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On Fuzzy Translations and Fuzzy Multiplications of Fuzzy Ideals and Subalgebras in Hilbert Algebras

Aiyared Iampan[1,*], Neelamegarajan Rajesh[2], K. Geetha[2]

[1] *Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand*
[2] *Department of Mathematics, Rajah Serfoji Government College, Thanjavur-613005, Tamil Nadu, India*

Abstract. The interplay between fuzzy structures and algebraic systems has been an active area of research due to its theoretical significance and potential applications in fuzzy logic, approximate reasoning, and non-classical algebraic frameworks. In this paper, we introduce and investigate the notions of fuzzy translations and fuzzy multiplications of fuzzy subalgebras and fuzzy ideals within Hilbert algebras. We establish fundamental properties, provide necessary and sufficient conditions for these operations to preserve algebraic structures, and examine their interrelations through the concept of fuzzy S-extensions. Several illustrative examples and counterexamples are presented to demonstrate that the converses of specific results do not hold in general. Our results not only extend previous studies on fuzzification in Hilbert algebras but also provide new structural insights that may apply to related algebraic systems and fuzzy logical models.

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Key Words and Phrases: Hilbert algebra, fuzzy subalgebra, fuzzy ideal, fuzzy translation, fuzzy multiplication, fuzzy S-extension.

1. Introduction

Zadeh [1] proposed the concept of fuzzy sets. The theory of fuzzy sets has numerous real-life applications, prompting many scholars to conduct further research. After the introduction of the concept of fuzzy sets, several research studies were performed on the generalizations of fuzzy sets. The integration between fuzzy sets and some uncertainty approaches, such as soft sets and rough sets, has been discussed in [2–4]. The concept of Hilbert algebra was introduced in the early 1950s by Henkin [5] for the purpose of investigating implication in intuitionistic and other non-classical logics. In the 1960s, these algebras were primarily studied by Diego [6] from an algebraic perspective. Diego

*Corresponding Author.
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Email addresses: aiyared.ia@up.ac.th (A. Iampan)
nrajesh_topology@yahoo.co.in (N. Rajesh), geeethaak@gmail.com (K. Geetha)

proved [6] that Hilbert algebras form a locally finite variety. Busneag [7, 8] treated Hilbert algebras, and Jun [9] recognized some of their filters as forming deductive systems. Dudek [10–12] explored the concept of fuzzification in subalgebras, ideals, and deductive systems within the framework of Hilbert algebras.

The concepts of fuzzy translation and fuzzy multiplication have been extensively investigated in various algebraic structures over the past decades. Chandramouleeswaran et al. [13] initiated the study in BF/BG-algebras, establishing foundational preservation results. Hameed and Alkurdi [14] later extended these notions to QS-algebras, while Guntasow et al. [15] explored them in the context of UP-algebras. Sowmiya and Jeyalakshmi [16] further refined the theory through the introduction of fuzzy α-translations and fuzzy β-multiplications in Z-algebras. Mary Jansi Rani and Alphonse Anitha [17] examined these operations in BV-algebras, providing a broader algebraic context, while Lee et al. [18] investigated fuzzy translations and fuzzy multiplications of BCK/BCI-algebras, contributing important foundational insights that influenced subsequent developments. Iampan et al. [19] further explored translations of bipolar fuzzy sets in Hilbert algebras, thereby establishing a direct link to the present study. More recently, Khonyong et al. [20] advanced the theory by introducing extensions and contractions of fuzzy translations in IUP-algebras, contributing a more generalized perspective on fuzzy structural transformations. In parallel, Senapati et al. [21] investigated Atanassov's intuitionistic fuzzy translations of intuitionistic fuzzy subalgebras in BG-algebras, emphasizing the interplay between intuitionistic and classical fuzzy frameworks. Other related investigations include those of Alshehri [22] in BRK-algebras, Shahoodh and Adwan [23] in D-algebras, Priya and Ramachandran [24] in PS-algebras, and Almuhaimeed [25] in KU-algebras, each establishing necessary structural preservation conditions. Earlier contributions by Jun [26] and Senapati [27] explored fuzzy and intuitionistic fuzzy translations in BCK/BCI-algebras and B-algebras, respectively, while the recent study by Nisha Devi et al. [28] offered a comprehensive analysis of interval-valued fuzzy translations and multiplications in Z-subalgebras. Collectively, these studies form a coherent and evolving framework for understanding fuzzy translations and fuzzy multiplications across a wide spectrum of algebraic systems, thus providing a strong motivation for extending these ideas to Hilbert algebras.

While the aforementioned studies have significantly advanced the theory of fuzzy translations and fuzzy multiplications across a variety of algebraic structures, including semigroups, BG-algebras, UP-algebras, Z-algebras, QS-algebras, BV-algebras, BRK-algebras, D-algebras, PS-algebras, KU-algebras, and others, a notable gap remains: the lack of a systematic treatment within the framework of Hilbert algebras. This is particularly striking given the close connection between Hilbert algebras and the algebraic semantics of certain non-classical propositional logics. Despite their theoretical importance, no prior work has examined how fuzzy α-translations and fuzzy γ-multiplications behave for fuzzy subalgebras and fuzzy ideals in this setting. The present study aims to fill this gap by extending these operations to Hilbert algebras and establishing necessary and sufficient conditions for the preservation of algebraic properties. Furthermore, we investigate their behavior under fuzzy S-extensions and fuzzy ideal extensions, thereby generalizing existing results from other algebraic contexts and uncovering new structural insights. These contributions not only strengthen the theoretical foundations of fuzzy algebraic systems but also open avenues for applications in fuzzy logic, approximate reasoning, and related domains.

2. Preliminaries

In this section, we recall some basic notions and results that will be used throughout the paper. We begin with the definition of Hilbert algebras and several related concepts, followed by the notions of fuzzy sets and their α-cuts. Special attention is given to the properties of fuzzy subalgebras and fuzzy ideals, as these play a central role in the subsequent sections. We also review the definitions of fuzzy α-translations and fuzzy γ-multiplications, which serve as the primary transformations under investigation in this work. These preliminaries are included to fix notation, clarify terminology, and provide a self-contained foundation for the main results presented in Sections 3 and 4.

