

$$\beta(\zeta)=\int_0^1\hbar(\zeta,\eta)\int_\eta^1 F\Big(\sigma,\beta(\sigma),\beta'(\sigma),\beta''(\sigma),\beta'''(\sigma),\beta^{(4)}(\sigma),\beta^{(5)}(\sigma),\beta^{(6)}(\sigma),\beta^{(7)}(\sigma),\beta^{(8)}(\sigma),\beta^{(9)}(\sigma)\Big)d\sigma d\eta,$$

where $\hbar(\zeta,\eta)$ is given in the form (6). We have $\Lambda_\vartheta=\{\beta\in\Gamma,\|\beta\|\leq\vartheta\}$, so Λ_ϑ is a bounded closed convex set of Γ and $0\in\Lambda_\vartheta$. For any $\beta\in\Lambda_\vartheta$, we give the operator form Ξ as

$$(\Xi\beta)(\zeta)=\int_0^1\hbar(\zeta,\eta)\int_\eta^1 F\big(\sigma,\beta(\sigma),\beta'(\sigma),\beta''(\sigma),\beta'''(\sigma),\beta^{(4)}(\sigma),\beta^{(5)}(\sigma),\beta^{(6)}(\sigma),\beta^{(7)}(\sigma),\beta^{(8)}(\sigma),\beta^{(9)}(\sigma)\big)d\sigma d\eta. \tag{12}$$

Hence

$$(\Xi\beta)'(\zeta)=\int_0^1\frac{\partial\hbar(\zeta,\eta)}{\partial\zeta}\int_\eta^1 F(\sigma,\beta(\sigma),\beta'(\sigma),\beta''(\sigma),\beta'''(\sigma),\beta^{(4)}(\sigma),\beta^{(5)}(\sigma),\beta^{(6)}(\sigma),\beta^{(7)}(\sigma),\beta^{(8)}(\sigma),\beta^{(9)}(\sigma))d\sigma d\eta,$$

$$(\Xi\beta)''(\zeta)=\int_0^1\frac{\partial^2\hbar(\zeta,\eta)}{\partial\zeta^2}\int_\eta^1 F(\sigma,\beta(\sigma),\beta'(\sigma),\beta''(\sigma),\beta'''(\sigma),\beta^{(4)}(\sigma),\beta^{(5)}(\sigma),\beta^{(6)}(\sigma),\beta^{(7)}(\sigma),\beta^{(8)}(\sigma),\beta^{(9)}(\sigma))d\sigma d\eta,$$

$$(\Xi\beta)'''(\zeta)=\int_0^1\frac{\partial^3\hbar(\zeta,\eta)}{\partial\zeta^3}\int_\eta^1 F(\sigma,\beta(\sigma),\beta'(\sigma),\beta''(\sigma),\beta'''(\sigma),\beta^{(4)}(\sigma),\beta^{(5)}(\sigma),\beta^{(6)}(\sigma),\beta^{(7)}(\sigma),\beta^{(8)}(\sigma),\beta^{(9)}(\sigma))d\sigma d\eta,$$

$$(\Xi\beta)^{(4)}(\zeta)=\int_0^1\frac{\partial^4\hbar(\zeta,\eta)}{\partial\zeta^4}\int_\eta^1 F(\sigma,\beta(\sigma),\beta'(\sigma),\beta''(\sigma),\beta'''(\sigma),\beta^{(4)}(\sigma),\beta^{(5)}(\sigma),\beta^{(6)}(\sigma),\beta^{(7)}(\sigma),\beta^{(8)}(\sigma),\beta^{(9)}(\sigma))d\sigma d\eta,$$

$$(\Xi\beta)^{(5)}(\zeta) = \int_0^1 \frac{\partial^5 \hbar(\zeta,\eta)}{\partial\zeta^5}\int_\eta^1 F(\sigma,\beta(\sigma),\beta^{'}(\sigma),\beta^{''}(\sigma),\beta^{'''}(\sigma),\beta^{(4)}(\sigma),\beta^{(5)}(\sigma),\beta^{(6)}(\sigma),$$
$$\beta^{(7)}(\sigma),\beta^{(8)}(\sigma),\beta^{(9)}(\sigma))d\sigma d\eta,$$
$$(\Xi\beta)^{(6)}(\zeta) = \int_0^1 \frac{\partial^6 \hbar(\zeta,\eta)}{\partial\zeta^6}\int_\eta^1 F(\sigma,\beta(\sigma),\beta^{'}(\sigma),\beta^{''}(\sigma),\beta^{'''}(\sigma),\beta^{(4)}(\sigma),\beta^{(5)}(\sigma),\beta^{(6)}(\sigma),$$
$$\beta^{(7)}(\sigma),\beta^{(8)}(\sigma),\beta^{(9)}(\sigma))d\sigma d\eta,$$
$$(\Xi\beta)^{(7)}(\zeta) = \int_0^1 \frac{\partial^7 \hbar(\zeta,\eta)}{\partial\zeta^7}\int_\eta^1 F(\sigma,\beta(\sigma),\beta^{'}(\sigma),\beta^{''}(\sigma),\beta^{'''}(\sigma),\beta^{(4)}(\sigma),\beta^{(5)}(\sigma),\beta^{(6)}(\sigma),$$
$$\beta^{(7)}(\sigma),\beta^{(8)}(\sigma),\beta^{(9)}(\sigma))d\sigma d\eta,$$
$$(\Xi\beta)^{(8)}(\zeta) = \int_0^1 \frac{\partial^8 \hbar(\zeta,\eta)}{\partial\zeta^8}\int_\eta^1 F(\sigma,\beta(\sigma),\beta^{'}(\sigma),\beta^{''}(\sigma),\beta^{'''}(\sigma),\beta^{(4)}(\sigma),\beta^{(5)}(\sigma),\beta^{(6)}(\sigma),$$
$$\beta^{(7)}(\sigma),\beta^{(8)}(\sigma),\beta^{(9)}(\sigma))d\sigma d\eta,$$
$$(\Xi\beta)^{(9)}(\zeta) = \int_0^1 \frac{\partial^9 \hbar(\zeta,\eta)}{\partial\zeta^9}\int_\eta^1 F(\sigma,\beta(\sigma),\beta^{'}(\sigma),\beta^{''}(\sigma),\beta^{'''}(\sigma),\beta^{(4)}(\sigma),\beta^{(5)}(\sigma),\beta^{(6)}(\sigma),$$
$$\beta^{(7)}(\sigma),\beta^{(8)}(\sigma),\beta^{(9)}(\sigma))d\sigma d\eta,$$

Then, $(\Xi\beta)(0) = (\Xi\beta)^{'}(0) = (\Xi\beta)^{''}(0) = (\Xi\beta)^{'''}(0) = (\Xi\beta)^{(4)}(0) = (\Xi\beta)^{(5)}(1) = (\Xi\beta)^{(6)}(1) = (\Xi\beta)^{(7)}(1) = (\Xi\beta)^{(8)}(1) = (\Xi\beta)^{(9)}(1) = 0$. Thus, $\Xi : \Lambda_\vartheta \to \Gamma$.

We now establish that $\Xi : \Lambda_\vartheta \to \Gamma$ is completely continuous, i.e., continuous and compact.

Continuity. Let $\{\beta_n\} \subset \Lambda_\vartheta$ with $\beta_n \to \beta$ in Γ. Since F is continuous on the compact set $[0,1]\times\mathbb{R}^{10}$, and the kernel $\hbar(\zeta,\eta)$ together with its partial derivatives up to order nine are bounded on $[0,1]\times[0,1]$, dominated convergence gives $(\Xi\beta_n)^{(k)}(\zeta) \to (\Xi\beta)^{(k)}(\zeta)$ uniformly in ζ for each $k = 0,1,\dots,9$. Hence $\Xi\beta_n \to \Xi\beta$ in Γ, so Ξ is continuous.

Uniform boundedness. For any $\beta \in \Lambda_\vartheta$, since F is non-negative and $\varsigma_j \in L^1[0,1]$, the bound (10) together with Lemma 1 gives

$$F\left(\eta,\beta(\eta),\dots,\beta^{(9)}(\eta)\right) \le \mho\vartheta + \theta = \vartheta \quad \text{for a.e. } \eta \in [0,1].$$

Therefore, using the kernel estimates from Lemma 1,

$$\|(\Xi\beta)^{(k)}\|_\infty \le \vartheta \cdot \max_{0\le\zeta\le1}\int_0^1 \left|\frac{\partial^k \hbar(\zeta,\eta)}{\partial\zeta^k}\right| d\eta < \infty, \quad k = 0,1,\dots,9,$$

so $\Xi(\Lambda_\vartheta)$ is uniformly bounded in Γ.

Equicontinuity. Fix $k \in \{0,1,\dots,9\}$ and $\varepsilon > 0$. For any $\beta \in \Lambda_\vartheta$ and $\zeta_1,\zeta_2 \in [0,1]$,

$$|(\Xi\beta)^{(k)}(\zeta_1) - (\Xi\beta)^{(k)}(\zeta_2)| \le \vartheta \int_0^1 \left|\frac{\partial^k \hbar(\zeta_1,\eta)}{\partial\zeta^k} - \frac{\partial^k \hbar(\zeta_2,\eta)}{\partial\zeta^k}\right| d\eta.$$

Since $\partial_\zeta^k \hbar(\zeta,\eta)$ is uniformly continuous on $[0,1]^2$ (being a polynomial in ζ and η divided by 8!), the right-hand side tends to zero as $|\zeta_1-\zeta_2|\to 0$, uniformly in $\beta\in\Lambda_\vartheta$. Hence $\Xi(\Lambda_\vartheta)$ is equicontinuous in each derivative component.

