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A Multitude of Centralizer Types on Semiprime Rings

Abu Zaid Ansari[1,*], Areej Almuhaimeed[2], Faiza Shujat[2]

[1] *Department of Mathematics, Faculty of Science, Islamic University of Madinah, Saudi Arabia*
[2] *Department of Mathematics, College of Science, Taibah University, Madinah, Saudi Arabia*

Abstract. The scientific study aims to demonstrate that an additive mapping within a semiprime ring can act as a φ-centralizer ($*$-centralizer) under a specific torsion restriction and certain algebraic identities. This compelling conclusion is supported by genuine examples that effectively substantiate and illuminate the theoretical findings with indisputable clarity and significance.

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1. Introduction

A primary objective of this work is to study associative rings S with identity e, with particular focus on those that become torsion-free under precisely specified conditions. A clear distinction is drawn between semiprime rings, characterized by a well-defined algebraic criterion, and related classes of rings possessing their distinguishing properties. The close correspondence established between these semiprime rings and the torsion-free condition is shown to yield an equivalence of structural significance; see [1] for further details. Johnson investigated the continuity of centralizers and clarified their role in the context of topological algebras, while also extending the analysis to certain classes of rings, as discussed in [2]. In addition, a variety of mappings is introduced and rigorously defined under different hypotheses, thereby providing a more comprehensive understanding of the underlying algebraic framework.

A left centralizer is an additive mapping from a ring S to itself that satisfies $\mathcal{G}(rs) = \mathcal{G}(r)s$ for all $s, r \in S$, while a right centralizer must comply with $\mathcal{G}(rs) = r\mathcal{G}(s)$ for all $s, r \in S$. A mapping that satisfies both conditions is simply called a centralizer. Similarly,

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Email addresses: ansari.abuzaid@gmail.com (A.Z. Ansari)
aamuhaimeed@taibahu.edu.sa (A. Almuhaimeed), faiza.shujat@gmail.com (F. Shujat)

additive mappings are called Jordan left (right) centralizers if they meet $\mathcal{G}(r^2) = \mathcal{G}(r)r$ ($\mathcal{G}(r^2) = r\mathcal{G}(r)$), differentiating between Jordan right and left centralizers, with a mapping being a Jordan centralizer if it is both. For details, see [3].

An additive mapping $\mathcal{G} : S \to S$ is termed a left (right) φ-centralizer if $\mathcal{G}(rs) = \mathcal{G}(r)\varphi(s)$ (resp. $\mathcal{G}(rs) = \varphi(r)\mathcal{G}(s)$) for all $s, r \in S$ and in particular, Jordan left (right) φ-centralizer if $\mathcal{G}(r^2) = \mathcal{G}(r)\varphi(r)$ (resp. $\mathcal{G}(r^2) = \varphi(r)\mathcal{G}(r)$) for all $r \in S$. If $\mathcal{G} : S \to S$ is (Jordan) left and (Jordan) right φ-centralizer, then it is known as (Jordan) φ-centralizer. Albas [4] established that every Jordan φ-centralizer implies a φ-centralizer. However, the reverse implication usually does not hold, except under certain torsion restrictions for semiprime rings. Furthermore, in [5], the authors proved that φ is a surjective endomorphism on a semiprime ring S, subject to 2-torsion-free conditions, and that $\mathcal{G} : S \to S$ is an additive mapping satisfying condition

$$2\mathcal{G}(r^2) = \mathcal{G}(r)\varphi(r) + \varphi(r)\mathcal{G}(r)$$

for every $r \in S$, it follows that $\mathcal{G}$ acts as a φ-centralizer on S.

Drawing from the preceding line of research, Ansari et al. in [6] significantly extended the scope of the earlier results. They precisely established that an additive mapping $\mathcal{G} : S \to S$ is a φ-centralizer on S, provided that it satisfies

$$2\mathcal{G}(r^{2q}) = \mathcal{G}(r^q)\varphi(r^q) + \varphi(r^q)\mathcal{G}(r^q),$$

for every $r \in S$. Motivated by the above line of investigations, this article extends the previous results to a more general setting under the specified condition

$$4\mathcal{G}(r^{q_1+q_2}) = \mathcal{G}(r^{q_2})\varphi(r^{q_1}) + \varphi(r^{q_2})\mathcal{G}(r^{q_1}) + \mathcal{G}(r^{q_1})\varphi(r^{q_2}) + \varphi(r^{q_1})\mathcal{G}(r^{q_2})$$

for all $r \in S$.

Lemma 1 ([5, Theorem 1.2])**.** *Given that φ is a surjective endomorphism on a semiprime ring S, subject to 2-torsion-free condition, and that $\mathcal{G} : S \to S$ is an additive mapping satisfying condition $2\mathcal{G}(r^2) = \mathcal{G}(r)\varphi(r) + \varphi(r)\mathcal{G}(r)$ for every $r \in S$, it follows that $\mathcal{G}$ functions as a φ-centralizer on S.*

Lemma 2. *Let S be a ring, subject to the $(q_1 + q_2)!$-torsion-free condition. Then S is $(q_1 + q_2)$-torsion-free as well as q_1q_2-torsion-free.*

Proof. Since S is $(q_1 + q_2)!$-torsion-free ring, then

$$[(q_1 + q_2)!]x = 0 \text{ implies } x = 0$$

This implies that

$$[(q_1 + q_2)(q_1 + q_2 - 1)(q_1 + q_2 - 1) \cdots (q_1 + 1)q_1(q_1 - 1)(q_1 - 2) \cdots 3 \cdot 2 \cdot 1]x = 0 \text{ implies } x = 0$$

Now, consider

$$(q_1 + q_2)x = 0$$

multiplying both side by $(q_1 + q_2 - 1)(q_1 + q_2 - 1) \cdots (q_1 + 1)q_1(q_1 - 1)(q_1 - 2) \cdots 3 \cdot 2 \cdot 1$, we obtain the following

$$[(q_1 + q_2)(q_1 + q_2 - 1)(q_1 + q_2 - 1) \cdots (q_1 + 1)q_1(q_1 - 1)(q_1 - 2) \cdots 3 \cdot 2 \cdot 1]x = 0$$

This implies $x = 0$. Hence, S is $(q_1 + q_2)$-torsion free ring.