Definition 1. *[6] A Hilbert algebra is a triplet with the formula $X = (X, *, 1)$, where X is a non-empty set, $*$ is a binary operation, and 1 is a fixed member of X that is true according to the axioms stated below:*

$$(\forall x, y \in X)(x * (y * x) = 1) \tag{1}$$

$$(\forall x, y, z \in X)((x * (y * z)) * ((x * y) * (x * z)) = 1) \tag{2}$$

$$(\forall x, y \in X)(x * y = 1, y * x = 1 \Rightarrow x = y) \tag{3}$$

Example 1. *[29] Let $X = \{1, a, b, c, d\}$ be a Hilbert algebra with the binary operation $*$ which is given in Table 1.*

Table 1: Cayley table for the binary operation $*$

$*$	a	b	c	d	1
a	1	1	1	1	1
b	a	1	c	1	1
c	a	b	1	1	1
d	a	b	c	1	1
1	a	b	c	d	1

In [10], the following conclusion was established.

Lemma 1. *Let $X = (X, *, 1)$ be a Hilbert algebra. Then*

(1) $(\forall x \in X)(x * x = 1)$

(2) $(\forall x \in X)(1 * x = x)$

(3) $(\forall x \in X)(x * 1 = 1)$

(4) $(\forall x, y, z \in X)(x * (y * z) = y * (x * z))$

(5) $(\forall x, y, z \in X)((x * z) * ((z * y) * (x * y)) = 1)$.

In a Hilbert algebra $X = (X, *, 1)$, the binary relation $\leq$ is defined by

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x * y = 1),$$

which is a partial order on X with 1 as the largest element.

Definition 2. *[30] A non-empty subset S of a Hilbert algebra $X = (X, *, 1)$ is called a subalgebra of X if $x * y \in S$ for all $x, y \in S$.*

Definition 3. *[31] A non-empty subset S of a Hilbert algebra $X = (X, *, 1)$ is called an ideal of X if the following conditions hold:*

(1) $1 \in S$

(2) $(\forall x, y \in X)(y \in S \Rightarrow x * y \in S)$

(3) $(\forall x, y_1, y_2 \in X)(y_1, y_2 \in S \Rightarrow (y_1 * (y_2 * x)) * x \in S)$.

A *fuzzy set* [1] in a nonempty set X is defined to be a function $\mu : X \to [0, 1]$, where $[0, 1]$ is the unit closed interval of real numbers.

Definition 4. *[32] A fuzzy set μ in a Hilbert algebra $X = (X, *, 1)$ is said to be a fuzzy subalgebra of X if the following condition holds:*

$$(\forall x, y \in X)(\mu(x * y) \geq \min\{\mu(x), \mu(y)\}).$$

Definition 5. *[11] A fuzzy set μ in a Hilbert algebra $X = (X, *, 1)$ is said to be a fuzzy ideal of X if the following conditions hold:*

(1) $(\forall x \in X)(\mu(1) \geq \mu(x))$

(2) $(\forall x, y \in X)(\mu(x * y) \geq \mu(y))$

(3) $(\forall x, y_1, y_2 \in X)(\mu((y_1 * (y_2 * x)) * x) \geq \min\{\mu(y_1), \mu(y_2)\})$.

3. Fuzzy translations and fuzzy multiplications of fuzzy subalgebras

In this section, we investigate the behavior of fuzzy subalgebras of a Hilbert algebra under two fundamental transformations: fuzzy α-translations and fuzzy γ-multiplications. The α-translation operation increases the membership degree of each element by a fixed constant α, while the γ-multiplication scales the membership degree of each element by a constant factor γ. Both transformations preserve the ordering of membership degrees and are monotonic in nature, which makes them natural candidates for structural preservation results. We establish necessary and sufficient conditions under which these operations preserve the fuzzy subalgebra property, and we examine how the corresponding α-cuts and γ-cuts of the transformed fuzzy sets relate to those of the original fuzzy subalgebra.

Several illustrative examples are provided to demonstrate the applicability of the obtained results.

In what follows, let $X = (X, *, 1)$ denote a Hilbert algebra.

Let μ be a fuzzy set in X. We define

$$\top = 1 - \sup\{\mu(x) : x \in X\}.$$

Intuitively, $\top$ represents the maximum amount by which the membership values of μ can be uniformly increased without exceeding 1. This quantity is crucial in defining fuzzy α-translations

$$\mu_\alpha^{\mathrm{T}}(x) = \mu(x) + \alpha,$$

since the condition $\alpha \in [0, \top]$ ensures that $\mu_\alpha^{\mathrm{T}}(x) \in [0, 1]$ for all $x \in X$.

Definition 6. *Let μ be a fuzzy set in X and let $\alpha \in [0, \top]$. A fuzzy α-translation of μ, denoted by μ_α^{T}, is defined to be a mapping $\mu_\alpha^{\mathrm{T}} : X \to [0, 1]$, $x \mapsto \mu(x) + \alpha$.*

Theorem 1. *Let μ be a fuzzy subalgebra of X and $\alpha \in [0, \top]$. Then the fuzzy α-translation μ_α^{T} of μ is a fuzzy subalgebra of X.*

Proof. Let $x, y \in X$. Then

$$\begin{array}{rcl} \mu_\alpha^{\mathrm{T}}(x * y) & = & \mu(x * y) + \alpha \\ & = & \min\{\mu(x), \mu(y)\} + \alpha \\ & = & \min\{\mu(x) + \alpha, \mu(y) + \alpha\} \\ & = & \min\{\mu_\alpha^{\mathrm{T}}(x), \mu_\alpha^{\mathrm{T}}(y)\}. \end{array}$$

Hence, μ_α^{T} is a fuzzy subalgebra of X.

Theorem 2. *Let μ be a fuzzy set in X such that the fuzzy α-translation μ_α^{T} of μ is a fuzzy subalgebra of X for some $\alpha \in [0, \top]$. Then μ is a fuzzy subalgebra of X.*

Proof. Assume that μ_α^{T} is a fuzzy subalgebra of X for some $\alpha \in [0, \top]$. Let $x, y \in X$. Then

$$\begin{array}{rcl} \mu(x * y) + \alpha & = & \mu_\alpha^{\mathrm{T}}(x * y) \\ & \geq & \min\{\mu_\alpha^{\mathrm{T}}(x), \mu_\alpha^{\mathrm{T}}(y)\} \\ & = & \min\{\mu(x) + \alpha, \mu(y) + \alpha\} \\ & = & \min\{\mu(x), \mu(y)\} + \alpha, \end{array}$$

which implies that $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$. Hence, μ is a fuzzy subalgebra of X.

Definition 7. *Let μ_1 and μ_2 be fuzzy sets in X. If $\mu_1(x) \leq \mu_2(x)$ for all $x \in X$, then we say that μ_2 is a fuzzy extension of μ_1.*

Definition 8. *Let μ_1 and μ_2 be fuzzy sets in X. Then μ_2 is called a fuzzy S-extension of μ_1 if the following assertions are valid:*

(1) μ_2 is a fuzzy extension of μ_1.