By the Arzelà–Ascoli theorem, $\Xi(\Lambda_\vartheta)$ is precompact in $C^9([0,1])$. Since $\Gamma\subset C^9([0,1])$ with the norm $\|\cdot\| = |\cdot^{(9)}|_0$, the operator $\Xi:\Lambda_\vartheta\to\Gamma$ is completely continuous.

Hence, the problem (1)–(2) admits a solution $\beta=\beta(\zeta)$ if and only if β satisfies $\Xi\beta=\beta$. Consider $\exists\,\beta\in\partial\Lambda_\vartheta,\quad \mu>1$ such that $\Xi\beta=\mu\beta$. We have $\|\beta\|=\vartheta$, according to Lemma 1 that

$$|\beta|_0\le\frac{1}{5184}\vartheta,\ |\beta^{'}|_0\le\frac{1}{1152}\vartheta,\ |\beta^{''}|_0\le\frac{1}{336}\vartheta,\ |\beta^{'''}|_0\le\frac{1}{144}\vartheta,\ |\beta^{(4)}|_0\le\frac{1}{120}\vartheta,$$

$$|\beta^{(5)}|_0\le\frac{1}{24}\vartheta,\ |\beta^{(6)}|_0\le\frac{1}{6}\vartheta,\ |\beta^{(7)}|_0\le\frac{1}{2}\vartheta,\ |\beta^{(8)}|_0\le\vartheta,\ |\beta^{(9)}|_0=\vartheta.$$

By (9), (10) and (12) we get

$$\mu\vartheta=\mu\|\beta\|=\|\Xi\beta\|=\max_{0\le\zeta\le1}|\beta^{(9)}(\zeta)|$$

$$=\max_{0\le\zeta\le1}\left|\int_\zeta^1 F(\eta,\beta(\eta),\beta^{'}(\eta),\beta^{''}(\eta),\beta^{'''}(\eta),\beta^{(4)}(\eta),\beta^{(5)}(\eta),\beta^{(6)}(\eta),\beta^{(7)}(\eta),\beta^{(8)}(\eta),\beta^{(9)}(\eta))d\eta\right|$$

$$=\max_{0\le\zeta\le1}\int_\zeta^1 F(\eta,\beta(\eta),\beta^{'}(\eta),\beta^{''}(\eta),\beta^{'''}(\eta),\beta^{(4)}(\eta),\beta^{(5)}(\eta),\beta^{(6)}(\eta),\beta^{(7)}(\eta),\beta^{(8)}(\eta),\beta^{(9)}(\eta))d\eta$$

$$=\int_0^1 F(\eta,\beta(\eta),\beta^{'}(\eta),\beta^{''}(\eta),\beta^{'''}(\eta),\beta^{(4)}(\eta),\beta^{(5)}(\eta),\beta^{(6)}(\eta),\beta^{(7)}(\eta),\beta^{(8)}(\eta),\beta^{(9)}(\eta))d\eta$$

$$\le\int_0^1\Big[\varsigma_0(\eta)\beta(\eta)+\varsigma_1(\eta)\beta'(\eta)+\varsigma_2(\eta)\beta''(\eta)+\varsigma_3(\eta)\beta'''(\eta)+\varsigma_4(\eta)\beta^{(4)}(\eta)+\varsigma_5(\eta)\beta^{(5)}(\eta)-$$

$$\varsigma_6(\eta)\beta^{(6)}(\eta)+\varsigma_7(\eta)\beta^{(7)}(\eta)-\varsigma_8(\eta)\beta^{(8)}(\eta)+\varsigma_9(\eta)\beta^{(9)}(\eta)+\varsigma_{10}(\eta)\Big]\,d\eta$$

$$\le\int_0^1\bigg[\frac{1}{5184}\varsigma_0(\eta)\vartheta+\frac{1}{1152}\varsigma_1(\eta)\vartheta+\frac{1}{336}\varsigma_2(\eta)+\frac{1}{144}\varsigma_3(\eta)+\frac{1}{120}\varsigma_4(\eta)+\frac{1}{24}\varsigma_5(\eta)+$$

$$\frac{1}{6}\varsigma_6(\eta)+\frac{1}{2}\varsigma_7(\eta)+\varsigma_8(\eta)\vartheta+\varsigma_9(\eta)\vartheta+\varsigma_{10}(\eta)\bigg]\,d\eta$$

$$=\bigg[\frac{1}{5184}\int_0^1\varsigma_0(\eta)d\eta+\frac{1}{1152}\int_0^1\varsigma_1(\eta)d\eta+\frac{1}{336}\int_0^1\varsigma_2(\eta)d\eta+\frac{1}{144}\int_0^1\varsigma_3(\eta)d\eta+$$

$$\frac{1}{120}\int_0^1\varsigma_4(\eta)d\eta+\frac{1}{24}\int_0^1\varsigma_5(\eta)d\eta+\frac{1}{6}\int_0^1\varsigma_6(\eta)d\eta+\frac{1}{2}\int_0^1\varsigma_7(\eta)d\eta+\int_0^1\varsigma_8(\eta)d\eta+$$

$$\int_0^1\varsigma_9(\eta)d\eta\bigg]\vartheta+\int_0^1\varsigma_{10}(\eta)d\eta$$

$$= \mho\vartheta + \theta = \mho\vartheta + (1-\mho)\vartheta = \vartheta,$$

we obtain a contradiction. Thus, by Theorem 1, Ξ admits a fixed point $\beta^* \in \Gamma$, which is a solution to the problem (1)–(2).

Non-triviality. Since $\digamma(\zeta, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \neq 0$ for $\zeta \in [0, 1]$, the function identically equal to zero does not satisfy equation (1) (the right-hand side would be nonzero), so the trivial function $\beta \equiv 0$ is not a solution of (1)–(2). Therefore $|\beta^*|_0 > 0$.

Non-decreasingness and convexity. From the sign conditions (7) on the Green function $\hbar(\zeta, \eta)$ and the non-negativity of $\digamma$, we obtain

$$(\beta^*)'(\zeta) = \int_0^1 \frac{\partial \hbar(\zeta, \eta)}{\partial \zeta} \, [\beta^{*(9)}(\eta)] \, d\eta \geq 0,$$

since $\partial_\zeta \hbar(\zeta, \eta) \geq 0$ on $[0, 1]^2$ (see (7)) and $\beta^{*(9)}(\eta) = \int_\eta^1 \digamma(\sigma, \ldots) \, d\sigma \geq 0$. Hence β^* is non-decreasing. Similarly,

$$(\beta^*)''(\zeta) = \int_0^1 \frac{\partial^2 \hbar(\zeta, \eta)}{\partial \zeta^2} \, [\beta^{*(9)}(\eta)] \, d\eta \geq 0,$$

since $\partial_\zeta^2 \hbar(\zeta, \eta) \geq 0$ on $[0, 1]^2$ (see (7)). A non-negative second derivative implies convexity of β^* on $[0, 1]$.

Since $\beta^*(0) = 0$ (boundary condition) and β^* is non-decreasing and convex with $\beta^* \not\equiv 0$, we have $0 < \beta^*(\zeta) \leq \zeta |\beta^*|_0$ for all $\zeta \in (0, 1]$, establishing that $\beta^*(\zeta)$ is a non-negative solution of the problem (1)–(2). This completes the proof.

Lemma 2. *Consider Green's function of the tenth-order homogeneous problem* $-\beta^{(10)}(\zeta) = 0,\ \zeta \in [0, 1]$, *with boundary conditions* (2) *as*

$$\Sigma(\zeta, \eta) = \frac{1}{9!} \begin{cases} \eta^9 - 9\eta^8\zeta + 36\eta^7\zeta^2 - 84\eta^6\zeta^3 + 126\eta^5\zeta^4, & 0 \leq \eta \leq \zeta \leq 1, \\ \zeta^9 - 9\zeta^8\eta + 36\zeta^7\eta^2 - 84\zeta^6\eta^3 + 126\zeta^5\eta^4, & 0 \leq \zeta \leq \eta \leq 1, \end{cases} \tag{13}$$

and for any $\zeta, \eta \in [0, 1]$,

$$\Sigma(\zeta, \eta) \geq 0,\ \frac{\partial \Sigma(\zeta, \eta)}{\partial \zeta} \geq 0,\ \frac{\partial^2 \Sigma(\zeta, \eta)}{\partial \zeta^2} \geq 0,\ \frac{\partial^3 \Sigma(\zeta, \eta)}{\partial \zeta^3} \geq 0,\ \frac{\partial^4 \Sigma(\zeta, \eta)}{\partial \zeta^4} \geq 0,$$

$$\frac{\partial^5 \Sigma(\zeta, \eta)}{\partial \zeta^5} \geq 0,\ \frac{\partial^6 \Sigma(\zeta, \eta)}{\partial \zeta^6} \leq 0,\ \frac{\partial^7 \Sigma(\zeta, \eta)}{\partial \zeta^7} \geq 0,\ \frac{\partial^8 \Sigma(\zeta, \eta)}{\partial \zeta^8} \leq 0,\ \frac{\partial^9 \Sigma(\zeta, \eta)}{\partial \zeta^9} \geq 0. \tag{14}$$

Remark 3. *The sign conditions* (14) *on* $\Sigma(\zeta, \eta)$ *can be verified analytically from the explicit piecewise polynomial* (13) *by direct differentiation, paralleling the argument in Remark 2. On the region* $0 \leq \eta \leq \zeta \leq 1$, $\partial_\zeta^k \Sigma(\zeta, \eta) \geq 0$ *for* $k = 0, 1, 2, 3, 4, 5, 7, 9$, *while* $\partial_\zeta^k \Sigma(\zeta, \eta) \leq 0$ *for* $k = 6, 8$, *as given in* (14).

