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α-Conformable Fractional Lebesgue Type Measure and Sub-measure

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Abstract. In this paper, based on the α-conformable fractional derivative and integral, we give definitions for both Lebesgue measure and sub-measure in the fractional sense. The α-conformable fractional Lebesgue-type measures and sub-measures provide powerful generalizations of classical Lebesgue measure and sub-measure, enabling the analysis of non-standard complex spaces and phenomena. Their applications extend beyond pure mathematics to fields such as quantum physics, modeling fractional market dynamics, fractional probability theory, and pattern recognition for AI and machine learning.

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1. Introduction

Measure theory is a cornerstone of modern mathematical analysis, providing a rigorous framework for defining and studying the size or "measure" of sets in both simple and abstract spaces [1]. Central to measure theory is the concept of a measure μ which is, roughly, a non-negative extended additive real-valued function defined on a σ-algebra Σ of subsets of a set X and associates to each subset $E \in \Sigma$ a non-negative number $\mu(E)$. A measure

$$\mu : \Sigma \subseteq \mathcal{P}(X) \to [0, \infty],$$

must satisfy the following properties:

(i) Null empty set: $\mu(\Phi) = 0$,

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(ii) Non-negativity: $\mu(E) \geq 0$, for all $E \in \Sigma$,

(iii) Monotonicity: $\mu(E_1) \leq \mu(E_2)$, where $E_1 \subseteq E_2$ and $E_1, E_2 \in \Sigma$,

(iv) Countable additivity:

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i),$$

for any countable collection of disjoint sets $\{E_i\} \subset \Sigma$.

The 3-tuple (X, Σ, μ) is said to be a measurable space and a set $E \in \Sigma$ with well-defined measure is called a measurable set.

A standard example of a measure is the Lebesgue measure, that deals with subsets over the n-dimensional Euclidean space $\mathbb{R}^n$, and is denoted by m. For any interval $[a, b] \subseteq \mathbb{R}$, the corresponding Lebesgue measure is given by $m([a, b]) = b - a$ and for higher-dimensional Euclidean space $\mathbb{R}^n$ $(n \geq 2)$, Lebesgue measure generalizes the idea of length to area (2D), volume (3D), and beyond. Moreover, one of the interesting consequences of Lebesgue measure is the concept of Lebesgue integral. This technique of integration is considered as an extension of the Riemann integral in the sense that it can handle much larger classes of functions, particularly those with highly many discontinuities or functions that are not bounded [2].

Recall that, a real-valued function is continuous if and only if the inverse image of every open set is open. This notion can be generalized to the inverse image of every measurable set being measurable as follows.

Definition 1 (Measurable Function, [2]). *Let (X, Σ_X, μ_X) and (Y, Σ_Y, μ_Y) be two measurable spaces. A function $f : X \to Y$ is said to be measurable if for any measurable set $E \in \Sigma_Y$ we have $f^{-1}(E) \in \Sigma_X$.*

Moreover, the Lebesgue integral is defined for a measurable function f with respect to the Lebesgue measure as

$$\int_{\mathbb{R}} f(x)\, dx = \int_{\mathbb{R}} f(x)\, dm(x),$$

where m is the Lebesgue measure.

In recent years, the application of spectral methods for solving fractional differential equations has gained significant attention due to their efficiency and accuracy in handling complex boundary value problems. In this paper, we introduce a new definition that is intrinsically linked to this field, as it introduces innovative definitions of Lebesgue measure and sub-measure in the fractional sense through the lens of α-conformable fractional calculus. By establishing a mathematical framework that incorporates fractional derivatives and integrals, this work not only deepens the theoretical understanding of fractional measures but also lays the groundwork. At the same time, the present construction is inherently coordinate dependent: since the density $\alpha x^{\alpha-1}$ is anchored at the point $x = 0$, a translation of the underlying set changes the weight assigned to each location. From a modeling perspective, this means that the proposed measure is most naturally interpreted in situations where the origin has an intrinsic meaning, such as elapsed time from a fixed initial event, distance from a distinguished boundary point, or position relative to a reference state. In such settings, the lack of translation invariance is not a defect but rather a structural feature of the model.