(2) If μ_1 is a fuzzy subalgebra of X, then μ_2 is a fuzzy subalgebra of X.

By means of the definition of fuzzy α-translation, we know that $\mu_\alpha^{\mathrm{T}}(x) \geq \mu(x)$ for all $x \in X$. Hence, we have the following theorem.

Theorem 3. *Let μ be a fuzzy subalgebra of X and $\alpha \in [0, \top]$. Then the fuzzy α-translation μ_α^{T} of μ is a fuzzy S-extension of μ.*

Proof. For every $x \in X$, by definition of fuzzy α-translation, we have

$$\mu_\alpha^{\mathrm{T}}(x) = \mu(x) + \alpha \geq \mu(x),$$

so μ_α^{T} is a fuzzy extension of μ. Since μ is a fuzzy subalgebra of X, Theorem 1 implies that μ_α^{T} is also a fuzzy subalgebra of X. By the definition of fuzzy S-extension, μ_α^{T} is therefore a fuzzy S-extension of μ.

The converse of Theorem 3 is not true in general, as seen in the following example.

Example 2. *Let $X = \{1, x, y, z, 0\}$ be a Hilbert algebra with the binary operation $*$ which is given in Table 2.*

Table 2: Cayley table for the binary operation $*$

$*$	1	x	y	z	0
1	1	x	y	z	0
x	1	1	y	z	0
y	1	x	1	z	z
z	1	1	y	1	y
0	1	1	1	1	1

Define a fuzzy set μ over X by $\mu = \begin{pmatrix} 1 & x & y & z & 0 \\ 0.7 & 0.5 & 0.5 & 0.7 & 0.4 \end{pmatrix}$. Then μ is a fuzzy subalgebra of X. Let ν be a fuzzy set in X given by $\nu = \begin{pmatrix} 1 & x & y & z & 0 \\ 0.74 & 0.55 & 0.54 & 0.76 & 0.44 \end{pmatrix}$. Then ν is a fuzzy S-extension of μ. But it is not the fuzzy α-translation μ_α^{T} of μ for all $\alpha \in [0, \top]$.

Remark 1. *Clearly, the intersection of fuzzy S-extensions of a fuzzy subalgebra μ is a fuzzy S-extension of μ. But the union of fuzzy S-extensions of a fuzzy subalgebra μ is not a fuzzy S-extension of μ as seen in the following example.*

Example 3. *Let $X = \{1, x, y, z, 0\}$ be a Hilbert algebra with the binary operation $*$ which is given in Table 2. Define a fuzzy set μ in X by $\mu = \begin{pmatrix} 1 & x & y & z & 0 \\ 0.7 & 0.5 & 0.5 & 0.3 & 0.2 \end{pmatrix}$. Then μ is a fuzzy subalgebra of X. Let ν and δ be fuzzy sets in X given by $\nu = \begin{pmatrix} 1 & x & y & z & 0 \\ 0.72 & 0.52 & 0.62 & 0.45 & 0.4 \end{pmatrix}$ and $\delta = \begin{pmatrix} 1 & x & y & z & 0 \\ 0.72 & 0.55 & 0.58 & 0.32 & 0.20 \end{pmatrix}$. Then ν and δ are fuzzy S-extensions of μ. But the union $\nu \cup \delta$ is not a fuzzy S-extension of μ since $(\nu \cup \delta)(y * 0) = 0.32 \ngeq 0.4 = \min\{(\nu \cup \delta)(y), (\nu \cup \delta)(0)\}$.*

For a fuzzy set μ of X, $\alpha \in [0, \top]$ and $t \in [0, 1]$ with $t \geq \alpha$, let $U_\alpha(\mu, t) = \{x \in X : \mu(x) \geq t - \alpha\}$.

If μ is a fuzzy subalgebra of X, then it is clear that $U_\alpha(\mu; t)$ is a subalgebra of X for all $t \in \mathrm{Im}(\mu)$ with $t \geq \alpha$. But, if we do not give a condition that μ is a fuzzy subalgebra of X, then $U_\alpha(\mu, t)$ is not a subalgebra of X as seen in the following example.

Example 4. *Let $X = \{1, x, y, z, 0\}$ be a Hilbert algebra with the binary operation $*$ which is given in Table 2. Define a fuzzy set μ in X by $\mu = \begin{pmatrix} 1 & x & y & z & 0 \\ 0.7 & 0.5 & 0.4 & 0.7 & 0.6 \end{pmatrix}$. Then μ is not a fuzzy subalgebra of X since $\mu(z * 0) = 0.5 \not\geq 0.6 = \min\{\mu(z), \mu(0)\}$. For $\alpha = 0.1$ and $t = 0.5$, we obtain $U_\alpha(\mu, t) = \{1, x, z, 0\}$, which is not a subalgebra of X since $1 * y = y \notin U_\alpha(\mu, t)$.*

Theorem 4. *Let μ be a fuzzy set in X and $\alpha \in [0, \top]$. Then the fuzzy α-translation μ_α^{T} of μ is a fuzzy subalgebra of X if and only if $U_\alpha(\mu, t)$ is a subalgebra of X for all $t \in \mathrm{Im}(\mu)$ with $t \geq \alpha$.*

Proof. Assume that μ_α^{T} is a fuzzy subalgebra of X. By definition, for all $x, y \in X$, we have

$$\mu_\alpha^{\mathrm{T}}(x * y) \geq \min\{\mu_\alpha^{\mathrm{T}}(x), \mu_\alpha^{\mathrm{T}}(y)\}. \tag{4}$$

Let $t \in [0, \top]$ and consider the α-cut of μ,

$$U_\alpha(\mu, t) = \{z \in X : \mu(z) \geq t\}.$$

We need to show that $U_\alpha(\mu, t)$ is a subalgebra of X. Suppose $x, y \in U_\alpha(\mu, t)$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$. Thus,

$$\mu_\alpha^{\mathrm{T}}(x) = \mu(x) + \alpha \geq t + \alpha \quad \text{and} \quad \mu_\alpha^{\mathrm{T}}(y) = \mu(y) + \alpha \geq t + \alpha.$$

Applying (4) to x and y gives

$$\mu_\alpha^{\mathrm{T}}(x * y) \geq \min\{\mu_\alpha^{\mathrm{T}}(x), \mu_\alpha^{\mathrm{T}}(y)\} \geq t + \alpha.$$

Again using the definition of μ_α^{T},

$$\mu(x * y) + \alpha \geq t + \alpha \quad \Rightarrow \quad \mu(x * y) \geq t.$$

Therefore, $x * y \in U_\alpha(\mu, t)$, which shows that $U_\alpha(\mu, t)$ is closed under $*$.