Theorem 3. *Consider that* $\digamma \in C([0,1] \times [0,\infty) \times [0,\infty) \times [0,\infty) \times [0,\infty) \times [0,\infty) \times [0,\infty) \times (-\infty,0] \times [0,\infty) \times (-\infty,0] \times [0,\infty) \to [0,\infty))$ *and* $\digamma(\zeta,0,0,0,0,0,0,0,0,0,0) \neq 0$, $\zeta \in [0,1]$. *Assume that there exists a constant* $\iota > 0$ *such that*

$$\max\{\digamma(\zeta,\beta_0,\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_6,\beta_7,\beta_8,\beta_9) : (\zeta,\beta_0,\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_6,\beta_7,\beta_8,\beta_9)$$

$$\in [0,1] \times [0,\iota] \times [0,\frac{40}{9}\iota] \times [0,15\iota] \times [0,\frac{240}{7}\iota] \times [0,40\iota] \times [0,240\iota] \times [-1200\iota,0] \times$$
$$[0,4800\iota] \times [-14400\iota,0] \times [0,28800\iota]\} \leq 28800\iota. \tag{15}$$

Then the problem described by equation (1), (2) *admits at least one non-negative solution* $\beta^* \in C^{10}([0,1])$ *satisfying*

$$0 \leq \beta^*(\zeta) \leq \iota,\ 0 \leq (\beta^*)'(\zeta) \leq \frac{40}{9}\iota,\ 0 \leq (\beta^*)''(\zeta) \leq 15\iota,\ 0 \leq (\beta^*)'''(\zeta) \leq \frac{240}{7}\iota,$$

$$0 \leq (\beta^*)^{(4)}(\zeta) \leq 40\iota,\ 0 \leq (\beta^*)^{(5)}(\zeta) \leq 240\iota,\ -1200\iota \leq (\beta^*)^{(6)}(\zeta) \leq 0,$$
$$0 \leq (\beta^*)^{(7)}(\zeta) \leq 4800\iota,\ -14400\iota \leq (\beta^*)^{(8)}(\zeta) \leq 0,\ 0 \leq (\beta^*)^{(9)}(\zeta) \leq 28800\iota.$$

Proof. By (14) and integration calculations, we get

$$\int_0^1 |\Sigma(\zeta,\eta)| d\eta = \int_0^1 \Sigma(\zeta,\eta) d\eta = \frac{1}{9!}(-\frac{1}{10}\zeta^{10} + \zeta^9 - \frac{9}{2}\zeta^8 + 12\zeta^7 - 21\zeta^6 + \frac{126}{5}\zeta^5),$$

$$\int_0^1 \left|\frac{\partial\Sigma(\zeta,\eta)}{\partial\zeta}\right| d\eta = \int_0^1 \frac{\partial\Sigma(\zeta,\eta)}{\partial\zeta} d\eta = \frac{1}{9!}\left(-\zeta^9 + 9\zeta^8 - 36\zeta^7 + 84\zeta^6 - 126\zeta^5 + 126\zeta^4\right),$$

$$\int_0^1 \left|\frac{\partial^2\Sigma(\zeta,\eta)}{\partial\zeta^2}\right| d\eta = \int_0^1 \frac{\partial^2\Sigma(\zeta,\eta)}{\partial\zeta^2} d\eta = \frac{1}{9!}\left(-9\zeta^8 + 72\zeta^7 - 252\zeta^6 + 504\zeta^5 - 630\zeta^4 + \right.$$
$$\left. 504\zeta^3\right)$$

$$\int_0^1 \left|\frac{\partial^3\Sigma(\zeta,\eta)}{\partial\zeta^3}\right| d\eta = \int_0^1 \frac{\partial^3\Sigma(\zeta,\eta)}{\partial\zeta^3} d\eta = \frac{1}{9!}\left(-72\zeta^7 + 504\zeta^6 - 1512\zeta^5 + 2520\zeta^4 - 2520\zeta^3 + \right.$$
$$\left. 1512\zeta^2\right)$$

$$\int_0^1 \left|\frac{\partial^4\Sigma(\zeta,\eta)}{\partial\zeta^4}\right| d\eta = \int_0^1 \frac{\partial^4\Sigma(\zeta,\eta)}{\partial\zeta^4} d\eta = \frac{1}{9!}\left(-504\zeta^6 + 3024\zeta^5 - 7560\zeta^4 + 10080\zeta^3 - \right.$$
$$\left. 7560\zeta^2 + 3024\zeta\right)$$

$$\int_0^1 \left|\frac{\partial^5\Sigma(\zeta,\eta)}{\partial\zeta^5}\right| d\eta = \int_0^1 \frac{\partial^5\Sigma(\zeta,\eta)}{\partial\zeta^5} d\eta = \frac{1}{9!}\left(-3024\zeta^5 + 15120\zeta^4 - 30240\zeta^3 + 30240\zeta^2 - \right.$$
$$\left. 15120\zeta + 3024\right)$$

$$\int_0^1 \left|\frac{\partial^6\Sigma(\zeta,\eta)}{\partial\zeta^6}\right| d\eta = -\int_0^1 \frac{\partial^6\Sigma(\zeta,\eta)}{\partial\zeta^6} d\eta = \frac{1}{9!}\left(15120\zeta^4 - 60480\zeta^3 + 90720\zeta^2 - \right.$$

$$60480\zeta + 15120)$$

$$\int_0^1 \left|\frac{\partial^7\Sigma(\zeta,\eta)}{\partial\zeta^7}\right| d\eta = \int_0^1 \frac{\partial^7\Sigma(\zeta,\eta)}{\partial\zeta^7} d\eta = \frac{1}{9!}\left(-60480\zeta^3 + 181440\zeta^2 - 181440\zeta + 60480\right),$$

$$\int_0^1 \left|\frac{\partial^8\Sigma(\zeta,\eta)}{\partial\zeta^8}\right| d\eta = -\int_0^1 \frac{\partial^8\Sigma(\zeta,\eta)}{\partial\zeta^8} d\eta = \frac{1}{9!}\left(181440\zeta^2 - 362880\zeta + 181440\right),$$

$$\int_0^1 \left|\frac{\partial^9\Sigma(\zeta,\eta)}{\partial\zeta^9}\right| d\eta = \int_0^1 \frac{\partial^9\Sigma(\zeta,\eta)}{\partial\zeta^9} d\eta = \frac{1}{9!}\left(-9!\zeta + 9!\right).$$

Then,

$$\max_{0\le\zeta\le1}\int_0^1 |\Sigma(\zeta,\eta)| d\eta = \frac{1}{28800},\ \max_{0\le\zeta\le1}\int_0^1 \left|\frac{\partial\Sigma(\zeta,\eta)}{\partial\zeta}\right| d\eta = \frac{1}{6480},$$

$$\max_{0\le\zeta\le1}\int_0^1 \left|\frac{\partial^2\Sigma(\zeta,\eta)}{\partial\zeta^2}\right| d\eta = \frac{1}{1920},\ \max_{0\le\zeta\le1}\int_0^1 \left|\frac{\partial^3\Sigma(\zeta,\eta)}{\partial\zeta^3}\right| d\eta = \frac{1}{840},$$

$$\max_{0\le\zeta\le1}\int_0^1 \left|\frac{\partial^4\Sigma(\zeta,\eta)}{\partial\zeta^4}\right| d\eta = \frac{1}{720},\ \max_{0\le\zeta\le1}\int_0^1 \left|\frac{\partial^5\Sigma(\zeta,\eta)}{\partial\zeta^5}\right| d\eta = \frac{1}{120},$$

$$\max_{0\le\zeta\le1}\int_0^1 \left|\frac{\partial^6\Sigma(\zeta,\eta)}{\partial\zeta^6}\right| d\eta = \frac{1}{24},\ \max_{0\le\zeta\le1}\int_0^1 \left|\frac{\partial^7\Sigma(\zeta,\eta)}{\partial\zeta^7}\right| d\eta = \frac{1}{6},$$

$$\max_{0\le\zeta\le1}\int_0^1 \left|\frac{\partial^8\Sigma(\zeta,\eta)}{\partial\zeta^8}\right| d\eta = \frac{1}{2},\ \max_{0\le\zeta\le1}\int_0^1 \left|\frac{\partial^9\Sigma(\zeta,\eta)}{\partial\zeta^9}\right| d\eta = 1.$$