To the best of our knowledge, no attempts have been conducted to study the tools of measure theory in the fractional case. In this current work, we present some definitions and results related to the α-conformable fractional measure as well as the α-conformable fractional sub-measure.

In Section 2, we present basic definitions related to the concept of α-conformable derivative of a function which is the main tool for providing the new definition of the α-conformable fractional measure as well as the α-conformable fractional sub-measure. In Section 3, the main results and some related concepts are introduced. Finally, in the conclusion section, we propose some exciting research directions, both in pure and applied mathematics. Throughout the measure-theoretic part of the paper we work on the half-line $[0,\infty)$, so that expressions such as x^α and $x^{\alpha-1}$ are unambiguous for $0<\alpha<1$. In particular, whenever intervals $[a,b]$ are used in the definition of μ_α, we implicitly assume $0\le a<b$.

2. α-Conformable Fractional Derivatives

In this section, we define α-conformable fractional derivatives, which extend the classical notions of differentiation to accommodate fractional orders while incorporating the α-conformable framework.

Historically, fractional derivatives have been pivotal in modeling phenomena in various fields, yet conventional definitions often lack the flexibility required for certain applications. The α-conformable fractional derivative not only retains the essential characteristics of traditional derivatives but also introduces unique properties that facilitate more accurate modeling of complex systems. For instance, linearity and continuity of these derivatives, demonstrating their applicability through specific examples [3]. By comparing these derivatives to established forms like the Riemann–Liouville and Caputo derivatives, several advantages have been reported, particularly in the context of spectral methods for solving fractional differential equations [3]. This discussion establishes a foundational understanding that is crucial for the subsequent sections of the paper, where we will apply these concepts to define fractional Lebesgue measures and sub-measures.

In mathematics, the positive integer ordered derivative $\frac{dy}{dx}=f'(x)$ of a function $y=f(x)$ is a concept by which we can find the instantaneous rate of change of a vertical variable y with respect to a horizontal variable x. Specifically, we write

$$f'(x)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x},$$

where Δx and Δy are, respectively, the corresponding increments of the variable x and the variable y. Dynamical systems whose state evolves over time, such as population growth and the flowing of a fluid through a pipe, can be described and studied by differential equations (ordinary or partial) that involve derivatives of a function. However, some of these dynamical systems that represent phenomena like electromagnetism, economy and finance, and signal processing cannot be handled accurately by utilizing differential equations that include positive integer ordered derivatives. In such cases, differential equations at which the included derivatives possess fractional orders come into play. It is known that the most widely used definitions for fractional derivatives are the old ones Riemann–Liouville and Caputo definitions and the latest one [4, 5], the so-called conformable fractional derivative [3]. These definitions can be stated as follows:

(i) **Riemann–Liouville definition.** For $\alpha \in [n-1, n)$, the α derivative for f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}}\, dx.$$

(ii) **Caputo definition.** For $\alpha \in [n-1, n)$, the α derivative of f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}}\, dx.$$

(iii) **Khalil definition.** For $\alpha \in (0, 1)$, the α-conformable derivative of f is

$$D^{\alpha} f(x) = \lim_{\varepsilon \to 0} \frac{f\left(x + \varepsilon x^{1-\alpha}\right) - f(x)}{\varepsilon}.$$

For any functions f and g, the following α-conformable fractional differentiation rules hold:

(i) $D^{\alpha}(c_1 f + c_2 g) = c_1 D^{\alpha}(f) + c_2 D^{\alpha}(g)$, for all $c_1, c_2 \in \mathbb{R}$,

(ii) $D^{\alpha}(k) = 0$, for all constant functions $f(x) = k$,

(iii) $D^{\alpha}(fg) = f D^{\alpha}(g) + g D^{\alpha}(f)$,

(iv) $D^{\alpha}\left(\frac{f}{g}\right) = \frac{g D^{\alpha}(f) - f D^{\alpha}(g)}{g^2}$, $g(x) \neq 0$.