Since t was arbitrary, we conclude that every α-cut $U_\alpha(\mu, t)$ is a subalgebra of X.

Conversely, assume that there exist $a, b \in X$ such that $\mu_\alpha^{\mathrm{T}}(a*b) < \beta \leq \min\{\mu_\alpha^{\mathrm{T}}(a), \mu_\alpha^{\mathrm{T}}(b)\}$. Then $\mu(a) \geq \beta - \alpha$ and $\mu(b) \geq \beta - \alpha$, but $\mu(a*b) < \beta - \alpha$. This shows that $a, b \in U_\alpha(\mu, \beta)$ and $a * b \notin U_\alpha(\mu, \beta)$. This is a contradiction, and so $\mu_\alpha^{\mathrm{T}}(x * y) \geq \min\{\mu_\alpha^{\mathrm{T}}(x), \mu_\alpha^{\mathrm{T}}(y)\}$ for all $x, y \in X$. Hence, μ_α^{T} is a fuzzy subalgebra of X.

Theorem 5. *Let μ be a fuzzy subalgebra of X and let $\alpha, \beta \in [0, \top]$. If $\alpha \geq \beta$, then the fuzzy α-translation μ_α^{T} of μ is a fuzzy S-extension of the fuzzy β-translation μ_β^{T} of μ.*

Proof. Since $\alpha \geq \beta$, we have

$$\mu_\alpha^{\mathrm{T}}(x) = \mu(x) + \alpha \geq \mu(x) + \beta = \mu_\beta^{\mathrm{T}}(x) \quad \text{for all } x \in X,$$

which means μ_α^{T} is a fuzzy extension of μ_β^{T}. From Theorem 1, both μ_α^{T} and μ_β^{T} are fuzzy subalgebras of X, hence μ_α^{T} is a fuzzy S-extension of μ_β^{T}.

For every fuzzy subalgebra μ of X and $\beta \in [0, \top]$, the fuzzy β-translation $\mu^T\beta$ of μ is a fuzzy subalgebra of X. If ν is a fuzzy S-extension of $\mu^T\beta$, then there exists $\alpha \in [0, \top]$ such that $\alpha \geq \beta$ and $\nu(x) \geq \mu_\alpha^{\mathrm{T}}(x)$ for all $x \in X$. Thus, we have the following theorem.

Theorem 6. *Let μ be a fuzzy subalgebra of X and $\beta \in [0, \top]$. For every fuzzy S-extension ν of the fuzzy β-translation μ_β^{T} of μ, there exists $\alpha \in [0, \top]$ such that $\alpha \geq \beta$ and ν is a fuzzy S-extension of the fuzzy α-translation μ_α^{T} of μ.*

Proof. Since ν is a fuzzy S-extension of μ_β^{T}, we have $\nu(x) \geq \mu_\beta^{\mathrm{T}}(x) = \mu(x) + \beta$ for all $x \in X$. Let

$$\alpha = \inf_{x \in X} \{\nu(x) - \mu(x)\}.$$

Then $\alpha \in [0, \top]$ and $\alpha \geq \beta$. Moreover, $\nu(x) \geq \mu(x) + \alpha = \mu_\alpha^{\mathrm{T}}(x)$ for all $x \in X$, so ν is a fuzzy extension of μ_α^{T}. Since μ_α^{T} is a fuzzy subalgebra of X (Theorem 1) and ν is also a fuzzy subalgebra by assumption, it follows that ν is a fuzzy S-extension of μ_α^{T}.

Definition 9. *Let μ be a fuzzy set in X and let $\gamma \in [0, 1]$. A fuzzy γ-multiplication of μ, denoted by μ_γ^{M}, is defined to be a mapping $\mu_\gamma^{\mathrm{M}} : X \to [0, 1]$, $x \mapsto \mu(x) \cdot \gamma$.*

For any fuzzy set μ of X, a fuzzy 0-multiplication μ_0^{M} of μ is clearly a fuzzy subalgebra of X.

Remark 2. *We note some extreme cases of the parameters α and γ that yield simplified forms of the fuzzy α-translation μ_α^{T} and the fuzzy γ-multiplication μ_γ^{M}:*

- ***Case** $\alpha = 0$: The α-translation reduces to the original fuzzy set, i.e.,*

  $$\mu_0^{\mathrm{T}}(x) = \mu(x) \quad \textit{for all } x \in X.$$

  Therefore, μ_0^{T} is identical to μ and inherits all its algebraic properties.

- ***Case** $\alpha = \top$: The α-translation increases all membership values by the maximal possible amount without exceeding 1. In particular, at least one element of X will attain membership value 1.*

- ***Case** $\gamma = 1$: The γ-multiplication reduces to the original fuzzy set:*

  $$\mu_1^{\mathrm{M}}(x) = \mu(x) \quad \textit{for all } x \in X.$$

  Thus, μ_1^{M} coincides with μ and preserves its algebraic structure.

- ***Case*** $\gamma = 0$***:*** *The* γ*-multiplication collapses all membership values to* 0*, i.e.,*

$$\mu_0^{\mathrm{M}}(x) = 0 \quad \textit{for all } x \in X.$$

This yields the least fuzzy set, which is trivially both a fuzzy subalgebra and a fuzzy ideal of X*.*

Theorem 7. *Let* μ *be a fuzzy subalgebra of* X *and* $\gamma \in [0,1]$*. Then the fuzzy* γ*-multiplication of* μ *is a fuzzy subalgebra of* X*.*

Proof. Since μ is a fuzzy subalgebra of X, we have

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\}, \quad \forall x, y \in X. \tag{5}$$

Let $x, y \in X$. Then

$$\mu_\gamma^{\mathrm{M}}(x) = \mu(x) \cdot \gamma \text{ and } \mu_\gamma^{\mathrm{M}}(y) = \mu(y) \cdot \gamma.$$

From (5), we obtain

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\}.$$

Thus,

$$\mu(x * y) \cdot \gamma \geq \min\{\mu(x), \mu(y)\} \cdot \gamma.$$

It follows that

$$\mu(x * y) \cdot \gamma \geq \min\{\mu(x) \cdot \gamma, \mu(y) \cdot \gamma\}.$$

By the definition of μ_γ^{M}, this is equivalent to

$$\mu_\gamma^{\mathrm{M}}(x * y) \geq \min\{\mu_\gamma^{\mathrm{M}}(x), \mu_\gamma^{\mathrm{M}}(y)\}.$$

Therefore, μ_γ^{M} is a fuzzy subalgebra of X.