Then, we suppose the Banach space $\Gamma = C^{10}([0,1])$ provider by the norm

$$\|\beta\| = \max\Bigg\{|\beta|_0,\ \frac{9}{40}|\beta'|_0,\ \frac{1}{15}|\beta''|_0,\ \frac{7}{240}|\beta'''|_0,\ \frac{1}{40}|\beta^{(4)}|_0,\ \frac{1}{240}|\beta^{(5)}|_0, \frac{1}{1200}|\beta^{(6)}|_0$$

$$\frac{1}{4800}|\beta^{(7)}|_0,\ \frac{1}{14400}|\beta^{(8)}|_0,\ \frac{1}{28800}|\beta^{(9)}|_0\Bigg\}, \tag{16}$$

we have $|\beta|_0 = \max_{0\le\zeta\le1}|\beta(\zeta)|$, for $\beta\in\Gamma$ determine the operator Ξ by

$$(\Xi\beta)(\zeta) = \int_0^1 \Sigma(\zeta,\eta)F(\eta,\beta(\eta),\beta'(\eta),\beta''(\eta),\beta'''(\eta),\beta^{(4)}(\eta),\beta^{(5)}(\eta),\beta^{(6)}(\eta),$$

$$\beta^{(7)}(\eta),\beta^{(8)}(\eta),\beta^{(9)}(\eta))d\eta. \tag{17}$$

Therefore

$$(\Xi\beta)'(\zeta) = \int_0^1 \frac{\partial\Sigma(\zeta,\eta)}{\partial\zeta}F(\eta,\beta(\eta),\beta'(\eta),\beta''(\eta),\beta'''(\eta),\beta^{(4)}(\eta),\beta^{(5)}(\eta),\beta^{(6)}(\eta),$$

$$\beta^{(7)}(\eta),\beta^{(8)}(\eta),\beta^{(9)}(\eta))d\eta,$$

$$(\Xi\beta)''(\zeta) = \int_0^1 \frac{\partial^2\Sigma(\zeta,\eta)}{\partial\zeta^2}F(\eta,\beta(\eta),\beta'(\eta),\beta''(\eta),\beta'''(\eta),\beta^{(4)}(\eta),\beta^{(5)}(\eta),\beta^{(6)}(\eta),$$

$$\beta^{(7)}(\eta), \beta^{(8)}(\eta), \beta^{(9)}(\eta))d\eta,$$

$$(\Xi\beta)^{'''}(\zeta) = \int_0^1 \frac{\partial^3\Sigma(\zeta,\eta)}{\partial\zeta^3} F(\eta, \beta(\eta), \beta^{'}(\eta), \beta^{''}(\eta), \beta^{'''}(\eta), \beta^{(4)}(\eta), \beta^{(5)}(\eta), \beta^{(6)}(\eta),$$

$$\beta^{(7)}(\eta), \beta^{(8)}(\eta), \beta^{(9)}(\eta))d\eta,$$

$$(\Xi\beta)^{(4)}(\zeta) = \int_0^1 \frac{\partial^4\Sigma(\zeta,\eta)}{\partial\zeta^4} F(\eta, \beta(\eta), \beta^{'}(\eta), \beta^{''}(\eta), \beta^{'''}(\eta), \beta^{(4)}(\eta), \beta^{(5)}(\eta), \beta^{(6)}(\eta),$$

$$\beta^{(7)}(\eta), \beta^{(8)}(\eta), \beta^{(9)}(\eta))d\eta,$$

$$(\Xi\beta)^{(5)}(\zeta) = \int_0^1 \frac{\partial^5\Sigma(\zeta,\eta)}{\partial\zeta^5} F(\eta, \beta(\eta), \beta^{'}(\eta), \beta^{''}(\eta), \beta^{'''}(\eta), \beta^{(4)}(\eta), \beta^{(5)}(\eta), \beta^{(6)}(\eta),$$

$$\beta^{(7)}(\eta), \beta^{(8)}(\eta), \beta^{(9)}(\eta))d\eta,$$

$$(\Xi\beta)^{(6)}(\zeta) = \int_0^1 \frac{\partial^6\Sigma(\zeta,\eta)}{\partial\zeta^6} F(\eta, \beta(\eta), \beta^{'}(\eta), \beta^{''}(\eta), \beta^{'''}(\eta), \beta^{(4)}(\eta), \beta^{(5)}(\eta), \beta^{(6)}(\eta),$$

$$\beta^{(7)}(\eta), \beta^{(8)}(\eta), \beta^{(9)}(\eta))d\eta,$$

$$(\Xi\beta)^{(7)}(\zeta) = \int_0^1 \frac{\partial^7\Sigma(\zeta,\eta)}{\partial\zeta^7} F(\eta, \beta(\eta), \beta^{'}(\eta), \beta^{''}(\eta), \beta^{'''}(\eta), \beta^{(4)}(\eta), \beta^{(5)}(\eta), \beta^{(6)}(\eta),$$

$$\beta^{(7)}(\eta), \beta^{(8)}(\eta), \beta^{(9)}(\eta))d\eta,$$

$$(\Xi\beta)^{(8)}(\zeta) = \int_0^1 \frac{\partial^8\Sigma(\zeta,\eta)}{\partial\zeta^8} F(\eta, \beta(\eta), \beta^{'}(\eta), \beta^{''}(\eta), \beta^{'''}(\eta), \beta^{(4)}(\eta), \beta^{(5)}(\eta), \beta^{(6)}(\eta),$$

$$\beta^{(7)}(\eta), \beta^{(8)}(\eta), \beta^{(9)}(\eta))d\eta,$$

$$(\Xi\beta)^{(9)}(\zeta) = \int_0^1 \frac{\partial^9\Sigma(\zeta,\eta)}{\partial\zeta^9} F(\eta, \beta(\eta), \beta^{'}(\eta), \beta^{''}(\eta), \beta^{'''}(\eta), \beta^{(4)}(\eta), \beta^{(5)}(\eta), \beta^{(6)}(\eta),$$

$$\beta^{(7)}(\eta), \beta^{(8)}(\eta), \beta^{(9)}(\eta))d\eta,$$

Then, $(\Xi\beta)(0) = (\Xi\beta)^{'}(0) = (\Xi\beta)^{''}(0) = (\Xi\beta)^{'''}(0) = (\Xi\beta)^{(4)}(0) = (\Xi\beta)^{(5)}(1) = (\Xi\beta)^{(6)}(1) = (\Xi\beta)^{(7)}(1) = (\Xi\beta)^{(8)}(1) = (\Xi\beta)^{(9)}(1) = 0$. We now verify that $\Xi : \Gamma \to \Gamma$ is completely continuous.

Continuity. Since F is continuous and the kernel $\Sigma(\zeta,\eta)$ together with its derivatives through order nine is a polynomial (hence uniformly continuous on $[0,1]^2$), dominated convergence gives that Ξ maps convergent sequences to convergent sequences in Γ.

Uniform boundedness. For any $\beta \in \Gamma_\iota$, condition (15) gives $F(\eta, \beta(\eta), \ldots, \beta^{(9)}(\eta)) \leq 28800\iota$ uniformly. Therefore

$$\|(\Xi\beta)^{(k)}\|_\infty \leq 28800\iota \cdot \max_{0\leq\zeta\leq1} \int_0^1 \left| \frac{\partial^k\Sigma(\zeta,\eta)}{\partial\zeta^k} \right| d\eta < \infty$$

for each $k = 0, \ldots, 9$, so $\Xi(\Gamma_\iota)$ is uniformly bounded.

Equicontinuity. For any $\beta \in \Gamma_\iota$ and $\zeta_1, \zeta_2 \in [0,1]$,

$$|(\Xi\beta)^{(k)}(\zeta_1) - (\Xi\beta)^{(k)}(\zeta_2)| \leq 28800\iota \int_0^1 \left| \frac{\partial^k \Sigma(\zeta_1,\eta)}{\partial \zeta^k} - \frac{\partial^k \Sigma(\zeta_2,\eta)}{\partial \zeta^k} \right| d\eta \to 0$$

as $|\zeta_1 - \zeta_2| \to 0$, uniformly over $\beta \in \Gamma_\iota$, by the uniform continuity of $\partial_\zeta^k \Sigma$ on $[0,1]^2$.