Since, $(q_1 + q_2)!$ is multiple of q_1 and q_2 both, then we can write $(q_1 + q_2)! = q_1 q_2 k$, where k is any integer. Since S is $(q_1 + q_2)!$-torsion-free ring, then

$$[(q_1 + q_2)!]x = 0 \text{ implies } x = 0,$$

which implies that

$$q_1 q_2 k x = 0 \text{ implies } x = 0.$$

Now, consider the following.

$$q_1 q_2 x = 0.$$

Multiplying both side by k, we obtain

$$q_1 q_2 k x = 0.$$

Which implies that $x = 0$. Hence, S is $q_1 q_2$-torsion-free ring.

Note: If S is any n-torsion free ring, where n is any composite integer and p is any positive integer such that p divides n, then S is p-torsion free ring.

2. Results on φ-centralizer

We start the analysis by considering the following problem:

Theorem 1. *Consider two fixed integers denoted as $q_2 > 1$ and $q_1 \geq 1$, and let S be a semiprime ring that is $(q_1 + q_2)!$-torsion free. If there exists an additive mapping $\mathcal{G}$ that operates on S to itself, strictly conforming to a designated algebraic identity $4\mathcal{G}(r^{q_1+q_2}) = \mathcal{G}(r^{q_2})\varphi(r^{q_1}) + \varphi(r^{q_2})\mathcal{G}(r^{q_1}) + \mathcal{G}(r^{q_1})\varphi(r^{q_2}) + \varphi(r^{q_1})\mathcal{G}(r^{q_2})$ for every element r in S, then $\mathcal{G}$ will function as a φ-centralizer in S, where φ is a surjective endomorphism on S.*

Proof. We are provided with

$$\begin{aligned} 4\mathcal{G}(r^{q_1+q_2}) &= \mathcal{G}(r^{q_2})\varphi(r^{q_1}) + \varphi(r^{q_2})\mathcal{G}(r^{q_1}) \\ &\quad + \mathcal{G}(r^{q_1})\varphi(r^{q_2}) + \varphi(r^{q_1})\mathcal{G}(r^{q_2}), \text{ for all } r \in S. \end{aligned} \tag{1}$$

Substituting $r+ns$ for r into equation (1), where n is a positive integer. We obtain

$4\mathcal{G}\Big(r^{q_1+q_2} + n\binom{q_1+q_2}{1}r^{q_1+q_2-1}s + n^2\binom{q_1+q_2}{2}r^{q_1+q_2-2}s^2 + \cdots + n^{q_1+q_2}s^{q_1+q_2}\Big) = \mathcal{G}\Big(r^{q_2} + n\binom{q_2}{1}r^{q_2-1}s+n^2\binom{q_2}{2}r^{q_2-2}s^2+\cdots+n^{q_2}s^{q_2}\Big)\Big(\varphi(r^{q_1})+n\binom{q_1}{1}\varphi(r^{q_1-1})\varphi(s)+n^2\binom{q_1}{2}\varphi(r^{q_1-2})\varphi(s^2)+\cdots+n^{q_1}\varphi(s^{q_1})\Big)+\Big(\varphi(r^{q_2})+n\binom{q_2}{1}\varphi(r^{q_2-1})\varphi(s)+n^2\binom{q_2}{2}\varphi(r^{q_2-2})\varphi(s^2)+\cdots+n^{q_2}\varphi(s^{q_2})\Big)\mathcal{G}\Big(r^{q_1}+n\binom{q_1}{1}r^{q_1-1}s + n^2\binom{q_1}{2}r^{q_1-2}s^2 + \cdots + n^{q_1}s^{q_1}\Big) + \mathcal{G}\Big(r^{q_1} + n\binom{q_1}{1}r^{q_1-1}s + n^2\binom{q_1}{2}r^{q_1-2}s^2 + \cdots + n^{q_1}s^{q_1}\Big)\Big(\varphi(r^{q_2}) + n\binom{q_2}{1}\varphi(r^{q_2-1})\varphi(s) + n^2\binom{q_2}{2}\varphi(r^{q_1-2})\varphi(s^2) + \cdots + n^{q_2}\varphi(s^{q_2})\Big) + \Big(\varphi(r^{q_1}) + n\binom{q_1}{1}\varphi(r^{q_1-1})\varphi(s)+n^2\binom{q_1}{2}\varphi(r^{q_1-2})\varphi(s^2)+\cdots+n^{q_1}\varphi(s^{q_1})\Big)\mathcal{G}\Big(r^{q_2}+n\binom{q_2}{1}r^{q_2-1}s+n^2\binom{q_2}{2}r^{q_2-2}s^2+\cdots + n^{q_2}s^{q_2}\Big).$