Theorem 8. *For any fuzzy set* μ *of* X*, the following are equivalent:*

(1) μ *is a fuzzy subalgebra of* X*.*

(2) $(\forall \gamma \in (0,1])(\mu_\gamma^{\mathrm{M}}$ *is a fuzzy subalgebra of* $X)$*.*

Proof. Necessity follows from Theorem 7. Let $\gamma \in (0,1]$ be such that μ_γ^{M} is a fuzzy subalgebra of X. Then $\mu(x*y)\cdot\gamma = \mu_\gamma^{\mathrm{M}}(x*y) \geq \min\{\mu_\gamma^{\mathrm{M}}(x), \mu_\gamma^{\mathrm{M}}(y)\} = \min\{\mu(x)\cdot\gamma, \mu(y)\cdot\gamma\} = \min\{\mu(x), \mu(y)\}\cdot\gamma$ for all $x, y \in X$, and so $\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$ since $\gamma \neq 0$. Hence, μ is a fuzzy subalgebra of X.

Theorem 9. *Let* μ *be a fuzzy set in* X*,* $\alpha \in [0, \top]$ *and* $\gamma \in (0,1]$*. Then every fuzzy* α*-translation* μ_α^{T} *of* μ *is a fuzzy* S*-extension of the fuzzy* γ*-multiplication* μ_γ^{M} *of* μ*.*

Proof. For every $x \in X$, we have $\mu_\alpha^{\mathrm{T}}(x) = \mu(x) + \alpha \geq \mu(x) \geq \mu(x) \geq \gamma = \mu_\gamma^{\mathrm{M}}(x)$, and so μ_α^{T} is a fuzzy extension of μ_γ^{M}. Assume that μ_γ^{M} is a fuzzy subalgebra of X. Then μ is a fuzzy subalgebra of X by Theorem 8. It follows from Theorem 1 that μ_α^{T} is a fuzzy subalgebra of X for all $\alpha \in [0, \top]$. Hence, every fuzzy α-translation μ_α^{T} is a fuzzy S-extension of the fuzzy γ-multiplication μ_γ^{M}.

Remark 3. *The concepts of fuzzy α-translation and fuzzy γ-multiplication are closely related in the context of Hilbert algebras. Indeed, both operations can be seen as transformations that preserve the underlying algebraic structure while adjusting the membership degrees. The α-translation uniformly increases the membership degree of each element by a constant shift α, whereas the γ-multiplication scales each membership degree by the factor γ. In the setting of Hilbert algebras, the preservation of subalgebra properties under these transformations reflects the stability of the algebraic structure with respect to monotonic modifications of the membership function. This connection suggests that certain results for α-translations can be adapted to γ-multiplications with appropriate modifications, and vice versa.*

4. Fuzzy translations and fuzzy multiplications of fuzzy ideals

In this section, we extend our investigation from fuzzy subalgebras to fuzzy ideals of a Hilbert algebra. Fuzzy ideals play a crucial role in the algebraic structure, serving as the fuzzy counterpart of crisp ideals and capturing important closure properties under the implication operation. We examine how fuzzy α-translations and fuzzy γ-multiplications affect the fuzzy ideal property, focusing on conditions that guarantee the preservation of idealness after transformation. Analogous to the subalgebra case, the α-translation uniformly shifts membership degrees upward, while the γ-multiplication proportionally rescales them. We provide necessary and sufficient conditions for these transformations to preserve fuzzy ideals, explore the relationships between the level cuts of the transformed fuzzy ideals and those of the original ones, and illustrate the results with concrete examples.

Theorem 10. *Let μ be a fuzzy ideal of X and $\alpha \in [0, \top]$. Then the fuzzy α-translation μ_α^{T} of μ is a fuzzy ideal of X.*

Proof. Let μ be a fuzzy ideal of X and $\alpha \in [0, \top]$. Then $\mu_\alpha^{\mathrm{T}}(1) = \mu(1)+\alpha \geq \mu(x)+\alpha = \mu_\alpha^{\mathrm{T}}(x)$. Let $x, y \in X$. Then $\mu_\alpha^{\mathrm{T}}(x*y) = \mu(x*y)+\alpha \geq \mu(y)+\alpha = \mu_\alpha^{\mathrm{T}}(y)$. Let $x, y_1, y_2 \in X$. Then

$$\begin{array}{rcl}\mu_\alpha^{\mathrm{T}}((y_1*(y_2*x))*x) & = & \mu((y_1*(y_2*x))*x)+\alpha \\ & \geq & \min\{\mu(y_1), \mu(y_2)\}+\alpha \\ & = & \min\{\mu(y_1)+\alpha, \mu(y_2)+\alpha\} \\ & = & \min\{\mu_\alpha^{\mathrm{T}}(y_1), \mu_\alpha^{\mathrm{T}}(y_2)\}.\end{array}$$

Hence, μ_α^{T} is a fuzzy ideal of X.

Theorem 11. *Let μ be a fuzzy set in X such that the fuzzy α-translation μ_α^{T} of μ is a fuzzy ideal of X for some $\alpha \in [0, \top]$. Then μ is a fuzzy ideal of X.*

Proof. Assume that μ_α^{T} is a fuzzy ideal of X for some $\alpha \in [0, \top]$. Let $x \in X$. Then $\mu(1) + \alpha = \mu_\alpha^{\mathrm{T}}(1) \geq \mu_\alpha^{\mathrm{T}}(x) = \mu(x) + \alpha$, and so $\mu(1) \geq \mu(x)$. Let $x, y \in X$. Then $\mu(x * y) + \alpha = \mu_\alpha^{\mathrm{T}}(x * y) \geq \mu_\alpha^{\mathrm{T}}(y) = \mu(y) + \alpha$, and so $\mu(x * y) \geq \mu(y)$. Let $x, y \in X$, we have

$$\begin{aligned} \mu((y_1 * (y_2 * x)) * x) + \alpha &= \mu_\alpha^{\mathrm{T}}((y_1 * (y_2 * x)) * x) \\ &\geq \min\{\mu_\alpha^{\mathrm{T}}(y_1), \mu_\alpha^{\mathrm{T}}(y_2)\} \\ &= \min\{\mu(y_1) + \alpha, \mu(y_2) + \alpha\} \\ &= \min\{\mu(y_1), \mu(y_2)\} + \alpha, \end{aligned}$$

which implies that $\mu((y_1 * (y_2 * x)) * x) \geq \min\{\mu(y_1), \mu(y_2)\}$ for all $x, y_1, y_2 \in X$. Hence, μ is a fuzzy ideal of X.