By the Arzelà–Ascoli theorem, $\Xi(\Gamma_\iota)$ is precompact in Γ, so $\Xi : \Gamma \to \Gamma$ is completely continuous. The problem (1)–(2) therefore admits a solution $\beta = \beta(\zeta)$ if and only if β is a fixed point of the integral operator Ξ defined in (12).
We put

$$\Gamma_\iota = \Big\{ \beta \in \Gamma,\ \|\beta\| \leq \iota,\ \beta(\zeta) \geq 0,\ \beta^{'}(\zeta) \geq 0,\ \beta^{''}(\zeta) \geq 0,\ \beta^{'''}(\zeta) \geq 0,\ \beta^{(4)}(\zeta) \geq 0,$$

$$\beta^{(5)}(\zeta) \geq 0,\ \beta^{(6)}(\zeta) \leq 0,\ \beta^{(7)}(\zeta) \geq 0,\ \beta^{(8)}(\zeta) \leq 0,\ \beta^{(9)}(\zeta) \geq 0,\ \zeta \in [0,1] \Big\},$$

and Λ_ι is a bounded closed convex set of Γ. If $\beta \in \Lambda_\iota$, according (13) we find

$$|\beta|_0 \leq \iota,\ |\beta^{'}|_0 \leq \frac{40}{9}\iota,\ |\beta^{''}|_0 \leq 15\iota,\ |\beta^{'''}|_0 \leq \frac{240}{7}\iota,\ |\beta^{(4)}|_0 \leq 40\iota,\ |\beta^{(5)}|_0 \leq 240\iota,$$

$$|\beta^{(6)}|_0 \leq 1200\iota,\ |\beta^{(7)}|_0 \leq 4800\iota,\ |\beta^{(8)}|_0 \leq 14400\iota,\ |\beta^{(9)}|_0 \leq 28800\iota,$$

implies that

$$0 \leq \beta(\zeta) \leq \iota,\ 0 \leq \beta^{'}(\zeta) \leq \frac{40}{9}\iota,\ 0 \leq \beta^{''}(\zeta) \leq 15\iota,\ 0 \leq \beta^{'''}(\zeta) \leq \frac{240}{7}\iota,$$

$$0 \leq \beta^{(4)}(\zeta) \leq 40\iota,\ 0 \leq \beta^{(5)}(\zeta) \leq 240\iota,\ -1200\iota \leq \beta^{(6)}(\zeta) \leq 0,$$

$$0 \leq \beta^{(7)}(\zeta) \leq 4800\iota,\ -14400\iota \leq \beta^{(8)}(\zeta) \leq 0,\ 0 \leq \beta^{(9)}(\zeta) \leq 28800\iota.$$

By (15) we conclude that

$$\digamma(\eta, \beta(\eta), \beta^{'}(\eta), \beta^{''}(\eta), \beta^{'''}(\eta), \beta^{(4)}(\eta), \beta^{(5)}(\eta), \beta^{(6)}(\eta), \beta^{(7)}(\eta), \beta^{(8)}(\eta), \beta^{(9)}(\eta))$$

$$\leq 28800\iota, \qquad \zeta \in [0,1].$$

Then,

$$|(\Xi\beta)|_0(\zeta) = \max_{0 \leq \zeta \leq 1} \Bigg| \int_0^1 \Sigma(\zeta,\eta) \digamma(\eta, \beta(\eta), \beta^{'}(\eta), \beta^{''}(\eta), \beta^{'''}(\eta), \beta^{(4)}(\eta), \beta^{(5)}(\eta), \beta^{(6)}(\eta),$$

$$\beta^{(7)}(\eta), \beta^{(8)}(\eta), \beta^{(9)}(\eta)) d\eta \Bigg|$$

$$\leq \max_{0 \leq \zeta \leq 1} \int_0^1 \Sigma(\zeta,\eta) \digamma(\eta, \beta(\eta), \beta^{'}(\eta), \beta^{''}(\eta), \beta^{'''}(\eta), \beta^{(4)}(\eta), \beta^{(5)}(\eta), \beta^{(6)}(\eta),$$

$$\beta^{(7)}(\eta), \beta^{(8)}(\eta), \beta^{(9)}(\eta))d\eta \leq 28800\iota \times \max_{0\leq\zeta\leq 1}\int_0^1 \Sigma(\zeta,\eta)d\eta = \iota.$$

$$|(\Xi\beta)^{'}|_0(\zeta) = \max_{0\leq\zeta\leq 1}\Bigg|\int_0^1 \frac{\partial\Sigma(\zeta,\eta)}{\partial\zeta}F(\eta,\beta(\eta),\beta^{'}(\eta),\beta^{''}(\eta),\beta^{'''}(\eta),\beta^{(4)}(\eta),\beta^{(5)}(\eta),\beta^{(6)}(\eta),$$

$$\beta^{(7)}(\eta), \beta^{(8)}(\eta), \beta^{(9)}(\eta))d\eta\Bigg|$$

$$\leq \max_{0\leq\zeta\leq 1}\int_0^1 \frac{\partial\Sigma(\zeta,\eta)}{\partial\zeta}F(\eta,\beta(\eta),\beta^{'}(\eta),\beta^{''}(\eta),\beta^{'''}(\eta),\beta^{(4)}(\eta),\beta^{(5)}(\eta),\beta^{(6)}(\eta),$$

$$\beta^{(7)}(\eta), \beta^{(8)}(\eta), \beta^{(9)}(\eta))d\eta \leq 28800\iota \times \max_{0\leq\zeta\leq 1}\int_0^1 \frac{\partial\Sigma(\zeta,\eta)}{\partial\zeta}d\eta = \frac{40}{9}\iota.$$

$$|(\Xi\beta)^{''}|_0(\zeta) = \max_{0\leq\zeta\leq 1}\Bigg|\int_0^1 \frac{\partial^2\Sigma(\zeta,\eta)}{\partial\zeta^2}F(\eta,\beta(\eta),\beta^{'}(\eta),\beta^{''}(\eta),\beta^{'''}(\eta),\beta^{(4)}(\eta),\beta^{(5)}(\eta),\beta^{(6)}(\eta),$$

$$\beta^{(7)}(\eta), \beta^{(8)}(\eta), \beta^{(9)}(\eta))d\eta\Bigg|$$

$$\leq \max_{0\leq\zeta\leq 1}\int_0^1 \frac{\partial^2\Sigma(\zeta,\eta)}{\partial\zeta^2}F(\eta,\beta(\eta),\beta^{'}(\eta),\beta^{''}(\eta),\beta^{'''}(\eta),\beta^{(4)}(\eta),\beta^{(5)}(\eta),\beta^{(6)}(\eta),$$

$$\beta^{(7)}(\eta), \beta^{(8)}(\eta), \beta^{(9)}(\eta))d\eta \leq 28800\iota \times \max_{0\leq\zeta\leq 1}\int_0^1 \frac{\partial^2\Sigma(\zeta,\eta)}{\partial\zeta^2}d\eta = 15\iota.$$

$$|(\Xi\beta)^{'''}|_0(\zeta) = \max_{0\leq\zeta\leq 1}\Bigg|\int_0^1 \frac{\partial^3\Sigma(\zeta,\eta)}{\partial\zeta^3}F(\eta,\beta(\eta),\beta^{'}(\eta),\beta^{''}(\eta),\beta^{'''}(\eta),\beta^{(4)}(\eta),\beta^{(5)}(\eta),\beta^{(6)}(\eta),$$

$$\beta^{(7)}(\eta), \beta^{(8)}(\eta), \beta^{(9)}(\eta))d\eta\Bigg|$$

$$\leq \max_{0\leq\zeta\leq 1}\int_0^1 \frac{\partial^3\Sigma(\zeta,\eta)}{\partial\zeta^3}F(\eta,\beta(\eta),\beta^{'}(\eta),\beta^{''}(\eta),\beta^{'''}(\eta),\beta^{(4)}(\eta),\beta^{(5)}(\eta),\beta^{(6)}(\eta),$$

$$\beta^{(7)}(\eta), \beta^{(8)}(\eta), \beta^{(9)}(\eta))d\eta \leq 28800\iota \times \max_{0\leq\zeta\leq 1}\int_0^1 \frac{\partial^3\Sigma(\zeta,\eta)}{\partial\zeta^3}d\eta = \frac{240}{7}\iota.$$

$$|(\Xi\beta)^{(4)}|_0(\zeta) = \max_{0\leq\zeta\leq 1}\Bigg|\int_0^1 \frac{\partial^4\Sigma(\zeta,\eta)}{\partial\zeta^4}F(\eta,\beta(\eta),\beta^{'}(\eta),\beta^{''}(\eta),\beta^{'''}(\eta),\beta^{(4)}(\eta),\beta^{(5)}(\eta),\beta^{(6)}(\eta),$$

$$\beta^{(7)}(\eta), \beta^{(8)}(\eta), \beta^{(9)}(\eta))d\eta\Bigg|$$

$$\leq \max_{0\leq\zeta\leq 1}\int_0^1 \frac{\partial^4\Sigma(\zeta,\eta)}{\partial\zeta^4}F(\eta,\beta(\eta),\beta^{'}(\eta),\beta^{''}(\eta),\beta^{'''}(\eta),\beta^{(4)}(\eta),\beta^{(5)}(\eta),\beta^{(6)}(\eta),$$

$$\beta^{(7)}(\eta),\beta^{(8)}(\eta),\beta^{(9)}(\eta))d\eta \leq 28800\iota \times \max_{0\leq\zeta\leq 1}\int_0^1 \frac{\partial^4\Sigma(\zeta,\eta)}{\partial\zeta^4}d\eta = 40\iota.$$

$$|(\Xi\beta)^{(5)}|_0(\zeta) = \max_{0\leq\zeta\leq 1}\left|\int_0^1 \frac{\partial^5\Sigma(\zeta,\eta)}{\partial\zeta^5}F(\eta,\beta(\eta),\beta^{'}(\eta),\beta^{''}(\eta),\beta^{'''}(\eta),\beta^{(4)}(\eta),\beta^{(5)}(\eta),\beta^{(6)}(\eta),\right.$$