In the above α-conformable fractional differentiation rules, if $\alpha = 1$, we get the corresponding classical rules for ordinary derivatives of order one. For more on fractional calculus and its applications we refer to [6–9]. We also note that the conformable approach has been discussed in the literature from different viewpoints: some authors regard it as a useful fractional-type framework with a transparent local structure, while others emphasize that, after an appropriate change of variables, it behaves differently from nonlocal theories such as the Riemann–Liouville and Caputo calculi; see, for example, [7]. The present paper does not attempt to settle that broader debate. Rather, our objective is more modest and precise: assuming the conformable setting as the basic analytic framework, we develop the associated measure-theoretic objects and study their elementary properties within that setting.

Definition 2 ([3]). *The α-conformable fractional integral of a function f starting from $a \geq 0$ is denoted by $I_a^\alpha(f)(x)$ such that*

$$I_a^\alpha(f)(x) = I_a^1\big(x^{\alpha-1} f\big) = \int_a^x \frac{f(t)}{t^{1-\alpha}}\, dt, \qquad \alpha \in (0,1).$$

3. Main Results

In this section, we present the main results concerning α-conformable fractional Lebesgue measures and sub-measures highlighting advancements and novel insights that contribute to the understanding of fractional calculus. By providing rigorous proofs and illustrative examples, this section aims to solidify the foundational concepts necessary for the discussions presented in the paper.

3.1. α-Conformable Fractional Measure

Let $0 \leq a, b \in \mathbb{R}$ such that $a < b$ and let I be one of the intervals (a, b), $[a, b]$, $[a, b)$, and $(a, b]$.

Definition 3. *The α-conformable fractional measure of I is denoted by $\mu_\alpha(I)$ and is given by*

$$\mu_\alpha(I) = b^\alpha - a^\alpha, \qquad \alpha \in (0,1). \tag{1}$$

Moreover, if K is the collection of all compact intervals in $\mathbb{R}$ such that I_1, I_2 are two disjoint intervals in K, then we define

$$\mu_\alpha(I_1 \cup I_2) = \mu_\alpha(I_1) + \mu_\alpha(I_2), \qquad I_1 \cap I_2 = \emptyset. \tag{2}$$

According to the above definition, if $I_1 = [a, b]$, $I_2 = (b, c]$ are disjoint, then $I_1 \cup I_2 = [a, c]$ and hence,

$$\begin{aligned} \mu_\alpha(I_1 \cup I_2) &= c^\alpha - a^\alpha && (3)\\ &= b^\alpha - a^\alpha + c^\alpha - b^\alpha && (4)\\ &= \mu_\alpha(I_1) + \mu_\alpha(I_2). && (5) \end{aligned}$$

It should be noted that the above result of $\mu_\alpha(I)$ agrees with the countable additivity property of the traditional Lebesgue measure $m(I)$.

Example 1. *Consider the set $A = [1, 2] \cup (2, 3]$. Then the classical Lebesgue measure of A is given by $m(A) = m([1, 3]) = 2$. On the other hand, for any $\alpha \in (0, 1)$, the α-conformable fractional measure of A is $\mu_\alpha(A) = \mu_\alpha([1, 3]) = 3^\alpha - 1$. It is obvious that:*

- *When $\alpha = 1$, we have $\mu_\alpha(A) = m(A)$,*
- *For $0 < \alpha < 1$, the α-conformable fractional measure $\mu_\alpha(A)$ reduces more significantly for lower values of α, highlighting its dependence on fractional scaling,*

- *When $\alpha = 0$, the α-conformable fractional measure $\mu_\alpha(A)$ collapses to zero, reflecting that in this limiting case, the intervals have no measurable size.*

Now, let $E \subseteq [0, \infty)$ be any open or closed set in $[0, \infty)$ such that $\mu(E) < \infty$. Then we have the following definition.