Theorem 12. *Let μ be a fuzzy ideal of X and $\alpha \in [0, \top]$. Then the fuzzy α-translation μ_α^{T} of μ is a fuzzy subalgebra of X.*

Proof. Let μ be any fuzzy ideal of X. Then

$$\begin{aligned} \mu_\alpha^{\mathrm{T}}(x * y) &= \mu(x * y) + \alpha \\ &= \mu(x * (y * 1)) + \alpha \\ &= \mu((x * (y * 1)) * 1) + \alpha \\ &\geq \min\{\mu(x), \mu(y)\} + \alpha \\ &= \min\{\mu_\alpha^{\mathrm{T}}(x), \mu_\alpha^{\mathrm{T}}(y)\}. \end{aligned}$$

Hence, μ is a fuzzy subalgebra of X.

The following example shows that the converse of Theorem 12 is not true.

Example 5. *Let $X = \{1, x, y, z, 0\}$ be a Hilbert algebra with the binary operation $*$ which is given in Table 2. Define a fuzzy set μ in X by $\mu = \begin{pmatrix} 1 & x & y & z & 0 \\ 0.7 & 0.5 & 0.4 & 0.3 & 0.3 \end{pmatrix}$. Then μ is a fuzzy subalgebra of X. But $\nu = \begin{pmatrix} 1 & x & y & z & 0 \\ 0.72 & 0.52 & 0.45 & 0.30 & 0.35 \end{pmatrix}$ is not a fuzzy subalgebra of X, since $\nu(y * 0) = 0.3 \ngeq 0.35 = \min\{\mu(y), \mu(0)\}$.*

Definition 10. *Let μ_1 and μ_2 be fuzzy sets in X. If $\mu_1(x) \leq \mu_2(x)$ for all $x \in X$, then we say that μ_2 is a fuzzy extension of μ_1.*

Definition 11. *Let μ_1 and μ_2 be fuzzy sets in X. Then μ_2 is called a fuzzy ideal extension of μ_1 if the following assertions are valid:*

(1) μ_2 is a fuzzy extension of μ_1.

(2) If μ_1 is a fuzzy ideal of X, then μ_2 is a fuzzy ideal of X.

Example 6. *Let $X = \{1, x, y, z, 0\}$ be a Hilbert algebra with the binary operation $*$ which is given in Table 2. Define a fuzzy set A in X by $\mu_1 = \begin{pmatrix} 1 & x & y & z & 0 \\ 0.7 & 0.5 & 0.4 & 0.3 & 0.2 \end{pmatrix}$. Then μ is a fuzzy subalgebra of X. But $\mu_2 = \begin{pmatrix} 1 & x & y & z & 0 \\ 0.75 & 0.55 & 0.45 & 0.35 & 0.25 \end{pmatrix}$. Then μ_2 is an ideal fuzzy extension of μ_1.*

By means of the definition of fuzzy α-translation, we know that $\mu_\alpha^{\mathrm{T}}(x) \geq \mu(x)$ for all $x \in X$. Hence, we have the following theorem.

Theorem 13. *Let μ be a fuzzy ideal of X and $\alpha \in [0, \top]$. Then the fuzzy α-translation μ_α^{T} of μ is a fuzzy ideal extension of μ.*

Proof. For every $x \in X$,

$$\mu_\alpha^{\mathrm{T}}(x) = \mu(x) + \alpha \geq \mu(x),$$

hence μ_α^{T} is a fuzzy extension of μ. Since μ is a fuzzy ideal of X, Theorem 10 ensures that μ_α^{T} is also a fuzzy ideal of X. Therefore, by the definition of fuzzy ideal extension, μ_α^{T} is a fuzzy ideal extension of μ.

Remark 4. *Clearly, the intersection of fuzzy ideal extensions of a fuzzy ideal μ is a fuzzy ideal extension of μ. But the union of fuzzy ideal extensions of a fuzzy ideal μ is not a fuzzy ideal extension of μ as seen in the following example.*

Example 7. *Let $X = \{1, x, y, z, 0\}$ be a Hilbert algebra with the binary operation $*$ which is given in Table 2. Define the fuzzy sets over X by $\mu = \begin{pmatrix} 1 & x & y & z & 0 \\ 0.7 & 0.5 & 0.4 & 0.3 & 0.3 \end{pmatrix}$, $\nu = \begin{pmatrix} 1 & x & y & z & 0 \\ 0.72 & 0.53 & 0.43 & 0.32 & 0.33 \end{pmatrix}$, and $\delta = \begin{pmatrix} 1 & x & y & z & 0 \\ 0.75 & 0.55 & 0.43 & 0.32 & 0.39 \end{pmatrix}$. Clearly, μ is a fuzzy ideal of X and ν and δ are fuzzy ideal extension of X. But their union is not a fuzzy ideal extension of X, since $(\nu \cup \delta)(y * 0) = 0.32 \ngeq 0.37 = \min\{(\nu \cup \delta)(y), (\nu \cup \delta)(0)\}$.*

For a fuzzy set μ of X, $\alpha \in [0, \top]$ and $t \in [0, 1]$ with $t \geq \alpha$, let $U_\alpha(\mu, t) = \{x \in X : \mu(x) \geq t - \alpha\}$.

If μ is a fuzzy ideal of X, then it is clear that $U_\alpha(\mu; t)$ is an ideal of X for all $t \in \mathrm{Im}(\mu)$ with $t > \alpha$. But if we do not give a condition that μ is a fuzzy ideal of X, then $U_\alpha(\mu, t)$ is not an ideal of X as seen in the following example.