Gathering the above expression in powers of n and applying (1), we obtain

$$
\begin{aligned}
&n\Big\{4\tbinom{q_1+q_2}{1}\mathcal{G}(r^{q_1+q_2-1}s) - \tbinom{q_1}{1}\mathcal{G}(r^{q_2})\varphi(r^{q_1-1})\varphi(s) - \tbinom{q_2}{1}\mathcal{G}(r^{q_2-1}s)\varphi(r^{q_1})\\
&- \tbinom{q_1}{1}\varphi(r^{q_2})\mathcal{G}(r^{q_1-1})\varphi(s) - \tbinom{q_2}{1}\varphi(r^{q_2-1})\varphi(s)\mathcal{G}(r^{q_1}) - \tbinom{q_2}{1}\mathcal{G}(r^{q_1})\varphi(r^{q_2-1})\varphi(s)\\
&- \tbinom{q_1}{1}\mathcal{G}(r^{q_1-1}s)\varphi(r^{q_2}) - \tbinom{q_2}{1}\varphi(r^{q_1})\mathcal{G}(r^{q_2-1}s) - \tbinom{q_1}{1}\varphi(r^{q_1-1})\varphi(s)\mathcal{G}(r^{q_2})\Big\}\\
&+ n^2\Big\{4\tbinom{q_1+q_2}{2}\mathcal{G}(r^{q_1+q_2-2}s^2) - \tbinom{q_1}{2}\mathcal{G}(r^{q_2})\varphi(r^{q_1-2})\varphi(s^2) - \tbinom{q_2}{1}\tbinom{q_1}{1}\mathcal{G}(r^{q_2-1}s)\varphi(r^{q_1-1})\varphi(s)\\
&- \tbinom{q_2}{2}\mathcal{G}(r^{q_2-2}s^2)\varphi(r^{q_1}) - \tbinom{q_1}{2}\varphi(r^{q_2})\mathcal{G}(r^{q_1-2}s^2) - \tbinom{q_2}{1}\tbinom{q_1}{1}\varphi(r^{q_2-1})\varphi(s)\mathcal{G}(r^{q_1-1}s)\\
&- \tbinom{q_2}{2}\varphi(r^{q_2-2})\varphi(s^2)\mathcal{G}(r^{q_1}) - \tbinom{q_2}{2}\mathcal{G}(r^{q_1})\varphi(r^{q_2-2})\varphi(s^2) - \tbinom{q_1}{1}\tbinom{q_2}{1}\mathcal{G}(r^{q_1-1}s)\varphi(r^{q_2-1})\varphi(s)\\
&- \tbinom{q_1}{2}\mathcal{G}(r^{q_1-2}s^2)\varphi(r^{q_2}) - \tbinom{q_2}{2}\varphi(r^{q_1})\mathcal{G}(r^{q_2-2}s^2) - \tbinom{q_1}{1}\tbinom{q_2}{1}\varphi(r^{q_1-1})\varphi(s)\mathcal{G}(r^{q_2-1}s)\\
&- \tbinom{q_1}{2}\varphi(r^{q_1-2})\varphi(s^2)\mathcal{G}(r^{q_2})\Big\}\\
&+\\
&\vdots\\
&+ n^{q_1+q_2-1}\Big\{4\tbinom{q_1+q_2}{q_1+q_2-1}\mathcal{G}(rs^{q_1+q_2-1}) - \tbinom{q_2}{q_2-1}\mathcal{G}(rs^{q_2-1})\varphi(s^{q_1}) - \tbinom{q_1}{q_1-1}\mathcal{G}(s^{q_2})\varphi(rs^{q_1-1})\\
&-\tbinom{q_2}{q_2-1}\varphi(rs^{q_2-1})\mathcal{G}(s^{q_1})-\tbinom{q_1}{q_1-1}\varphi(s^{q_2})\mathcal{G}(rs^{q_1-1})-\tbinom{q_1}{q_1-1}\mathcal{G}(rs^{q_1-1})\varphi(s^{q_2})-\tbinom{q_2}{q_2-1}\mathcal{G}(s^{q_1})\varphi(rs^{q_2-1})-\\
&\tbinom{q_1}{q_1-1}\varphi(rs^{q_1-1})\mathcal{G}(s^{q_2}) - \tbinom{q_2}{q_2-1}\varphi(s^{q_1})\mathcal{G}(rs^{q_1-1})\Big\}.
\end{aligned}
$$

Rewrite the expression in a simpler form as

$$n\mathcal{H}_1(r,s) + n^2\mathcal{H}_2(r,s) + \cdots + n^{q_1+q_2-1}\mathcal{H}_{q_1+q_2-1}(r,s) = 0,$$

where $\mathcal{H}_i(r,s)$ are the coefficients of n^i for each $i = 1,2,\cdots,(q_1+q_2-1)$. The substitution of $1,2,\cdots,(q_1+q_2-1)$ in place of n leads to the formulation of a system comprising (q_1+q_2-1) homogeneous equations, which therefore results in the formation of a Vandermonde matrix.

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^{q_1+q_2-1} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ (q_1+q_2-2) & (q_1+q_2-2)^2 & \cdots & (q_1+q_2-2)^{q_1+q_2-1} \\ (q_1+q_2-1) & (q_1+q_2-1)^2 & \cdots & (q_1+q_2-1)^{q_1+q_2-1} \end{bmatrix}.$$

Since the determinant of the matrix equals a product of positive integers, each strictly less than n, it follows that $\mathcal{H}_i(r,s)=0$, it holds for every r,s within the context of S as well as for $i=1,2,..,(q_1+q_2-1)$. Specifically, $\mathcal{H}_1(r,s)=0$ results in

$$\begin{aligned} 0 \;=\; & 4\tbinom{q_1+q_2}{1}\mathcal{G}(r^{q_1+q_2-1}s)-\tbinom{q_1}{1}\mathcal{G}(r^{q_2})\varphi(r^{q_1-1})\varphi(s)-\tbinom{q_2}{1}\mathcal{G}(r^{q_2-1}s)\varphi(r^{q_1}) \\ & -\tbinom{q_1}{1}\varphi(r^{q_2})\mathcal{G}(r^{q_1-1})\varphi(s)-\tbinom{q_2}{1}\varphi(r^{q_2-1})\varphi(s)\mathcal{G}(r^{q_1})-\tbinom{q_2}{1}\mathcal{G}(r^{q_1})\varphi(r^{q_2-1})\varphi(s) \\ & -\tbinom{q_1}{1}\mathcal{G}(r^{q_1-1}s)\varphi(r^{q_2})-\tbinom{q_2}{1}\varphi(r^{q_1})\mathcal{G}(r^{q_2-1}s)-\tbinom{q_1}{1}\varphi(r^{q_1-1})\varphi(s)\mathcal{G}(r^{q_2}). \end{aligned}$$