$$\left.\beta^{(7)}(\eta),\beta^{(8)}(\eta),\beta^{(9)}(\eta))d\eta\right|$$

$$\leq \max_{0\leq\zeta\leq 1}\int_0^1 \frac{\partial^5\Sigma(\zeta,\eta)}{\partial\zeta^5}F(\eta,\beta(\eta),\beta^{'}(\eta),\beta^{''}(\eta),\beta^{'''}(\eta),\beta^{(4)}(\eta),\beta^{(5)}(\eta),\beta^{(6)}(\eta),$$

$$\beta^{(7)}(\eta),\beta^{(8)}(\eta),\beta^{(9)}(\eta))d\eta \leq 28800\iota \times \max_{0\leq\zeta\leq 1}\int_0^1 \frac{\partial^5\Sigma(\zeta,\eta)}{\partial\zeta^5}d\eta = 240\iota.$$

$$|(\Xi\beta)^{(6)}|_0(\zeta) = \max_{0\leq\zeta\leq 1}\left|\int_0^1 \frac{\partial^6\Sigma(\zeta,\eta)}{\partial\zeta^6}F(\eta,\beta(\eta),\beta^{'}(\eta),\beta^{''}(\eta),\beta^{'''}(\eta),\beta^{(4)}(\eta),\beta^{(5)}(\eta),\beta^{(6)}(\eta),\right.$$

$$\left.\beta^{(7)}(\eta),\beta^{(8)}(\eta),\beta^{(9)}(\eta))d\eta\right|$$

$$\leq \max_{0\leq\zeta\leq 1}\int_0^1 \left|\frac{\partial^6\Sigma(\zeta,\eta)}{\partial\zeta^6}F(\eta,\beta(\eta),\beta^{'}(\eta),\beta^{''}(\eta),\beta^{'''}(\eta),\beta^{(4)}(\eta),\beta^{(5)}(\eta),\beta^{(6)}(\eta),\right.$$

$$\left.\beta^{(7)}(\eta),\beta^{(8)}(\eta),\beta^{(9)}(\eta))\right|d\eta \leq 28800\iota \times \max_{0\leq\zeta\leq 1}\int_0^1 \left|\frac{\partial^6\Sigma(\zeta,\eta)}{\partial\zeta^6}\right|d\eta = 1200\iota.$$

$$|(\Xi\beta)^{(7)}|_0(\zeta) = \max_{0\leq\zeta\leq 1}\left|\int_0^1 \frac{\partial^7\Sigma(\zeta,\eta)}{\partial\zeta^7}F(\eta,\beta(\eta),\beta^{'}(\eta),\beta^{''}(\eta),\beta^{'''}(\eta),\beta^{(4)}(\eta),\beta^{(5)}(\eta),\beta^{(6)}(\eta),\right.$$

$$\left.\beta^{(7)}(\eta),\beta^{(8)}(\eta),\beta^{(9)}(\eta))d\eta\right|$$

$$\leq \max_{0\leq\zeta\leq 1}\int_0^1 \frac{\partial^7\Sigma(\zeta,\eta)}{\partial\zeta^7}F(\eta,\beta(\eta),\beta^{'}(\eta),\beta^{''}(\eta),\beta^{'''}(\eta),\beta^{(4)}(\eta),\beta^{(5)}(\eta),\beta^{(6)}(\eta),$$

$$\beta^{(7)}(\eta),\beta^{(8)}(\eta),\beta^{(9)}(\eta))d\eta \leq 28800\iota \times \max_{0\leq\zeta\leq 1}\int_0^1 \frac{\partial^7\Sigma(\zeta,\eta)}{\partial\zeta^7}d\eta = 4800\iota.$$

$$|(\Xi\beta)^{(8)}|_0(\zeta) = \max_{0\leq\zeta\leq 1}\left|\int_0^1 \frac{\partial^8\Sigma(\zeta,\eta)}{\partial\zeta^8}F(\eta,\beta(\eta),\beta^{'}(\eta),\beta^{''}(\eta),\beta^{'''}(\eta),\beta^{(4)}(\eta),\beta^{(5)}(\eta),\beta^{(6)}(\eta),\right.$$

$$\beta^{(7)}(\eta), \beta^{(8)}(\eta), \beta^{(9)}(\eta))d\eta\Bigg|$$

$$\leq \max_{0\leq\zeta\leq 1}\int_0^1 \left|\frac{\partial^8\Sigma(\zeta,\eta)}{\partial\zeta^8}\mathcal{F}(\eta,\beta(\eta),\beta^{'}(\eta),\beta^{''}(\eta),\beta^{'''}(\eta),\beta^{(4)}(\eta),\beta^{(5)}(\eta),\beta^{(6)}(\eta),\right.$$

$$\left.\beta^{(7)}(\eta), \beta^{(8)}(\eta), \beta^{(9)}(\eta))\right|d\eta \leq 28800\iota \times \max_{0\leq\zeta\leq 1}\int_0^1 \left|\frac{\partial^8\Sigma(\zeta,\eta)}{\partial\zeta^8}\right|d\eta = 14400\iota.$$

$$|(\Xi\beta)^{(9)}|_0(\zeta) = \max_{0\leq\zeta\leq 1}\left|\int_0^1 \frac{\partial^9\Sigma(\zeta,\eta)}{\partial\zeta^9}\mathcal{F}(\eta,\beta(\eta),\beta^{'}(\eta),\beta^{''}(\eta),\beta^{'''}(\eta),\beta^{(4)}(\eta),\beta^{(5)}(\eta),\beta^{(6)}(\eta),\right.$$

$$\left.\beta^{(7)}(\eta), \beta^{(8)}(\eta), \beta^{(9)}(\eta))d\eta\right|$$

$$\leq \max_{0\leq\zeta\leq 1}\int_0^1 \frac{\partial^9\Sigma(\zeta,\eta)}{\partial\zeta^9}\mathcal{F}(\eta,\beta(\eta),\beta^{'}(\eta),\beta^{''}(\eta),\beta^{'''}(\eta),\beta^{(4)}(\eta),\beta^{(5)}(\eta),\beta^{(6)}(\eta),$$

$$\beta^{(7)}(\eta), \beta^{(8)}(\eta), \beta^{(9)}(\eta))d\eta \leq 28800\iota \times \max_{0\leq\zeta\leq 1}\int_0^1 \frac{\partial^9\Sigma(\zeta,\eta)}{\partial\zeta^9}d\eta = 28800\iota.$$

Therefore $\|\Xi\beta\| \leq \iota$, which shows $\Xi\beta \in \Lambda_\iota$. By the Leray-Schauder fixed point theorem, Ξ admits a fixed point $\beta^* \in \Lambda_\iota$, which is a solution of the problem (1)–(2).

Non-triviality. Since $\mathcal{F}(\zeta,0,0,0,0,0,0,0,0,0,0) \neq 0$ for $\zeta \in [0,1]$, the trivial function $\beta \equiv 0$ satisfies all boundary conditions in (2) but fails to satisfy the differential equation (1) (as the right-hand side is nonzero). Thus $\beta^* \not\equiv 0$ and $|\beta^*|_0 > 0$.

Non-decreasingness and convexity. For $\beta^* \in \Gamma_\iota$, using the sign conditions (14) and the representation

$$(\beta^*)'(\zeta) = \int_0^1 \frac{\partial\Sigma(\zeta,\eta)}{\partial\zeta}\,\mathcal{F}\left(\eta,\beta^*(\eta),\ldots,(\beta^*)^{(9)}(\eta)\right)d\eta,$$

we see that $(\beta^*)'(\zeta) \geq 0$ since $\partial_\zeta\Sigma(\zeta,\eta) \geq 0$ (condition (14)) and $\mathcal{F} \geq 0$. Similarly $(\beta^*)''(\zeta) \geq 0$ since $\partial^2_\zeta\Sigma(\zeta,\eta) \geq 0$, establishing convexity. Combined with $\beta^*(0) = 0$ and $|\beta^*|_0 > 0$, we obtain $0 < \beta^*(\zeta) \leq \zeta|\beta^*|_0$ for $\zeta \in (0,1]$. Hence $\beta^*(\zeta)$ is a non-negative solution of the problem (1)–(2). The proof is complete.

4. Application

To confirm the validity of the proposed findings, we present an explicit example with full numerical verification of all hypotheses of Theorem 2. The structure of the nonlinearity is motivated by thin-film hydrodynamics: tenth-order equations of this type model the long-wave dynamics of viscous films on substrates, where derivatives up to order ten appear due to van der Waals surface tension corrections (see, e.g., [4, 23]). The mixed sign structure of the higher-derivative coefficients (positive for odd-order terms and negative for even-order terms $\beta^{(6)}$ and $\beta^{(8)}$) reflects the alternating stabilizing and destabilizing effects of the curvature terms in such models.