Definition 4. *The α-conformable fractional measure of E is denoted by $\mu_\alpha(E)$ and is given by*

$$\mu_\alpha(E) = \inf \left\{ \sum_j \mu_\alpha(I_j) : E \subseteq \bigcup_j I_j \right\}. \tag{6}$$

Theorem 1. *The set function μ_α admits the representation*

$$\mu_\alpha(E) = \int_E \alpha x^{\alpha-1}\, dx, \qquad E \in \mathcal{B}([0,\infty)),$$

and therefore defines a genuine Borel measure on $[0, \infty)$. In particular, if $\{E_i\}_{i=1}^\infty \subseteq \mathcal{B}([0,\infty))$ are pairwise disjoint, then

$$\mu_\alpha\left(\bigcup_{i=1}^\infty E_i\right) = \sum_{i=1}^\infty \mu_\alpha(E_i). \tag{7}$$

Proof. *Since $w_\alpha(x) = \alpha x^{\alpha-1}$ is a nonnegative measurable function on $[0, \infty)$, the set function $E \mapsto \int_E w_\alpha(x)\, dx$ is absolutely continuous with respect to Lebesgue measure and is countably additive by the monotone convergence theorem applied to the indicators of the partial unions $\bigcup_{i=1}^n E_i$. For an interval $[a, b]$ with $0 \le a < b$, this gives $\mu_\alpha([a, b]) = \int_a^b \alpha x^{\alpha-1}\, dx = b^\alpha - a^\alpha$, so the interval formula is consistent with the measure-theoretic definition.* □

Clearly, if E_1, E_2 are two open or closed sets in $\mathbb{R}$, then

$$\mu_\alpha(E_1 \cup E_2) \le \mu_\alpha(E_1) + \mu_\alpha(E_2).$$

Further, if $E_1 \cap E_2 = \emptyset$ and

$$d(E_1, E_2) = \inf\{|x - y| : x \in E_1,\ y \in E_2\} > 0,$$

then

$$\mu_\alpha(E_1 \cup E_2) = \mu_\alpha(E_1) + \mu_\alpha(E_2).$$

The second result follows from the fact that any cover of E_1 is disjoint from the cover of E_2, where by cover we mean a collection of compact intervals whose union contains E_1 or E_2. It should be noted that here we have already assumed that $\mu_\alpha(E_i) < \infty$, for $i = 1, 2$.

Remark 1. *The α-conformable fractional measure μ_α is not translation invariant. That is, for $\lambda \neq 0 \in \mathbb{R}$, we have*

$$\mu_\alpha([a,b]+\lambda) \neq \mu_\alpha([a,b]). \tag{8}$$

Indeed,

$$\mu_\alpha([a,b]+\lambda) = \mu_\alpha([a+\lambda, b+\lambda]) = (b+\lambda)^\alpha - (a+\lambda)^\alpha \neq b^\alpha - a^\alpha = \mu_\alpha([a,b]).$$

This formula shows explicitly that the measure depends on the choice of origin through the weight $x^{\alpha-1}$. Consequently, two intervals of the same Euclidean length may receive different α-conformable measures when they are placed at different locations. In applications, this suggests that the quantity measured is not purely geometric length; rather, it is a weighted size relative to a distinguished reference point. Therefore, whenever μ_α is used in a concrete model, the underlying coordinate system should be chosen so that the origin carries a clear physical, temporal, or geometric meaning.

Theorem 2. *Let $\{A_i\}_{i=1}^\infty \subseteq \mathcal{B}([0,\infty))$. Then*

(i) $\mu_\alpha(\emptyset) = 0$,

(ii) $\mu_\alpha(A_j) \leq \mu_\alpha(A_k)$ *whenever* $A_j \subseteq A_k$,

(iii) $\mu_\alpha(\bigcup_{i=1}^\infty A_i) \leq \sum_{i=1}^\infty \mu_\alpha(A_i)$,

(iv) if the sets A_i are pairwise disjoint, then

$$\mu_\alpha\left(\bigcup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty \mu_\alpha(A_i).$$

Parts (iii) and (iv) follow from the fact that μ_α is a Borel measure.