Let us consider the implementation of $r=e$ and utilize $\varphi(e)=e$ for demonstration.

$$2(q_1+q_2)\mathcal{G}(s)=q_1\mathcal{G}(e)\varphi(s)+q_2\varphi(s)\mathcal{G}(e)+q_2\mathcal{G}(e)\varphi(s)+q_1\varphi(s)\mathcal{G}(e)+q_1\mathcal{G}(s)+q_2\mathcal{G}(s).$$

This implies that

$$2(q_1+q_2)\mathcal{G}(s)=(q_1+q_2)\mathcal{G}(e)\varphi(s)+(q_1+q_2)\varphi(s)\mathcal{G}(e).$$

That can be written as

$$(q_1+q_2)\big[2\mathcal{G}(s)-\mathcal{G}(e)\varphi(s)-\varphi(s)\mathcal{G}(e)\big]=0.$$

Applying Lemma 2, it is deduced that

$$2\mathcal{G}(s)-\mathcal{G}(e)\varphi(s)-\varphi(s)\mathcal{G}(e)=0.$$

It indicates that

$$2\mathcal{G}(s)=\mathcal{G}(e)\varphi(s)+\varphi(s)\mathcal{G}(e). \tag{2}$$

Next, we analyze in detail the implications of the condition $\mathcal{H}_2(r,s)=0$, namely,

$$\begin{aligned} 0 \;=\; & 4\tbinom{q_1+q_2}{2}\mathcal{G}(r^{q_1+q_2-2}s^2)-\tbinom{q_1}{2}\mathcal{G}(r^{q_2})\varphi(r^{q_1-2})\varphi(s^2) \\ & -\tbinom{q_2}{1}\tbinom{q_1}{1}\mathcal{G}(r^{q_2-1}s)\varphi(r^{q_1-1})\varphi(s)-\tbinom{q_2}{2}\mathcal{G}(r^{q_2-2}s^2)\varphi(r^{q_1}) \\ & -\tbinom{q_1}{2}\varphi(r^{q_2})\mathcal{G}(r^{q_1-2}s^2)-\tbinom{q_2}{1}\tbinom{q_1}{1}\varphi(r^{q_2-1})\varphi(s)\mathcal{G}(r^{q_1-1}s) \\ & -\tbinom{q_2}{2}\varphi(r^{q_2-2})\varphi(s^2)\mathcal{G}(r^{q_1})-\tbinom{q_2}{2}\mathcal{G}(r^{q_1})\varphi(r^{q_2-2})\varphi(s^2) \\ & -\tbinom{q_1}{1}\tbinom{q_2}{1}\mathcal{G}(r^{q_1-1}s)\varphi(r^{q_2-1})\varphi(s)-\tbinom{q_1}{2}\mathcal{G}(r^{q_1-2}s^2)\varphi(r^{q_2}) \\ & -\tbinom{q_2}{2}\varphi(r^{q_1})\mathcal{G}(r^{q_2-2}s^2)-\tbinom{q_1}{1}\tbinom{q_2}{1}\varphi(r^{q_1-1})\varphi(s)\mathcal{G}(r^{q_2-1}s) \\ & -\tbinom{q_1}{2}\varphi(r^{q_1-2})\varphi(s^2)\mathcal{G}(r^{q_2}),\ \text{for all } r,s \text{ in } S. \end{aligned}$$

Reformulate the aforementioned equation and utilizing e in place of r, we have

$$\begin{aligned} 4\binom{q_1+q_2}{2}\mathcal{G}(s^2) \quad = \quad & \binom{q_1}{2}\mathcal{G}(e)\varphi(s^2) + \binom{q_2}{1}\binom{q_1}{1}\mathcal{G}(s)\varphi(s) + \binom{q_2}{2}\mathcal{G}(s^2) \\ & + \binom{q_1}{2}\mathcal{G}(s^2) + \binom{q_2}{1}\binom{q_1}{1}\varphi(s)\mathcal{G}(s) + \binom{q_2}{2}\varphi(s^2)\mathcal{G}(e) \\ & + \binom{q_2}{2}\mathcal{G}(e)\varphi(s^2) + \binom{q_1}{1}\binom{q_2}{1}\mathcal{G}(s)\varphi(s) + \binom{q_1}{2}\mathcal{G}(s^2) \\ & + \binom{q_2}{2}\mathcal{G}(s^2) + \binom{q_1}{1}\binom{q_2}{1}\varphi(s)\mathcal{G}(s) + \binom{q_1}{2}\varphi(s^2)\mathcal{G}(e) \text{ for all } s \in S. \end{aligned}$$

This implies that

$$\begin{aligned} 2(q_1+q_2)(q_1+q_2-1)\mathcal{G}(s^2) \quad = \quad & \tfrac{q_1(q_1-1)}{2}\mathcal{G}(e)\varphi(s^2) + q_1q_2\mathcal{G}(s)\varphi(s) \\ & + \tfrac{q_2(q_2-1)}{2}\mathcal{G}(s^2) + \tfrac{q_1(q_1-1)}{2}\mathcal{G}(s^2) \\ & + q_1q_2\varphi(s)\mathcal{G}(s) + \tfrac{q_2(q_2-1)}{2}\varphi(s^2)\mathcal{G}(e) \\ & + \tfrac{q_2(q_2-1)}{2}\mathcal{G}(e)\varphi(s^2) + q_1q_2\mathcal{G}(s)\varphi(s) \\ & + \tfrac{q_1(q_1-1)}{2}\mathcal{G}(s^2) + \tfrac{q_2(q_2-1)}{2}\mathcal{G}(s^2) \\ & + q_1q_2\varphi(s)\mathcal{G}(s) + \tfrac{q_1(q_1-1)}{2}\varphi(s^2)\mathcal{G}(e). \end{aligned}$$