Example 8. *Let $X = \{1, x, y, z, 0\}$ be a Hilbert algebra with the binary operation $*$ which is given in Table 2. Define the fuzzy sets over X by $\mu = \begin{pmatrix} 1 & x & y & z & 0 \\ 0.7 & 0.5 & 0.4 & 0.3 & 0.2 \end{pmatrix}$. But $U_\alpha(\mu, t) = \{1, x, y\}$, which is not an ideal of X.*

Theorem 14. *Let μ be a fuzzy set in X and $\alpha \in [0, \top]$. Then the fuzzy α-translation μ_α^{T} of μ is a fuzzy ideal of X if and only if $U_\alpha(\mu, t)$ is an ideal of X for all $t \in \mathrm{Im}(\mu)$ with $t > \alpha$.*

Proof. Assume that the α-translation μ_α^{T} of μ is a fuzzy ideal of X and let $t \in \mathrm{Im}(\mu)$ be such that $t > \alpha$, where $\alpha \in [0, \top]$. Let $x \in U_\alpha(\mu, t)$. Then $\mu(1) + \alpha = \mu_\alpha^{\mathrm{T}}(1) \geq \mu_\alpha^{\mathrm{T}}(x) = \mu(x) + \alpha \geq t$. Hence, $1 \in U_\alpha(\mu, t)$. Let $x, y \in X$ be such that $y \in U_\alpha(\mu, t)$. Then $\mu(x * y) + \alpha = \mu_\alpha^{\mathrm{T}}(x * y) \geq \mu_\alpha^{\mathrm{T}}(y) = \mu(y) + \alpha \geq t$. Hence, $x * y \in U_\alpha(\mu, t)$. Let $x, y_1, y_2 \in X$

be such that $y_1, y_2 \in U_\alpha(\mu, t)$. Then $\mu(y_1) \geq t-\alpha$ and $\mu(y_2) \geq t-\alpha$, that is, $\mu(y_1)+\alpha \geq t$ and $\mu(y_2)+\alpha \geq t$. Since μ_α^{T} of μ is a fuzzy ideal of X, we have

$$\begin{aligned}\mu((y_1 * (y_2 * x)) * x) + \alpha &= \mu_\alpha^{\mathrm{T}}((y_1 * (y_2 * x)) * x)\\ &\geq \min\{\mu_\alpha^{\mathrm{T}}(y_1), \mu_\alpha^{\mathrm{T}}(y_2)\}\\ &\geq t,\end{aligned}$$

that is, $\mu((y_1 * (y_2 * x)) * x) \geq t-\alpha$ so that $(y_1 * (y_2 * x)) * x \in U_\alpha(\mu, t)$. Hence, $U_\alpha(\mu, t)$ is an ideal of X.

Conversely, assume that there exist $a, b \in X$ such that $\mu_\alpha^{\mathrm{T}}(a*b) < \beta \leq \min\{\mu_\alpha^{\mathrm{T}}(a), \mu_\alpha^{\mathrm{T}}(b)\}$. Then $\mu(a) \geq \beta - \alpha$ and $\mu(b) \geq \beta-\alpha$, but $\mu(a*b) < \beta-\alpha$. This shows that $a, b \in U_\alpha(\mu, \beta)$ and $a * b \notin U_\alpha(\mu, \beta)$. This is a contradiction, and so $\mu_\alpha^{\mathrm{T}}((y_1 * (y_2 * x)) * x) \geq \min\{\mu_\alpha^{\mathrm{T}}(y_1), \mu_\alpha^{\mathrm{T}}(y_2)\}$ for all $x, y_1, y_2 \in X$. Hence, μ_α^{T} is a fuzzy ideal of X.

Remark 5. *In Theorem 14, if $t \leq \alpha$, then $U_\alpha(\mu, t) = X$.*

Theorem 15. *Let μ be a fuzzy ideal of X and let $\alpha, \beta \in [0, \top]$. If $\alpha \geq \beta$, then the fuzzy α-translation μ_α^{T} of μ is a fuzzy ideal extension of the fuzzy β-translation μ_β^{T} of μ.*

Proof. If $\alpha \geq \beta$, then for all $x \in X$,

$$\mu_\alpha^{\mathrm{T}}(x) = \mu(x) + \alpha \geq \mu(x) + \beta = \mu_\beta^{\mathrm{T}}(x),$$

so μ_α^{T} is a fuzzy extension of μ_β^{T}. By Theorem 10, both μ_α^{T} and μ_β^{T} are fuzzy ideals of X, hence μ_α^{T} is a fuzzy ideal extension of μ_β^{T}.

For every fuzzy ideal μ of X and $\beta \in [0, \top]$, the fuzzy β-translation μ_β^{T} of μ is a fuzzy ideal of X. If ν is a fuzzy ideal extension of μ_β^{T}, then there exists $\alpha \in [0, \top]$ such that $\alpha \geq \beta$ and $\nu(x) \geq \mu_\alpha^{\mathrm{T}}(x)$ for all $x \in X$. Thus, we have the following theorem.

Theorem 16. *Let μ be a fuzzy ideal of X and $\beta \in [0, \top]$. For every fuzzy ideal extension ν of the fuzzy β-translation μ_β^{T} of μ, there exists $\alpha \in [0, \top]$ such that $\alpha \geq \beta$ and ν is a fuzzy ideal extension of the fuzzy α-translation μ_α^{T} of μ.*

Proof. Since ν is a fuzzy ideal extension of μ_β^{T}, we have $\nu(x) \geq \mu_\beta^{\mathrm{T}}(x) = \mu(x) + \beta$ for all $x \in X$. Let

$$\alpha = \inf_{x \in X}\{\nu(x) - \mu(x)\}.$$

Then $\alpha \in [0, \top]$ and $\alpha \geq \beta$. Furthermore, $\nu(x) \geq \mu_\alpha^{\mathrm{T}}(x)$ for all $x \in X$, so ν is a fuzzy extension of μ_α^{T}. By Theorem 10, μ_α^{T} is a fuzzy ideal of X, and since ν is also a fuzzy ideal by assumption, it follows that ν is a fuzzy ideal extension of μ_α^{T}.

Theorem 17. *Let μ be a fuzzy ideal of X and $\gamma \in [0, 1]$. Then the fuzzy γ-multiplication of μ is a fuzzy ideal of X.*

Proof. Let $x \in X$. Since μ is a fuzzy ideal of X, $\mu(1) \geq \mu(x)$. Thus, $\mu(1) \cdot \gamma \geq \mu(x) \cdot \gamma$. That is, $\mu^{\mathrm{M}}_{\gamma}(1) \geq \mu^{\mathrm{M}}_{\gamma}(x)$. Let $x, y \in X$. Since μ is a fuzzy ideal of X, $\mu(x * y) \geq \mu(y)$. Thus, $\mu(x * y) \cdot \gamma \geq \mu(y) \cdot \gamma$, that is, $\mu^{\mathrm{M}}_{\gamma}(x * y) \geq \mu^{\mathrm{M}}_{\gamma}(y)$. Let $x, y_1, y_2 \in X$. Since μ is a fuzzy ideal of X, we have $\mu((y_1 * (y_2 * x)) * x) \geq \min\{\mu(y_1), \mu(y_2)\}$. Thus,

$$\mu((y_1 * (y_2 * x)) * x) \cdot \gamma \geq \min\{\mu(y_1), \mu(y_2)\} \cdot \gamma.$$

Since $\gamma \geq 0$, we have $\min\{\mu(y_1), \mu(y_2)\} \cdot \gamma = \min\{\mu(y_1) \cdot \gamma, \mu(y_2) \cdot \gamma\}$. Therefore,

$$\mu^{\mathrm{M}}_{\gamma}((y_1 * (y_2 * x)) * x) \geq \min\{\mu^{\mathrm{M}}_{\gamma}(y_1), \mu^{\mathrm{M}}_{\gamma}(y_2)\}.$$

Hence, $\mu^{\mathrm{M}}_{\gamma}$ is a fuzzy ideal of X.