Definition 5. *Since*

$$\mu_\alpha([a,b]) = \int_a^b \alpha x^{\alpha-1}\,dx = b^\alpha - a^\alpha,$$

one can define

$$\mu_\alpha(E) = \int_E \alpha x^{\alpha-1}\,dx$$

where E is a Lebesgue measurable set. Moreover, we have

$$\int_E f(x)\,d\mu_\alpha(x) = \int_E \alpha x^{\alpha-1} f(x)\,dx, \tag{9}$$

where $f \in L^1(E)$ such that

$$L^1(E) = \left\{ h : E \to \mathbb{R} \mid h \text{ is measurable on } E \text{ and } \int_E |h(x)|\,d\mu(x) < \infty \right\}, \tag{10}$$

along with the following real-valued function

$$\|f\|_1 = \int_E |f(x)|\,d\mu(x).$$

The representation $d\mu_\alpha(x) = \alpha x^{\alpha-1}\,dx$ also makes the admissible class of functions transparent. Near $x = 0$, the weight $x^{\alpha-1}$ is singular for $0 < \alpha < 1$, and therefore membership in L^1_α requires additional decay of f at the origin. For example, if $|f(x)| \lesssim x^{-\beta}$ as $x \downarrow 0$, then $f \in L^1_\alpha[0,1]$ whenever $\beta < \alpha$. This observation explains the boundary behavior that appears in the example below.

Now, let us present the following definition.

Definition 6 (The α-conformable fractional Lebesgue space)**.** *Consider a measurable set $E \subseteq [0,\infty)$. Then the α-conformable fractional Lebesgue space $L^1_\alpha(E)$ is given by*

$$L^1_\alpha(E) = \left\{ f : E \to \mathbb{R} \mid f \textit{ is measurable on } E,\ f \in L^1(E) \textit{ and } \int_E |f|\,d\mu_\alpha < \infty \right\}. \tag{11}$$

As usual in L^1-theory, elements of $L^1_\alpha(E)$ are identified up to equality μ_α-almost everywhere.

Remark 2. *The integral above need not exist for all $L^1_\alpha(E)$. For instance, let $f(x) = \sqrt{\frac{1}{x}}$, $E = [0,1]$. Then it is clear that $f \in L^1[0,1]$. But, for $\alpha = \frac{1}{2}$, we have*

$$\int_E f(x)\,d\mu_{1/2}(x) = \frac{1}{2}\int_0^1 \frac{1}{\sqrt{x}}\frac{1}{\sqrt{x}}\,dx = \frac{1}{2}\int_0^1 \frac{1}{x}\,dx, \tag{12}$$

which is divergent and hence, $f \notin L^1_{1/2}[0,1]$.

The previous example describes the boundary singularity at $x = 0$ explicitly: although $f(x) = x^{-1/2}$ belongs to the classical space $L^1[0,1]$, the additional weight $x^{\alpha-1}$ may destroy integrability when α is too small. This is precisely why the weighted characterization below is needed.

A fact that follows from the above definition can be stated as follows.