It follows that

$$\begin{aligned} 2(q_1+q_2)(q_1+q_2-1)\mathcal{G}(s^2) \quad = \quad & \left[\tfrac{q_1(q_1-1)}{2} + \tfrac{q_2(q_2-1)}{2}\right]\mathcal{G}(e)\varphi(s^2) \\ & + 2\left[\tfrac{q_1(q_1-1)}{2} + \tfrac{q_2(q_2-1)}{2}\right]\mathcal{G}(s^2) \\ & + 2q_1q_2\mathcal{G}(s)\varphi(s) + 2q_1q_2\varphi(s)\mathcal{G}(s) \\ & + \left[\tfrac{q_1(q_1-1)}{2} + \tfrac{q_2(q_2-1)}{2}\right]\varphi(s^2)\mathcal{G}(e). \end{aligned}$$

Consequently,

$$\begin{aligned} \{2(q_1+q_2)(q_1+q_2-1)\}\,\mathcal{G}(s^2) \quad = \quad & \left[\tfrac{q_1(q_1-1)}{2} + \tfrac{q_2(q_2-1)}{2}\right]\left\{\mathcal{G}(e)\varphi(s^2)\varphi(s^2)\mathcal{G}(e)\right\} \\ & + \left[q_1(q_1-1) + q_2(q_2-1)\right]\mathcal{G}(s^2) \\ & + 2q_1q_2\left[\mathcal{G}(s)\varphi(s) + \varphi(s)\mathcal{G}(s)\right]. \end{aligned} \tag{3}$$

By substituting s^2 in place of s in expression (2), we obtain

$$2\mathcal{G}(s^2) = \mathcal{G}(e)\varphi(s^2) + \varphi(s^2)\mathcal{G}(e). \tag{4}$$

Multiplying from both side by $\left[\frac{q_1(q_1-1)}{2} + \frac{q_2(q_2-1)}{2}\right]$ in the above relation, we get

$$\left[q_1(q_1-1) + q_2(q_2-1)\right]\mathcal{G}(s^2) = \left[\frac{q_1(q_1-1)}{2} + \frac{q_2(q_2-1)}{2}\right]\left\{\mathcal{G}(e)\varphi(s^2) + \varphi(s^2)\mathcal{G}(e)\right\}. \tag{5}$$

Applying (5) in (3), we get

$$\begin{aligned} \{2(q_1+q_2)(q_1+q_2-1)\}\,\mathcal{G}(s^2) \quad = \quad & 2\left[q_1(q_1-1) + q_2(q_2-1)\right]\mathcal{G}(s^2) \\ & + 2q_1q_2\left[\mathcal{G}(s)\varphi(s) + \varphi(s)\mathcal{G}(s)\right]. \end{aligned}$$

Applying the torsion condition, we obtain

$$\begin{aligned}\{(q_1+q_2)(q_1+q_2-1)\}\,\mathcal{G}(s^2) &= [q_1(q_1-1)+q_2(q_2-1)]\,\mathcal{G}(s^2)\\ &\quad +q_1q_2\,[\mathcal{G}(s)\varphi(s)+\varphi(s)\mathcal{G}(s)]\,.\end{aligned}$$

This implies that

$$\{(q_1+q_2)(q_1+q_2-1)-[q_1(q_1-1)+q_2(q_2-1)]\}\,\mathcal{G}(s^2) = q_1q_2\,[\mathcal{G}(s)\varphi(s)+\varphi(s)\mathcal{G}(s)]\,.$$

Using simple calculation, we arrive at the following

$$2q_1q_2\mathcal{G}(s^2)=q_1q_2\mathcal{G}(s)\varphi(s)+q_1q_2\varphi(s)\mathcal{G}(s).$$

Applying Lemma 2, we obtain

$$2\,\mathcal{G}(s^2)=\mathcal{G}(s)\,\varphi(s)+\varphi(s)\,\mathcal{G}(s).$$

Employing Lemma 1, the desired conclusion is thereby established.

The following example illustrates the clear and significant impact of the above results.

Example 1. *Consider a ring* $S=\left\{\begin{pmatrix} r & s\\ 0 & t\end{pmatrix} \mid r,s,t\in 2\mathbb{Z}_8\right\}$. *Define two mappings* $\mathcal{G}$ *and* φ *from* S *to itself, by* $\mathcal{G}\left[\begin{pmatrix} r & s\\ 0 & t\end{pmatrix}\right]=\begin{pmatrix} 0 & s\\ 0 & 0\end{pmatrix}$ *and* $\varphi\left[\begin{pmatrix} r & s\\ 0 & t\end{pmatrix}\right]=\begin{pmatrix} r & -s\\ 0 & t\end{pmatrix}$. *Since,* $\begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix} r & s\\ 0 & t\end{pmatrix}\begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}=\begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}$ *but* $\begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}\neq\begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}$, *then it is clear that the ring* S *is not semiprime, but* $\mathcal{G}$ *satisfies the algebraic identity in Theorem 1 and* $\mathcal{G}$ *fails to fulfill as a* φ*-centralizer. Consequently, the assumption of semiprimeness is not meaningless in this theorem.*

