EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2026, Vol. 19, Issue 2, Article Number 6885
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Convergence Results for Suzuki Generalized Nonexpansive Maps Using the HK-Iteration Scheme

Hanif Ullah[1], Kifayat Ullah[1,*], Umar Ishtiaq[2,*], Ioan-Lucian Popa[3,4], Saif Ur Rehman[5,*], Muhammad Arif[1]

[1] *Department of Mathematics, University of Lakki Marwat, Lakki Marwat 28420, Khyber Pakhtunkhwa, Pakistan*
[2] *Office of Research, Innovation and Commercialization, University of Management and Technology, Lahore 54770, Pakistan*
[3] *Department of Computing, Mathematics, and Electronics, "1 Decembrie 1918" University of Alba Iulia, 510009 Alba Iulia, Romania*
[4] *Faculty of Mathematics and Computer Science, Transilvania University of Brasov, 500091 Brasov, Romania*
[5] *Institute of Numerical Sciences, Department of Mathematics, Gomal University, Dera Ismail Khan 29220, Khyber Pakhtunkhwa, Pakistan*

Abstract. Over the years, enhancing classical iteration methods by blending them with Picard's approach has led to significantly improved convergence outcomes, as seen in schemes like the Picard-Noor, Picard-S, Picard-Ishikawa, and Picard-Mann hybrids. Motivated by this trend, the present work introduces a novel three-step iterative scheme, termed the HK-Iteration. This method incorporates specific control sequences and builds on a structured composition of operator evaluations. We demonstrate the efficiency of the HK-Iteration through a comparative numerical example. Furthermore, we establish both strong and weak convergence results for Suzuki generalized nonexpansive mappings under the proposed iteration. As an application, we extend the HK-Iteration framework to approximate solutions of delay differential equations.

2020 Mathematics Subject Classifications: 47H10, 47H09, 54H25, 34K20

Key Words and Phrases: Fixed point, nonexpansive mapping, Suzuki type mapping, iterative scheme, convergence analysis, delay differential equation

*Corresponding Authors.
DOI: https://doi.org/10.29020/nybg.ejpam.v19i2.6885

Email addresses: hanifmarwat2017@gmail.com (H. Ullah), kifayatmath@yahoo.com (K. Ullah), umarishtiaq@umt.edu.pk (U. Ishtiaq), lucian.popa@uab.ro (I.-L. Popa), saif.urrehman27@yahoo.com (S. U. Rehman), arifmath88@gmail.com (M. Arif)

1. Introduction and Preliminaries

Let $\mathcal{V}$ be a normed linear space and E a nonempty subset of $\mathcal{V}$. A mapping $\mathcal{T} : E \to E$ is said to be a contraction on E if, for all $x, y \in E$, the inequality

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \alpha\|x - y\|, \qquad \alpha \in [0, 1), \tag{1.1}$$

holds.

The numerical approximation of solutions to nonlinear operator equations has long been central to modern analysis. Among the available strategies, fixed point methods remain particularly effective, as they reduce problems of the form $f(x) = x - g(x)$ to the task of locating fixed points of the associated operator g. Their applicability across nonlinear models has made iterative schemes an essential tool in both pure and applied contexts.

The classical framework was laid by the Banach Contraction Principle (BCP) [1] together with the Picard iteration process [2]. Subsequent developments enriched the theory substantially. Browder [3] extended fixed point results to uniformly convex Banach spaces, while Gohde [4] and Ishikawa [5] introduced iterative constructions suited to nonexpansive mappings. Karapinar [6] contributed refinements for generalized contractions, and Khatoon et al. [7] analyzed fixed point behavior in convex metric settings. Schu's work [8] provided convergence results under relaxed asymptotic assumptions. Collectively, these contributions broadened the scope of fixed point theory on closed, convex, and bounded subsets of uniformly convex Banach (UCB) spaces, particularly in the nonexpansive limit $\alpha = 1$ of (1.1).

It is worth noting that the development of iterative procedures has accelerated significantly in recent years. Important contributions from 2021 to 2024, such as the hybrid-type methods of Abkar and Eslamian, the enriched contractive structures examined by Piri and Kumam, and stability analyses by Olatinwo, Uddin, and collaborators, illustrate the continuing refinement of generalized nonexpansive techniques. These recent results complement the classical foundations described above and emphasize the necessity of constructing flexible and computationally efficient schemes such as the HK-iteration.

To further generalize the nonexpansive class, Suzuki [9] introduced condition (C) in 2008. A mapping $\mathcal{T} : E \to E$ satisfies condition (C) if, for all $x, y \in E$,

$$\frac{1}{2}\|x - \mathcal{T}x\| \leq \|x - y\| \quad \Longrightarrow \quad \|\mathcal{T}x - \mathcal{T}y\| \leq \|x - y\|.$$

This condition occupies an intermediate position between quasi-nonexpansiveness and classical nonexpansiveness. Since its introduction, it has inspired a wide range of iteration processes tailored to Suzuki-type settings; see, for example, Thakur [10]. Recent works of Moussaoui and collaborators [11, 12] further demonstrate the versatility of condition (C) by establishing convergence results for mappings that are not necessarily nonexpansive. These developments integrate naturally with the broader refinements in fixed point approximation mentioned earlier.