Theorem 18. *For any fuzzy set μ of X, the following are equivalent:*

(1) μ is a fuzzy ideal of X.

(2) $(\forall \gamma \in (0, 1])(\mu^{\mathrm{M}}_{\gamma}$ is a fuzzy ideal of $X)$.

Proof. Necessity follows from Theorem 17. Let $\gamma \in (0, 1]$ be such that $\mu^{\mathrm{M}}_{\gamma}$ is a fuzzy ideal of X. Then for every $x \in X$, we have $\mu(1) \cdot \gamma = \mu^{\mathrm{M}}_{\gamma}(1) \geq \mu^{\mathrm{M}}_{\gamma}(x) = \mu(x) \cdot \gamma$, and so $\mu(1) = \mu(x)$ since $\gamma \neq 0$. Let $x, y \in X$. Then $\mu(x * y) \cdot \gamma = \mu^{\mathrm{M}}_{\gamma}(x * y) \geq \mu^{\mathrm{M}}_{\gamma}(y) = \mu(y) \cdot \gamma$, and so $\mu(x * y) \geq \mu(y)$ since $\gamma \neq 0$. Let $x, y_1, y_2 \in X$. Then $\mu((y_1 * (y_2 * x)) * x) \cdot \gamma = \mu^{\mathrm{M}}_{\gamma}((y_1 * (y_2 * x)) * x) \geq \min\{\mu^{\mathrm{M}}_{\gamma}(x), \mu^{\mathrm{M}}_{\gamma}(y)\} = \min\{\mu(y_1) \cdot \gamma, \mu(y_2) \cdot \gamma\} = \min\{\mu(y_1), \mu(y_2)\} \cdot \gamma$, and so $\mu((y_1 * (y_2 * x)) * x) \geq \min\{\mu(y_1), \mu(y_2)\}$ for all $x, y_1, y_2 \in X$ since $\gamma \neq 0$. Hence, μ is a fuzzy ideal of X.

Theorem 19. *Let μ be a fuzzy set in X, $\alpha \in [0, \top]$ and $\gamma \in (0, 1]$. Then every fuzzy α-translation $\mu^{\mathrm{T}}_{\alpha}$ of μ is a fuzzy ideal extension of the fuzzy γ-multiplication $\mu^{\mathrm{M}}_{\gamma}$ of μ.*

Proof. For every $x \in X$, we have $\mu^{\mathrm{T}}_{\alpha}(x) = \mu(x) + \alpha \geq \mu(x) \geq \mu(x) \cdot \gamma = \mu^{\mathrm{M}}_{\gamma}(x)$, and so $\mu^{\mathrm{T}}_{\alpha}$ is a fuzzy extension of $\mu^{\mathrm{M}}_{\gamma}$. Assume that $\mu^{\mathrm{M}}_{\gamma}$ is a fuzzy ideal of X. Then μ is a fuzzy ideal of X by Theorem 8. It follows from Theorem 1 that $\mu^{\mathrm{T}}_{\alpha}$ is a fuzzy ideal of X for all $\alpha \in [0, \top]$. Hence, every fuzzy α-translation $\mu^{\mathrm{T}}_{\alpha}$ is a fuzzy ideal extension of the fuzzy γ-multiplication $\mu^{\mathrm{M}}_{\gamma}$.

Remark 6. *The interplay between fuzzy α-translations and fuzzy γ-multiplications is also evident for fuzzy ideals in Hilbert algebras. While α-translations shift all membership degrees upward, potentially turning non-members into members of higher-level cuts, γ-multiplications proportionally rescale the degrees without altering the relative ordering of elements. In both cases, the preservation of the fuzzy ideal property depends on the monotonicity of the transformation and the closure conditions of the underlying crisp ideal. Thus, results established for fuzzy α-translations can often be reformulated for fuzzy γ-multiplications, which points to a deeper structural harmony between these two operations in the theory of fuzzy ideals.*

Table 3: Summary of "if and only if" characterizations in the paper

Theorem(s)	**Statement**	**Equivalence Condition**
1 + 2	μ_α^{T} is a fuzzy subalgebra of X for some $\alpha \in [0, \top]$	μ is a fuzzy subalgebra of X
4	μ_α^{T} is a fuzzy subalgebra of X	$U_\alpha(\mu, t)$ is a subalgebra of X for all $t \in \mathrm{Im}(\mu)$ with $t \geq \alpha$
8	μ is a fuzzy subalgebra of X	μ_γ^{M} is a fuzzy subalgebra of X for all $\gamma \in (0, 1]$
10 + 11	μ_α^{T} is a fuzzy ideal of X for some $\alpha \in [0, \top]$	μ is a fuzzy ideal of X
14	μ_α^{T} is a fuzzy ideal of X	$U_\alpha(\mu, t)$ is an ideal of X for all $t \in \mathrm{Im}(\mu)$ with $t > \alpha$
18	μ is a fuzzy ideal of X	μ_γ^{M} is a fuzzy ideal of X for all $\gamma \in (0, 1]$

5. Conclusion

In this work, we introduce and study the concepts of fuzzy translations and fuzzy multiplications for fuzzy subalgebras and fuzzy ideals in Hilbert algebras. We have established a series of theorems that characterize when these operations preserve the subalgebraic or ideal structure, and we have identified precise relationships between fuzzy translations, fuzzy S-extensions, and fuzzy multiplications. In particular, we proved equivalence conditions for a fuzzy set to be a subalgebra (or an ideal) in terms of its γ-multiplications, and we showed that every fuzzy α-translation is a fuzzy S-extension of a corresponding fuzzy γ-multiplication. Several examples and counterexamples were provided to illustrate the necessity of the stated conditions and to demonstrate that certain converses do not hold in general.

These results extend previous studies on fuzzification in Hilbert algebras by providing a unified framework that links translation-type and multiplication-type operations. The structural properties obtained here may serve as a basis for further investigations into other fuzzy algebraic systems, including generalized Hilbert algebras, BCK/BCI-algebras, and their applications in fuzzy logic and approximate reasoning. Future research may also investigate the computational aspects of these operations, as well as their role in modeling uncertainty within logical systems.

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