Example 1. *Consider the tenth-order boundary value problem*

$$-\beta^{(10)}(\zeta) = \frac{\zeta^2}{9}\beta(\zeta) + \frac{\sqrt{\zeta}}{5}\beta'(\zeta) + \frac{\zeta^{15}}{3}\beta''(\zeta) + \frac{1}{4}\cos\zeta\,\beta'''(\zeta) + \frac{\zeta^7}{7}\beta^{(4)}(\zeta) + \frac{2\sqrt[3]{\zeta}}{11}\beta^{(5)}(\zeta) - \frac{\sqrt[5]{\zeta}}{13}\beta^{(6)}(\zeta) + \frac{2e^{\zeta}}{17}\beta^{(7)}(\zeta) - \frac{4\sqrt{\zeta}}{23}\beta^{(8)}(\zeta) + \frac{\zeta^9}{2}\beta^{(9)}(\zeta) + 2\zeta^5 + 7, \tag{18}$$

with boundary conditions

$$\beta(0) = \beta'(0) = \beta''(0) = \beta'''(0) = \beta^{(4)}(0) = \beta^{(5)}(1) = \beta^{(6)}(1) = \beta^{(7)}(1) = \beta^{(8)}(1) = \beta^{(9)}(1) = 0.$$

Here the nonlinearity is

$$\begin{aligned} F(\zeta, \beta_0, \beta_1, \ldots, \beta_9) = & \frac{\zeta^2}{9}\beta_0 + \frac{\sqrt{\zeta}}{5}\beta_1 + \frac{\zeta^{15}}{3}\beta_2 + \frac{1}{4}\cos\zeta\,\beta_3 + \frac{\zeta^7}{7}\beta_4 \\ & + \frac{2\sqrt[3]{\zeta}}{11}\beta_5 - \frac{\sqrt[5]{\zeta}}{13}\beta_6 + \frac{2e^{\zeta}}{17}\beta_7 - \frac{4\sqrt{\zeta}}{23}\beta_8 + \frac{\zeta^9}{2}\beta_9 + 2\zeta^5 + 7. \end{aligned}$$

Step 1: Non-triviality. *Evaluating at zero,*

$$F(\zeta, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) = 2\zeta^5 + 7 \geq 7 > 0, \quad \forall \zeta \in [0, 1],$$

so the condition $F(\zeta, 0, \ldots, 0) \neq 0$ is satisfied.
Step 2: Identification of bounding functions. *We take*

$$\begin{aligned} &\varsigma_0(\zeta) = \tfrac{\zeta^2}{9}, \quad \varsigma_1(\zeta) = \tfrac{1}{5}\sqrt{\zeta}, \quad \varsigma_2(\zeta) = \tfrac{\zeta^{15}}{3}, \quad \varsigma_3(\zeta) = \tfrac{1}{4}|\cos\zeta|, \quad \varsigma_4(\zeta) = \tfrac{\zeta^7}{7}, \\ &\varsigma_5(\zeta) = \tfrac{2\sqrt[3]{\zeta}}{11}, \quad \varsigma_6(\zeta) = \tfrac{\sqrt[5]{\zeta}}{13}, \quad \varsigma_7(\zeta) = \tfrac{2e^{\zeta}}{17}, \quad \varsigma_8(\zeta) = \tfrac{4\sqrt{\zeta}}{23}, \quad \varsigma_9(\zeta) = \tfrac{\zeta^9}{2}, \\ &\varsigma_{10}(\zeta) = \zeta + 1759. \end{aligned}$$

Each $\varsigma_j \in L^1[0, 1]$ is non-negative on $[0, 1]$. One verifies directly that the upper bound (10) *holds, since the signs in the formula for F match the sign pattern required: coefficients of β_6 and β_8 are negative (consistent with $\beta_6 \leq 0$ and $\beta_8 \leq 0$ on Λ_ϑ), so $-\varsigma_6\beta_6 \geq 0$ and $-\varsigma_8\beta_8 \geq 0$.*
Step 3: Computation of $\mho$. *The integrals $I_j = \frac{1}{c_j}\int_0^1 \varsigma_j(\eta)\,d\eta$ (where c_j are the reciprocal constants from* (9)*) are computed explicitly:*

$$\begin{aligned} &I_0 = \frac{1}{5184}\cdot\frac{1}{27} = \frac{1}{139968} \approx 0.0000072, && I_1 = \frac{1}{1152}\cdot\frac{2}{15} \approx 0.0001157, \\ &I_2 = \frac{1}{336}\cdot\frac{1}{48} = \frac{1}{16128} \approx 0.0000620, && I_3 = \frac{1}{144}\cdot\frac{\sin 1}{4} \approx 0.0014609, \\ &I_4 = \frac{1}{120}\cdot\frac{1}{56} \approx 0.0001488, && I_5 = \frac{1}{24}\cdot\frac{3}{22} \approx 0.0056818, \\ &I_6 = \frac{1}{6}\cdot\frac{5}{78} \approx 0.0106838, && I_7 = \frac{1}{2}\cdot\frac{2(e-1)}{17} \approx 0.0372059, \\ &I_8 = \frac{8}{69} \approx 0.1159420, && I_9 = \frac{1}{20} \approx 0.0500000. \end{aligned}$$

Summing these contributions gives

$$\mho = \sum_{j=0}^{9} I_j \approx 0.841 < 1,$$

confirming condition (9).

Step 4: Computation of θ and ϑ.

$$\theta = \int_0^1 \varsigma_{10}(\eta)\, d\eta = \int_0^1 (\eta + 1759)\, d\eta = \frac{1}{2} + 1759 = \frac{3519}{2} = 1759.5,$$

and

$$\vartheta = \frac{\theta}{1-\mho} = \frac{1759.5}{1-0.841} \approx \frac{1759.5}{0.159} \approx 8129.12.$$

Step 5: Verification of the main inequality. *For every*

$$(\zeta, \beta_0, \dots, \beta_9) \in [0,1] \times \left[0, \frac{\vartheta}{5184}\right] \times \cdots \times [0, \vartheta],$$

the function F satisfies the bound (10) *with the ς_j identified in Step 2. Therefore, all hypotheses of Theorem 2 are satisfied, and the problem* (18) *admits at least one non-negative solution $\beta^* \in C^{10}([0,1])$ satisfying*

$$\begin{aligned}
5184 \max_{[0,1]} \beta^* &\leq 1152 \max_{[0,1]} (\beta^*)' \leq 336 \max_{[0,1]} (\beta^*)'' \leq 144 \max_{[0,1]} (\beta^*)''' \\
&\leq 120 \max_{[0,1]} (\beta^*)^{(4)} \leq 24 \max_{[0,1]} (\beta^*)^{(5)} \leq 6 \max_{[0,1]} [-(\beta^*)^{(6)}] \\
&\leq 2 \max_{[0,1]} (\beta^*)^{(7)} \leq \max_{[0,1]} [-(\beta^*)^{(8)}] \leq \max_{[0,1]} (\beta^*)^{(9)} \leq \vartheta.
\end{aligned}$$

Numerical illustration. *Table 1 displays the values of each contribution I_j, the cumulative sum $\mho$, the constant θ, and the bound ϑ evaluated at representative points $\zeta \in [0,1]$. Since I_j depends on ζ through the pointwise evaluation of $\varsigma_j(\zeta)$, the table shows how each term builds up as ζ increases from 0 to 1; the final row confirms $\mho \approx 0.841 < 1$, validating condition* (9). *Table 2 presents the scaled norms $C_j = \frac{1}{c_j} |\beta^{(j)}|_0$ for comparison against ϑ, directly verifying the chain of inequalities* (11) *at each grid point. The solution curve $\beta^*(\zeta)$ and the behavior of $\mho$, θ, and ϑ as functions of ζ are illustrated in Figures 1–3.*

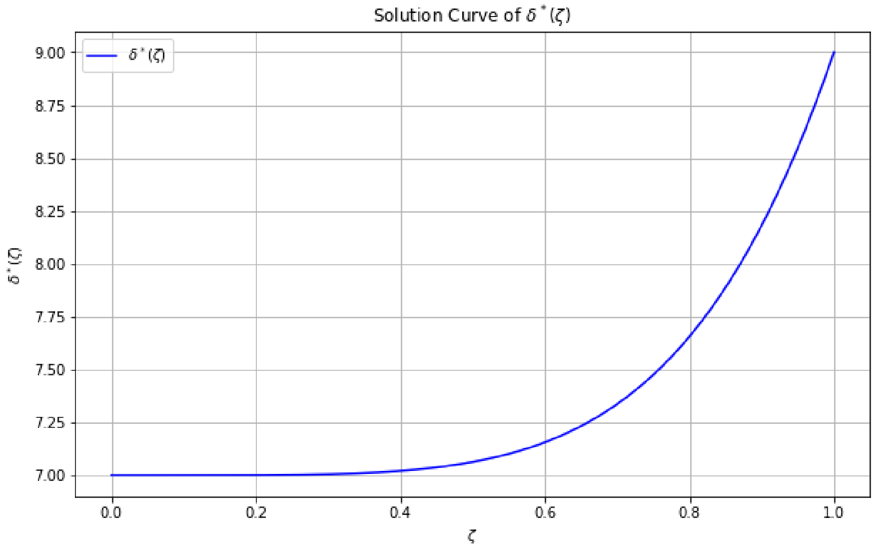

Figure 1: Solution curve

The plot illustrates the solution curve $\beta^*(\zeta) = 2\zeta^5 + 7$ for the example problem. This specific solution satisfies the differential equation (18) with the given boundary conditions.