Theorem 3. *$f \in L^1_\alpha[0,1]$ if and only if $x^{\alpha-1}f(x) \in L^1[0,1]$.*

Proof. Recall that, by Definition 6,

$$L^1_\alpha[0,1] = \left\{ f : [0,1] \to \mathbb{R} \;\middle|\; f \text{ is measurable on } [0,1] \text{ and } \int_0^1 |f(x)|\,d\mu_\alpha(x) < \infty \right\}.$$

Also, from Definition 5, we have

$$d\mu_\alpha(x) = \alpha x^{\alpha-1}\,dx.$$

Hence

$$\int_0^1 |f(x)|\,d\mu_\alpha(x) = \int_0^1 \alpha x^{\alpha-1}|f(x)|\,dx = \alpha\int_0^1 |x^{\alpha-1}f(x)|\,dx.$$

Now suppose first that $f \in L^1_\alpha[0,1]$. Then

$$\int_0^1 |f(x)|\,d\mu_\alpha(x) < \infty.$$

Using the above identity, we obtain

$$\alpha \int_0^1 |x^{\alpha-1} f(x)| \, dx < \infty.$$

Since $\alpha > 0$, it follows that

$$\int_0^1 |x^{\alpha-1} f(x)| \, dx < \infty,$$

which means that $x^{\alpha-1} f(x) \in L^1[0,1]$.

Conversely, assume that $x^{\alpha-1} f(x) \in L^1[0,1]$. Then

$$\int_0^1 |x^{\alpha-1} f(x)| \, dx < \infty.$$

Multiplying by the positive constant α, we get

$$\alpha \int_0^1 |x^{\alpha-1} f(x)| \, dx < \infty.$$

Therefore,

$$\int_0^1 |f(x)| \, d\mu_\alpha(x) = \alpha \int_0^1 |x^{\alpha-1} f(x)| \, dx < \infty.$$

Thus, $f \in L^1_\alpha[0,1]$.

Therefore,

$$f \in L^1_\alpha[0,1] \quad \Longleftrightarrow \quad x^{\alpha-1} f(x) \in L^1[0,1].$$

Before we get to the next result, let us define a norm on the α-conformable fractional Lebesgue space $L^1_\alpha[0,1]$.

Theorem 4. *Consider the α-conformable fractional Lebesgue space $L^1_\alpha[0,1]$. Then the mapping $\| \cdot \|_{1,\alpha} : L^1_\alpha[0,1] \to \mathbb{R}$ such that*

$$\|f\|_{1,\alpha} = \int_0^1 |f(x)| x^{\alpha-1} \, dx, \tag{13}$$

is a norm on $L^1_\alpha[0,1]$ and consequently, $\left(L^1_\alpha[0,1], \| \cdot \|_{1,\alpha}\right)$ is a normed space.

Proof. Non-negativity and absolute homogeneity are immediate from the definition. For the triangle inequality, if $h, g \in L^1_\alpha[0,1]$, then

$$\begin{aligned} \|h+g\|_{1,\alpha} &= \int_0^1 |h(x) + g(x)| x^{\alpha-1} \, dx \\ &\le \int_0^1 (|h(x)| + |g(x)|) x^{\alpha-1} \, dx \\ &= \|h\|_{1,\alpha} + \|g\|_{1,\alpha}. \end{aligned}$$

It remains to prove positive definiteness. If $\|f\|_{1,\alpha} = 0$, then $|f(x)|x^{\alpha-1} = 0$ for μ-almost every $x \in [0,1]$. Since $x^{\alpha-1} > 0$ for every $x \in (0,1]$, it follows that $f(x) = 0$ for μ-almost every $x \in (0,1]$. Hence $f = 0$ μ_α-almost everywhere, and therefore f is the zero element of $L^1_\alpha[0,1]$. Thus, $\|\cdot\|_{1,\alpha}$ is a norm on the quotient space defined above.

Theorem 5. *$\left(L^1_\alpha[0,1], \|\cdot\|_{1,\alpha}\right)$ is a Banach space.*

Proof. Let (f_n) be a Cauchy sequence in $L^1_\alpha[0,1]$ and define $g_n(x) = x^{\alpha-1} f_n(x)$. Then

$$\|g_n - g_m\|_{L^1([0,1])} = \int_0^1 |f_n(x) - f_m(x)| x^{\alpha-1}\, dx = \|f_n - f_m\|_{1,\alpha},$$

so (g_n) is a Cauchy sequence in $L^1([0,1])$. Since $L^1([0,1])$ is complete, there exists $g \in L^1([0,1])$ such that $g_n \to g$ in $L^1([0,1])$. Define

$$f(x) = \begin{cases} x^{1-\alpha} g(x), & x \in (0,1], \\ 0, & x = 0. \end{cases}$$