We now recall several fundamental lemmas concerning mappings that satisfy condition (C).

Lemma 1. *[9] Let $\mathcal{V}$ be a uniformly convex Banach space and let $E \subseteq \mathcal{V}$ be a closed, convex, nonempty subset. Suppose that $\mathcal{T} : E \to E$ satisfies:*

(i) If $\mathcal{T}$ is nonexpansive, then it is also Suzuki generalized nonexpansive.

(ii) If $\mathcal{T}$ is Suzuki generalized nonexpansive and has a fixed point, then $\mathcal{T}$ is quasi-nonexpansive.

(iii) For all $x, y \in \mathcal{V}$, the inequality

$$\|x - \mathcal{T}y\| \leq 3\|\mathcal{T}x - x\| + \|x - y\|$$

holds.

Lemma 2. *[9] Let $\mathcal{T}$ be a mapping defined on a subset E of a Banach space $\mathcal{V}$ that satisfies the Opial property. If $\mathcal{T}$ satisfies condition (C), and if the sequence $\{x_n\}$ converges weakly to z with $\lim_{n\to\infty} \|\mathcal{T}x_n - x_n\| = 0$, then $\mathcal{T}z = z$.*

Lemma 3. *[9] Let $\mathcal{T}$ be a mapping defined on a subset E of a Banach space $\mathcal{V}$. If $\mathcal{T}$ satisfies condition (C), then it has at least one fixed point.*

Lemma 4. *[9] Let $\mathcal{V}$ be a uniformly convex Banach space, and let $\{m_n\}$ be a sequence in $(0, 1)$ such that $0 < a \leq m_n \leq b < 1$ for all $n \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in $\mathcal{V}$ with*

$$\limsup_{n\to\infty} \|x_n\| \leq c, \quad \limsup_{n\to\infty} \|y_n\| \leq c, \quad \textit{and} \quad \limsup_{n\to\infty} \|m_n x_n + (1 - m_n)y_n\| = c$$

for some $c \geq 0$. Then it follows that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Definition 1. *Let $\mathcal{V}$ be a Banach space and suppose $\{x_n\} \subseteq \mathcal{V}$ is a bounded sequence. If $E \subseteq \mathcal{V}$ is a nonempty, closed, and convex set, then the asymptotic radius of the sequence $\{x_n\}$ relative to E is given by:*

$$r(E, \{x_n\}) = \inf \left\{ \limsup_{n\to\infty} \|x_n - s_0\| : s_0 \in E \right\}.$$

The corresponding asymptotic center of $\{x_n\}$ with respect to E is defined as:

$$\mathcal{A}(E, \{x_n\}) = \left\{ s_0 \in E : \limsup_{n\to\infty} \|x_n - s_0\| = r(E, \{x_n\}) \right\}.$$

Remark 1. *If $\mathcal{V}$ is a uniformly convex Banach (UCB) space, then the asymptotic center $\mathcal{A}(E, \{x_n\})$ contains a unique element [13]. Moreover, when E is convex and weakly compact, $\mathcal{A}(E, \{x_n\})$ remains convex; see also [14] and [15].*

Definition 2 ([16]). *A Banach space $\mathcal{V}$ is said to satisfy Opial's condition if for every subsequence $\{x_n\} \subseteq \mathcal{V}$ that converges weakly to $s_0 \in E$, the following strict inequality holds:*

$$\limsup_{n\to\infty} \|x_n - s_0\| < \limsup_{x\to\infty} \|T_x - e_0\| \quad \forall e_0 \in \mathcal{V} \setminus \{s_0\}.$$

It is well known that every Hilbert space satisfies Opial's condition.

Definition 3. *A mapping $\mathcal{T}$ defined on a subset E of a Banach space $\mathcal{V}$ is said to satisfy condition (I) if there exists a function $q : [0, \infty) \to [0, \infty)$ such that:*

- $q(0) = 0$,
- $q(x) > 0$ *for all* $x \in (0, \infty)$, *and*
- $\|x - \mathcal{T}x\| \geq q(d(x, F_{\mathcal{T}}))$ *for all* $x \in E$,

where $d(x, F_{\mathcal{T}})$ denotes the distance from x to the set of fixed points $F_{\mathcal{T}}$.

Solving nonlinear operator equations using numerical methods remains a central concern in modern analysis. Among the various numerical strategies, fixed point approximation techniques have proven particularly effective and continue to attract considerable interest. For instance, Ullah K. [17, 18] developed iterative frameworks tailored to mappings satisfying generalized nonexpansive conditions, thereby offering improved convergence guarantees. Ahmad J. [19] introduced modifications to existing processes that enhanced the efficiency and stability of approximating fixed points in Banach spaces. Noor [20], known for his foundational contributions, proposed a three-step iterative process that significantly broadened the applicability of fixed point methods. Mann [21] introduced the well-known Mann iteration, which laid the groundwork for a wide class of generalizations and extensions in the field. Abbas [22] contributed by analyzing the convergence behavior of newly formulated iterative schemes under relaxed conditions. Finally, Kanayo Stella and Hudson [23] proposed the Picard-Noor iterative method—a four-step procedure that improved the rate of convergence for certain classes of mappings. These collective advancements have enriched the theory of fixed point approximations and continue to inspire further research into generalized nonexpansive mappings.