3. Results on $*$-centralizer

An additive mapping $*: S\to S$ is characterized as an involution if it meets the criteria of $(xy)^*=y^*x^*$ and $(x^*)^*=x$ for every $x,y\in S$. A ring equipped with such an involution is referred to as a $*$-ring, or more generally as a ring with involution. In this framework, an additive mapping $\mathcal{G}: S\to S$ stands out as a left (right) $*$-centralizer if and only if it fulfills the conditions of $\mathcal{G}(rs)=\mathcal{G}(r)s^*$ (respectively, $\mathcal{G}(rs)=r^*\mathcal{G}(s)$) for all $s,r\in S$. Notably, it is designated as a Jordan left (right) $*$-centralizer when it satisfies $\mathcal{G}(r^2)=\mathcal{G}(r)r^*$ (or $\mathcal{G}(r^2)=r^*\mathcal{G}(r)$) for all $r\in S$. Additional investigation in the text demonstrates that if $\mathcal{G}: S\to S$ exhibits as both a (Jordan) left and a (Jordan) right $*$-centralizer, it obtains characterization of a (Jordan) $*$-centralizer. Ashraf and Mozumder [7] have established that every Jordan $*$-centralizer is $*$-centralizer under specific torsion constraints in semiprime rings.

Drawing substantive findings and theoretical implications from current scholarly research, Ansari and Shujat in [3] have demonstrated that an additive mapping $G : S \to S$ functions as a $*$-centralizer on S, if G satisfies $2\mathcal{G}(r^{2q}) = \mathcal{G}(r^q)(r^*)^q + (r^*)^q\mathcal{G}(r^q)$ for each r in S. Expanding on this framework, the present article advances this line of investigation by extending the earlier conclusions under the condition $4\mathcal{G}(r^{q_1+q_2}) = \mathcal{G}(r^{q_2})(r^*)^{q_1} + (r^*)^{q_2}\mathcal{G}(r^{q_1}) + \mathcal{G}(r^{q_1})(r^*)^{q_2} + (r^*)^{q_1}\mathcal{G}(r^{q_2})$ for all $r \in S$.

In a remarkable extension of the complementary research outlined by the authors in [8–10]. The researchers in these studies effectively resolve the torsion constraint within the domain by employing the advanced methodologies offered by the Vandermonde determinant framework. To establish the validity of our principal findings, we must use the following pivotal lemmas:

Lemma 3 ([11, Lemma 2.1]). *Assuming that S constitutes a ring with an involution $*$, and that e represents the unity of S, it follows that $e^* = e$.*

Lemma 4 ([7, Corollary 2.1]). *Suppose that $*$ stands as an involution on a semiprime ring S under the condition of being 2-torsion free, while $\mathcal{G} : S \to S$ represents an additive mapping that satisfies $2\mathcal{G}(r^2) = \mathcal{G}(r)r^* + r^*\mathcal{G}(r)$ for every $r \in S$. Consequently, it emerges that $\mathcal{G}$ operates as a $*$-centralizer on S.*

We begin by addressing the pivotal conclusion.

Theorem 2. *Let $q_2 > 1$ and $q_1 \geq 1$ denote any two fixed integers and consider S as a semiprime ring with $(q_1 + q_2)!$-torsion freeness. If $\mathcal{G}$ is an additive mapping from S to itself that satisfies the algebraic equation $4\mathcal{G}(r^{q_1+q_2}) = \mathcal{G}(r^{q_2})(r^*)^{q_1} + (r^*)^{q_2}\mathcal{G}(r^{q_1}) + \mathcal{G}(r^{q_1})(r^*)^{q_2} + (r^*)^{q_1}\mathcal{G}(r^{q_2})$ for every r in S, then $\mathcal{G}$ constitutes a $*$-centralizer on S.*

Proof. Since

$$\begin{aligned} 4\mathcal{G}(r^{q_1+q_2}) &= \mathcal{G}(r^{q_2})(r^*)^{q_1} + (r^*)^{q_2}\mathcal{G}(r^{q_1}) \\ &\quad +\mathcal{G}(r^{q_1})(r^*)^{q_2} + (r^*)^{q_1}\mathcal{G}(r^{q_2}), \text{ for all } r \in S. \end{aligned} \tag{6}$$

Putting $r + ns$ for r in (6), where n is a positive integer and using Lemma 3, we find

$4\mathcal{G}\Big(r^{q_1+q_2} + n\binom{q_1+q_2}{1}r^{q_1+q_2-1}s + n^2\binom{q_1+q_2}{2}r^{q_1+q_2-2}s^2 + \cdots + n^{q_1+q_2}s^{q_1+q_2}\Big) = \mathcal{G}\Big(r^{q_2} + n\binom{q_2}{1}r^{q_2-1}s + n^2\binom{q_2}{2}r^{q_2-2}s^2 + \cdots + n^{q_2}s^{q_2}\Big)\Big((r^*)^{q_1} + n\binom{q_1}{1}(r^*)^{q_1-1}(s^*) + n^2\binom{q_1}{2}(r^*)^{q_1-2}(s^*)^2 + \cdots + n^{q_1}(s^*)^{q_1}\Big) + \Big((r^*)^{q_2} + n\binom{q_2}{1}(r^*)^{q_2-1}(s^*) + n^2\binom{q_2}{2}(r^*)^{q_2-2}(s^*)^2 + \cdots + n^{q_2}(s^*)^{q_2}\Big)\mathcal{G}\Big(r^{q_1} + n\binom{q_1}{1}r^{q_1-1}s + n^2\binom{q_1}{2}r^{q_1-2}s^2 + \cdots + n^{q_1}s^{q_1}\Big) + \mathcal{G}\Big(r^{q_1} + n\binom{q_1}{1}r^{q_1-1}s + n^2\binom{q_1}{2}r^{q_1-2}s^2 + \cdots + n^{q_1}s^{q_1}\Big)\Big((r^*)^{q_2} + n\binom{q_2}{1}(r^*)^{q_2-1}(s^*) + n^2\binom{q_2}{2}(r^*)^{q_1-2}(s^*)^2 + \cdots + n^{q_2}(s^*)^{q_2}\Big) + \Big((r^*)^{q_1} +$

$n\binom{q_1}{1}(r^*)^{q_1-1}(s^*)+n^2\binom{q_1}{2}(r^*)^{q_1-2}(s^*)^2+\cdots+n^{q_1}(s^*)^{q_1}\Big)\mathcal{G}\Big(r^{q_2}+n\binom{q_2}{1}r^{q_2-1}s+n^2\binom{q_2}{2}r^{q_2-2}s^2+\cdots+n^{q_2}s^{q_2}\Big)$.