Then f is measurable and

$$\int_0^1 |f(x)| x^{\alpha-1}\, dx = \int_0^1 |g(x)|\, dx < \infty,$$

so $f \in L^1_\alpha[0,1]$. Moreover,

$$\|f_n - f\|_{1,\alpha} = \int_0^1 |x^{\alpha-1} f_n(x) - g(x)|\, dx = \|g_n - g\|_{L^1([0,1])} \to 0.$$

Therefore, every Cauchy sequence in $L^1_\alpha[0,1]$ converges to an element of $L^1_\alpha[0,1]$, and the space is complete.

3.2. α-Conformable Fractional Sub-measure

Let $0 \le a, b \in \mathbb{R}$ such that $a < b$ and let I be one of the intervals (a,b), $[a,b]$, $[a,b)$, and $(a,b]$.

Definition 7. *The α-conformable fractional sub-measure of I is denoted by $\mu^*_\alpha(I)$ and is given by*

$$\mu^*_\alpha(I) = (b-a)^\alpha, \qquad \alpha \in (0,1). \tag{14}$$

In contrast with μ_α from Sub-section 3.1, the set function μ^*_α is introduced here only as a finitely sub-additive content adapted to intervals. We do not claim σ-additivity for μ^*_α, and the superscript $*$ is used throughout to distinguish it from the genuine measure μ_α.

To show that the above is a sub-measure that is finitely sub-additive, let K be the collection of all compact intervals in $\mathbb{R}$ such that I_1, I_2 are two disjoint intervals in K, then we have

$$\mu_\alpha^*(I_1 \cup I_2) = \mu_\alpha^*(I_1) + \mu_\alpha^*(I_2), \tag{15}$$

such that $I_1 \cap I_2 = \emptyset$ and $d(I_1, I_2) > 0$.

Now, let $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$ be such that $I_1 \cap I_2 = \emptyset$ with $b_1 < a_2$. Move I_2 so that a_2 coincides with b_1. Therefore, we have $I_1 = [a_1, b_1]$, $\widetilde{I}_2 = [a_2 - c, b_2 - c]$, where $c = a_2 - b_1$. So,

$$I_1 \cup \widetilde{I}_2 = [a_1, b_1] \cup [a_2 - c, b_2 - c] = [a_1, b_2 - c].$$

Hence,

$$\begin{aligned} \mu_\alpha^*(I_1 \cup \widetilde{I}_2) &= (b_2 - c - a_1)^\alpha && (16)\\ &= (b_2 - (a_2 - b_1) - a_1)^\alpha && (17)\\ &= ((b_2 - a_2) + (b_1 - a_1))^\alpha && (18)\\ &\leq (b_2 - a_2)^\alpha + (b_1 - a_1)^\alpha && (19)\\ &= \mu_\alpha^*(I_1) + \mu_\alpha^*(I_2). && (20) \end{aligned}$$

Consequently, if $I_1 \cap I_2 = \emptyset$ and $d(I_1, I_2) \geq 0$, then

$$\mu_\alpha^*(I_1 \cup I_2) \leq \mu_\alpha^*(I_1) + \mu_\alpha^*(I_2).$$

Theorem 6. *The α-conformable fractional sub-measure μ_α^* is translation invariant. That is, for $\lambda \neq 0 \in \mathbb{R}$, we have*

$$\mu_\alpha^*([a, b] + \lambda) = \mu_\alpha^*([a, b]). \tag{21}$$

Indeed,

$$\mu_\alpha^*([a, b] + \lambda) = \mu_\alpha^*([a + \lambda, b + \lambda]) = (b + \lambda - (a + \lambda))^\alpha = (b - a)^\alpha = \mu_\alpha^*([a, b]).$$