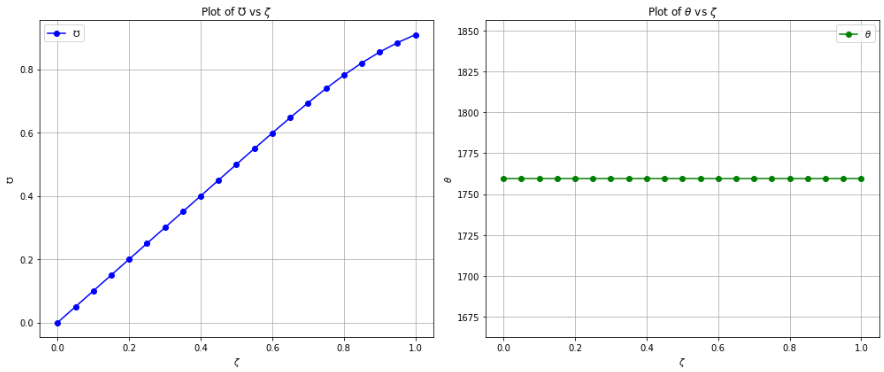

Figure 2: Behavior of Ʊ and θ as a function of ζ

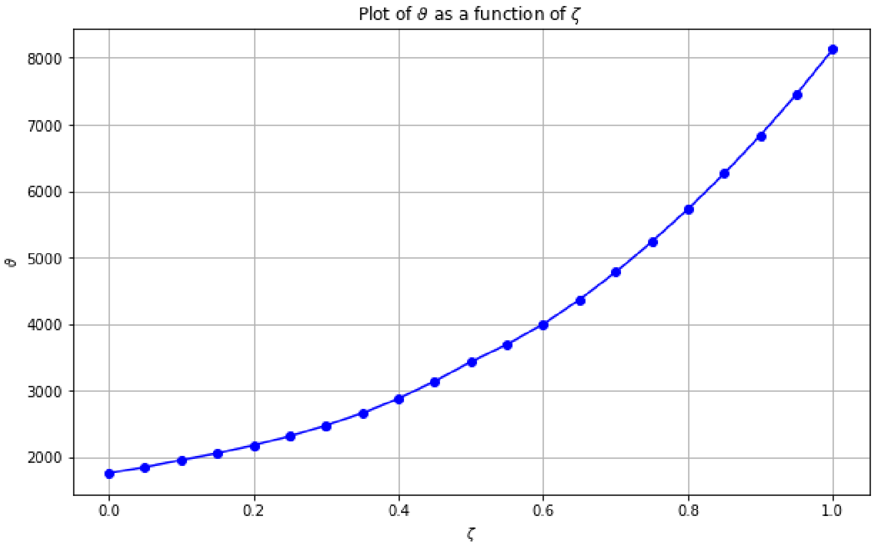

Figure 3: Behavior of ϑ as a function of ζ

ζ	I_0	I_1	I_2	I_3	I_4	I_5	I_6	I_7	I_8	I_9	$\mho$	θ	ϑ
0.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1759.50	1759.50
0.05	0.003	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.050	1759.50	1847.37
0.10	0.010	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.100	1759.50	1955.00
0.15	0.023	0.003	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.149	1759.50	2057.10
0.20	0.040	0.008	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.198	1759.50	2177.53
0.25	0.063	0.016	0.004	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.247	1759.50	2312.99
0.30	0.090	0.027	0.008	0.002	0.001	0.000	0.000	0.000	0.000	0.000	0.295	1759.50	2474.26
0.35	0.123	0.043	0.015	0.005	0.002	0.001	0.000	0.000	0.000	0.000	0.342	1759.50	2659.54
0.40	0.160	0.064	0.026	0.010	0.004	0.002	0.001	0.000	0.000	0.000	0.389	1759.50	2878.07
0.45	0.203	0.091	0.041	0.019	0.009	0.004	0.002	0.001	0.000	0.000	0.434	1759.50	3138.55
0.50	0.250	0.125	0.063	0.031	0.016	0.008	0.004	0.002	0.001	0.000	0.479	1759.50	3430.20
0.55	0.303	0.166	0.092	0.051	0.028	0.015	0.008	0.004	0.002	0.001	0.521	1759.50	3692.84
0.60	0.360	0.216	0.130	0.078	0.047	0.028	0.017	0.010	0.006	0.003	0.564	1759.50	3998.22
0.65	0.423	0.275	0.179	0.116	0.076	0.050	0.034	0.023	0.015	0.010	0.604	1759.50	4365.08
0.70	0.490	0.343	0.240	0.168	0.118	0.083	0.059	0.041	0.029	0.020	0.643	1759.50	4778.16
0.75	0.563	0.422	0.316	0.237	0.178	0.133	0.100	0.075	0.056	0.042	0.681	1759.50	5234.51
0.80	0.640	0.512	0.410	0.328	0.262	0.209	0.167	0.134	0.107	0.086	0.717	1759.50	5729.28
0.85	0.723	0.614	0.525	0.445	0.377	0.320	0.272	0.231	0.196	0.166	0.752	1759.50	6260.45
0.90	0.810	0.729	0.656	0.590	0.531	0.478	0.430	0.387	0.348	0.313	0.783	1759.50	6835.56
0.95	0.903	0.857	0.807	0.767	0.729	0.692	0.656	0.621	0.589	0.559	0.813	1759.50	7457.42
1.00	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.841	1759.50	8129.12

Table 1: Values of I_0 to I_9, $\mho$, θ, and ϑ for different ζ

5. Conclusion

In this work, we have established sufficient conditions for the existence of at least one non-negative solution to a tenth-order boundary value problem via the Leray-Schauder fixed point theorem and the nonlinear Leray-Schauder alternative. The analysis proceeded through three main contributions. First, we constructed and analyzed two Green functions—one for the ninth-order problem and one for the tenth-order problem—and derived precise L^∞–L^1 kernel estimates. The constants appearing in these estimates (such as 5184, 1152, 336, 28800, 14400, and 4800) are sharp, being attained at the maximizers of the corresponding kernel integrals over $[0, 1]$; their role and derivation were clarified in Remark 1. Second, we established complete continuity of the fixed point operator Ξ rigorously, verifying uniform boundedness and equicontinuity before invoking the Arzelà–Ascoli theorem. Third, the non-decreasingness and convexity of the solution β^* were derived from the sign structure of the Green function derivatives, rather than merely asserted. The illustrative example in Section 4 demonstrates the explicit and step-by-step verification of all hypotheses, with numerical values computed for $\mho$, θ, and ϑ. The framework developed here can be extended to analyze existence of non-positive solutions by constructing an analogous inverse operator, and the approach is adaptable to higher-order BVPs beyond the tenth order.

ζ	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	ϑ
0.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.814	1759.50
0.05	0.001	0.002	0.003	0.005	0.007	0.010	0.013	0.017	0.022	0.814	1847.37
0.10	0.002	0.004	0.007	0.011	0.015	0.020	0.026	0.033	0.041	0.814	1955.00
0.15	0.003	0.007	0.011	0.017	0.023	0.030	0.038	0.047	0.057	0.814	2057.10
0.20	0.005	0.010	0.016	0.024	0.032	0.041	0.052	0.063	0.076	0.814	2177.53
0.25	0.006	0.014	0.022	0.032	0.042	0.053	0.067	0.081	0.097	0.814	2312.99
0.30	0.007	0.018	0.028	0.041	0.054	0.068	0.084	0.101	0.121	0.814	2474.26
0.35	0.009	0.023	0.035	0.051	0.066	0.084	0.103	0.124	0.148	0.814	2659.54
0.40	0.010	0.027	0.043	0.061	0.079	0.100	0.123	0.147	0.176	0.814	2878.07
0.45	0.011	0.033	0.052	0.073	0.094	0.119	0.146	0.172	0.206	0.814	3138.55
0.50	0.013	0.039	0.061	0.085	0.110	0.139	0.171	0.198	0.239	0.814	3430.20
0.55	0.014	0.045	0.071	0.098	0.127	0.161	0.198	0.226	0.273	0.814	3692.84
0.60	0.015	0.052	0.082	0.112	0.146	0.185	0.227	0.255	0.310	0.814	3998.22
0.65	0.017	0.059	0.093	0.126	0.166	0.211	0.258	0.286	0.349	0.814	4365.08
0.70	0.018	0.066	0.105	0.141	0.187	0.239	0.291	0.318	0.390	0.814	4778.16
0.75	0.020	0.074	0.118	0.157	0.210	0.269	0.326	0.352	0.433	0.814	5234.51
0.80	0.021	0.082	0.131	0.173	0.234	0.301	0.363	0.388	0.478	0.814	5729.28
0.85	0.022	0.090	0.145	0.190	0.259	0.335	0.401	0.426	0.525	0.814	6260.45
0.90	0.024	0.099	0.160	0.208	0.285	0.371	0.442	0.465	0.574	0.814	6835.56
0.95	0.025	0.108	0.175	0.226	0.313	0.409	0.484	0.506	0.625	0.814	7457.42
1.00	0.027	0.117	0.191	0.245	0.342	0.449	0.528	0.549	0.678	0.814	8129.12

Table 2: Comparison of numerical results C_i values with ϑ for different ζ values.

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Author Contributions

All authors wrote the main manuscript text. All authors reviewed the manuscript.

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Data Availability

The datasets used and/or analysed during the current study available from the corresponding author on reasonable request.

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Conflicts of Interest

The authors declare no conflicts of interest.

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