Consider the preceding statement by incorporating the equation (6) as

$$n\mathcal{H}'_1(r,s)+n^2\mathcal{H}'_2(r,s)+\cdots+n^{q_1+q_2-1}\mathcal{H}'_{q_1+q_2-1}(r,s)=0,$$

where $\mathcal{H}'_i(r,s)$ symbolize the pivotal coefficients of n^i's for each $i=1,2,\cdots,(q_1+q_2-1)$. By substituting $1,2,\cdots,(q_1+q_2-1)$ for n, a system of (q_1+q_2-1) homogeneous equations emerges. It reveals a distinguished Vandermonde matrix, which conclusively determines that $\mathcal{H}'_i(r,s)=0$ for every r,s in S and for $i=1,2,\cdots,(q_1+q_2-1)$, as in the last theorem. Particularly, $\mathcal{H}'_1(r,s)=0$ implies that

$$\begin{aligned}0 \;=\;& 4\binom{q_1+q_2}{1}\mathcal{G}(r^{q_1+q_2-1}s)-\binom{q_1}{1}\mathcal{G}(r^{q_2})(r^*)^{q_1-1}(s^*)-\binom{q_2}{1}\mathcal{G}(r^{q_2-1}s)(r^*)^{q_1}\\&-\binom{q_1}{1}(r^*)^{q_2}\mathcal{G}(r^{q_1-1}s)-\binom{q_2}{1}(r^*)^{q_2-1}(s^*)\mathcal{G}(r^{q_1})-\binom{q_2}{1}\mathcal{G}(r^{q_1})(r^*)^{q_2-1}(s^*)\\&-\binom{q_1}{1}\mathcal{G}(r^{q_1-1}s)(r^*)^{q_2}-\binom{q_2}{1}(r^*)^{q_1}\mathcal{G}(r^{q_2-1}s)-\binom{q_1}{1}(r^*)^{q_1-1}(s^*)\mathcal{G}(r^{q_2}).\end{aligned}$$

We shall implement $r=e$ while employing Lemma 2 to achieve

$$2(q_1+q_2)\mathcal{G}(s)=q_1\mathcal{G}(e)(s^*)+q_2(s^*)\mathcal{G}(e)+q_2\mathcal{G}(e)(s^*)+q_1(s^*)\mathcal{G}(e)+q_1\mathcal{G}(s)+q_2\mathcal{G}(s).$$

On simplification, we obtain

$$2(q_1+q_2)\mathcal{G}(s)=(q_1+q_2)\mathcal{G}(e)s^*+(q_1+q_2)s^*\mathcal{G}(e).$$

Applying Lemma 2, we obtain

$$2\mathcal{G}(s)=\mathcal{G}(e)s^*+s^*\mathcal{G}(e),\ \text{for all } s \ \text{ in } \ S. \tag{7}$$

Subsequently, $\mathcal{H}'_2(r,s)=0$ gives the following

$$\begin{aligned}0 \;=\;& 4\binom{q_1+q_2}{2}\mathcal{G}(r^{q_1+q_2-2}s^2)-\binom{q_1}{2}\mathcal{G}(r^{q_2})(r^*)^{q_1-2}(s^*)^2\\&-\binom{q_2}{1}\binom{q_1}{1}\mathcal{G}(r^{q_2-1}s)(r^*)^{q_1-1}s^*-\binom{q_2}{2}\mathcal{G}(r^{q_2-2}s^2)(r^*)^{q_1}\\&-\binom{q_1}{2}(r^*)^{q_2}\mathcal{G}(r^{q_1-2}s^2)-\binom{q_2}{1}\binom{q_1}{1}(r^*)^{q_2-1}(s^*)\mathcal{G}(r^{q_1-1}s)\\&-\binom{q_2}{2}(r^*)^{q_2-2}(s^*)^2\mathcal{G}(r^{q_1})-\binom{q_2}{2}\mathcal{G}(r^{q_1})(r^*)^{q_2-2}(s^*)^2\\&-\binom{q_1}{1}\binom{q_2}{1}\mathcal{G}(r^{q_1-1}s)(r^*)^{q_2-1}s^*-\binom{q_1}{2}\mathcal{G}(r^{q_1-2}s^2)(r^*)^{q_2}\\&-\binom{q_2}{2}(r^*)^{q_1}\mathcal{G}(r^{q_2-2}s^2)-\binom{q_1}{1}\binom{q_2}{1}(r^*)^{q_1-1}s^*\mathcal{G}(r^{q_2-1}s)\\&-\binom{q_1}{2}(r^*)^{q_1-2}(s^*)^2\mathcal{G}(r^{q_2})\ \text{for all } r,s \text{ in } S.\end{aligned}$$