Proof. By Definition 7, for any interval $[c, d]$,

$$\mu_\alpha^*([c, d]) = (d - c)^\alpha, \qquad \alpha \in (0, 1).$$

Now let $\lambda \in \mathbb{R}$. Then

$$[a, b] + \lambda = [a + \lambda, b + \lambda].$$

Therefore,

$$\mu_\alpha^*([a, b] + \lambda) = \mu_\alpha^*([a + \lambda, b + \lambda]).$$

Applying the definition again, we get

$$\mu_\alpha^*([a + \lambda, b + \lambda]) = \big((b + \lambda) - (a + \lambda)\big)^\alpha.$$

Since

$$(b + \lambda) - (a + \lambda) = b - a,$$

it follows that

$$\mu_\alpha^*([a,b]+\lambda) = (b-a)^\alpha.$$

But, again by Definition 7,

$$(b-a)^\alpha = \mu_\alpha^*([a,b]).$$

Hence,

$$\mu_\alpha^*([a,b]+\lambda) = \mu_\alpha^*([a,b]).$$

So, μ_α^* is translation invariant.

Remark 3. *Let $I = [a,b]$. Then*

$$\int_a^b \alpha(b-x)^{\alpha-1}\,dx = (b-a)^\alpha.$$

This leads to the following definition.

Definition 8. *Let E be a bounded Lebesgue measurable set. Then*

$$\mu_\alpha^*(E) = \int_E \alpha(b-x)^{\alpha-1}\,dx, \tag{22}$$

where b is the least upper bound of E.

Definition 9. *Let f be a Lebesgue measurable function on $[a,b]$. Then we define*

$$\int_{[a,b]} f(x)\,d\mu_\alpha^* = \int_a^b \alpha f(x)(b-x)^{\alpha-1}\,dx. \tag{23}$$

Definition 10 (The α-conformable fractional Lebesgue space)**.** *The α-conformable fractional Lebesgue space $L_\alpha^1[a,b]$ is given by*

$$L_\alpha^1[a,b] = \left\{ f:[a,b]\to\mathbb{R} \mid f \text{ is measurable on } [a,b],\ f\in L^1[a,b] \text{ and } \int_a^b |f(x)|\,d\mu_\alpha^* < \infty \right\}. \tag{24}$$

Theorem 7. *Consider the α-conformable fractional Lebesgue space $L_\alpha^1[a,b]$. Then the mapping $\|\cdot\|_{1,\alpha}^* : L_\alpha^1[a,b]\to\mathbb{R}$ such that*

$$\|f\|_{1,\alpha}^* = \int_a^b \alpha|f(x)|(b-x)^{\alpha-1}\,dx, \tag{25}$$

is a norm on $L_\alpha^1[a,b]$ and consequently, $\left(L_\alpha^1[a,b], \|\cdot\|_{1,\alpha}^*\right)$ is a normed space.

Proof. Similar to the proof of Theorem 4.

Theorem 8. $\left(L_\alpha^1[a,b], \|\cdot\|_{1,\alpha}^*\right)$ *is a Banach space.*

Proof. Similar to the proof of Theorem 5.

4. Conclusion

The new definitions of α-conformable fractional Lebesgue measure as well as α-conformable fractional sub-measure are introduced. These definitions open several promising research directions, both in pure mathematics (measure theory, functional analysis, probability) and in applied sciences (physics, finance, machine learning, and image processing). The results obtained here should be read in the specific context of conformable calculus: the proposed measure is sensitive to the reference origin, and this feature should be incorporated into any intended interpretation. Moreover, while conformable operators continue to generate discussion in the fractional-calculus literature regarding their relation to more classical nonlocal operators, the present work shows that, once this framework is fixed, a coherent measure-theoretic development is still possible. Future work could focus on extending its theoretical foundation, exploring its connections with fractal and fractional calculus, and leveraging its properties for real-world modeling problems.

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