To achieve the desired outcome, transform the aforementioned equation by substituting e in place of r, we obtain

$$\begin{aligned}4\binom{q_1+q_2}{2}\mathcal{G}(s^2) \;=\;& \binom{q_1}{2}\mathcal{G}(e)(s^*)^2+\binom{q_2}{1}\binom{q_1}{1}\mathcal{G}(s)s^*+\binom{q_2}{2}\mathcal{G}(s^2)\\&+\binom{q_1}{2}\mathcal{G}(s^2)+\binom{q_2}{1}\binom{q_1}{1}s^*\mathcal{G}(s)+\binom{q_2}{2}(s^*)^2\mathcal{G}(e)\\&+\binom{q_2}{2}\mathcal{G}(e)(s^*)^2+\binom{q_1}{1}\binom{q_2}{1}\mathcal{G}(s)s^*+\binom{q_1}{2}\mathcal{G}(s^2)\\&+\binom{q_2}{2}\mathcal{G}(s^2)+\binom{q_1}{1}\binom{q_2}{1}s^*\mathcal{G}(s)+\binom{q_1}{2}(s^*)^2\mathcal{G}(e)\ \text{for all } s\in S.\end{aligned}$$

A straightforward computation yields

$$\begin{aligned}[2(q_1+q_2)(q_1+q_2-1)]\mathcal{G}(s^2) \quad = \quad & \tfrac{1}{2}[q_1(q_1-1)+q_2(q_2-1)][\mathcal{G}(e)(s^*)^2+\mathcal{G}(e)(s^*)^2] \\ & +[q_2(q_2-1)-q_1(q_1-1)]\mathcal{G}(s^2) \\ & +2q_1q_2[\mathcal{G}(s)s^*+s^*\mathcal{G}(s)].\end{aligned}$$

Now replacing s by s^2 in (7) and using the parallel steps that we did in the last theorem, we find

$$2q_1q_2\mathcal{G}(s^2) = q_1q_2\mathcal{G}(s)s^* + q_1q_2s^*\mathcal{G}(s), \text{ for all } s \in S.$$

Using Lemma 2, we find

$$2\mathcal{G}(s^2) = \mathcal{G}(s)s^* + s^*\mathcal{G}(s).$$

Utilizing Lemma 4, we derive the necessary conclusion.

The following example highlights the importance of the results discussed above.

Example 2. *Examine a ring* $S = \left\{ \begin{pmatrix} r & s \\ 0 & t \end{pmatrix} \mid r, s, t \in 2\mathbb{Z}_8 \right\}$. *Identify two mappings* $\mathcal{G}$ *and* $*$ *from* S *to itself by* $\mathcal{G}\left[\begin{pmatrix} r & s \\ 0 & t \end{pmatrix}\right] = \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix}$ *and* $\begin{pmatrix} r & s \\ 0 & t \end{pmatrix}^* = \begin{pmatrix} t & -s \\ 0 & r \end{pmatrix}$. *It becomes evident that the ring* S *lacks semiprimeness while* $\mathcal{G}$ *fulfills the algebraic identity characterized by the Theorem. However, it is clear that* $\mathcal{G}$ *fails to function as a* $*$-*centralizer, thus highlighting the essential role of the semiprimeness hypothesis in ensuring the validity of this theorem.*

4. Conclusions

In the present study, we conclude that the linear mappings on semiprime rings act as φ-centralizer and $*$-centralizer under a specific torsion restriction and certain algebraic identities. The study of generalized φ-centralizer and $*$-centralizer processes on rings, along with the CSL subalgebra of the von Neumann algebra, offers a rich field of research. This work ambitiously seeks to reveal continuity theorems across various algebraic structures, from Banach and semi-simple Banach algebras to Lie and C^* algebras, highlighting its considerable prospects. Readers are encouraged to examine functional identities associated with specific derivation types, such as φ-centralizer and $*$-centralizer in the CSL subalgebra of the von Neumann algebra.

This exploration of additive maps for rings and their subsets demonstrates deep mathematical and theoretical insight. Also, these kinds of structural investigations are essential to many applications in computer science, engineering, and other fields. These structures have developed much interest among researchers, making it possible for future work to efficiently organize, compare, and optimize data, offering crucial tools for solving challenging issues across a limited range of fields.

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References

[1] A Z Ansari. Characterization of additive mappings on semiprime rings. *Rend. Circ. Mat. Palermo, II. Ser*, 73:899–906, 2024.

[2] B E Johnson. An introduction to the theory of centralizers. 14:299–320, 1964.

[3] A Z Ansari and F Shujat. Additive mappings on semiprime rings functioning as centralizers. *Aust. J. Math. Anal. Appl.*, 19(2):1–9, 2022.

[4] E Albas. On τ -centralizers of semiprime rings. *Siberian Math. J.*, 48(2):191–196, 2007.

[5] M N Daif and M S Tammam El-Sayiad. On θ-centralizers of semiprime rings. *St. Petersburg Math. J.*, 21(1):43–52, 2010.

[6] A Z Ansari F Shujat A Kamel and A Fallatah. Jordan ϕ-centralizers on Semiprime and Involution Rings. *European Journal of Pure and Applied Mathematics*, 18(1):5493–5493, 2025.

[7] M Ashraf and M S Mozumder. On Jordan α-$*$-centralizers in semiprime rings with involution. *Int. J. Contemp. Math. Sciences*, 23(7):1103–1112, 2012.

[8] A Z Ansari and F Shujat. Extension of φ-centralizers on semiprime rings. *International Journal of Mathematics and Computer Science*, 19(2):435–438, 2024.

[9] N Rehman and A Z Ansari. On additive mappings in $*$-ring with identity element. *Vietnam J. Math.*, 43(4):819–828, 2015.

[10] F Shujat and S Alharbi. On generalized $(\alpha, *)$-derivations and α-centralizers on rings. *Eur. J. Pure Appl. Math.*, 18(2):6044, 2025.

[11] N Rehman A Z Ansari and T Bano. On generalized Jordan $*$-derivations in rings. *J. Egyp. Math. Soc.*, 22:11–13, 2014.