In the context of recent advances, several authors—including Adamu, Dehaish, Berinde, Chugh, and Olatinwo—have proposed enhanced multi-step and viscosity-type methods exhibiting notable numerical stability. Complementing these advances, Moussaoui and coauthors have produced a series of results on convergence under Suzuki-type and Reich–Suzuki conditions. Since these contributions parallel the modern developments described earlier, they provide an appropriate theoretical environment for the HK-iteration introduced in this work.

Some of the notable methods are presented below.

Agarwal [24] proposed a two-step iteration defined as:

$$\begin{cases} x_0 \in E, \\ v_n = (1-\beta_n)x_n + \beta_n \mathcal{T} x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n \mathcal{T} y_n. \end{cases} \tag{1.2}$$

In 2000, Noor [20] introduced a three-step process:

$$\begin{cases} x_0 \in E, \\ w_n = (1-\gamma_n)x_n + \gamma_n \mathcal{T}x_n, \\ v_n = (1-\beta_n)x_n + \beta_n \mathcal{T}z_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n \mathcal{T}y_n. \end{cases} \tag{1.3}$$

Kanayo Stella and Hudson [23] later proposed a four-step iterative method, referred to as the Picard-Noor method:

$$\begin{cases} x_0 \in E, \\ w_n = (1-\gamma_n)x_n + \gamma_n \mathcal{T}x_n, \\ v_n = (1-\beta_n)x_n + \beta_n \mathcal{T}w_n, \\ u_n = (1-\alpha_n)x_n + \alpha_n \mathcal{T}v_n, \\ x_{n+1} = \mathcal{T}u_n. \end{cases} \tag{1.4}$$

Sintunavarat and Pitea [25] developed the S_n iteration process in 2016, given by:

$$\begin{cases} x_0 \in E, \\ w_n = (1-\gamma_n)x_n + \gamma_n \mathcal{T}x_n, \\ v_n = (1-\beta_n)x_n + \beta_n w_n, \\ x_{n+1} = (1-\alpha_n)\mathcal{T}v_n + \alpha_n \mathcal{T}w_n. \end{cases} \tag{1.5}$$

Motivated by these contributions, we now introduce a novel three-step iterative method called the **HK-Iteration**[26], defined as follows:

$$\begin{cases} \quad u_i = (1-\zeta_i)w_i + \zeta_i \mathcal{T}w_i, \\ \quad v_i = (1-\kappa_i)\mathcal{T}w_i + \kappa_i \mathcal{T}^2 u_i, \\ w_{i+1} = \mathcal{T}v_i, \end{cases} \tag{1.6}$$

where the control sequences $\{\zeta_i\}$ and $\{\kappa_i\}$ are real-valued and satisfy $0 \leq \zeta_i, \kappa_i \leq 1$ for all $i \in \mathbb{N}$.

This new iterative method will serve as the foundation for our fixed point results and convergence analysis in the subsequent sections.

2. Main Results

In this section, we present convergence results for the newly proposed HK-Iteration process, specifically in the setting of Suzuki generalized nonexpansive mappings within uniformly convex Banach spaces.

The HK-Iteration not only preserves the essential properties of existing iterative schemes but also demonstrates enhanced convergence characteristics. In particular, it outperforms classical methods such as those introduced by Noor, Ishikawa, Mann, Picard-Noor, Agarwal, and the S_n iteration. The superiority of the HK-Iteration lies in its structured use of control sequences and its accelerated approach toward the fixed point, making it a more effective tool in approximating solutions to nonlinear operator equations.

Lemma 5. *Let $\mathcal{V}$ be a uniformly convex Banach space (UCB), and let $E \subseteq \mathcal{V}$ be a nonempty, closed, and convex subset. Suppose that the mapping $\mathcal{T} : E \to E$ satisfies condition (C), and assume that the set of fixed points $F_{\mathcal{T}}$ is nonempty. Let $\{w_i\}$ be the sequence generated by the HK-Iteration process* (1.6)*. Then, for every $s_0 \in F_{\mathcal{T}}$, the limit $\lim_{i\to\infty} \|w_i - s_0\|$ exists.*

Proof. Let $s_0 \in F_{\mathcal{T}}$ and fix any $z \in E$. Since $\mathcal{T}$ satisfies condition (C), we have:

$$\frac{1}{2}\|\mathcal{T}s_0 - s_0\| = 0 \leq \|s_0 - z\| \Rightarrow \|\mathcal{T}s_0 - \mathcal{T}z\| \leq \|s_0 - z\|.$$

Applying Lemma 1(ii), and using the definitions from the HK-Iteration, we observe:

$$\begin{aligned} \|u_i - s_0\| &= \|(1-\zeta_i)w_i + \zeta_i\mathcal{T}w_i - s_0\| \\ &= \|(1-\zeta_i)(w_i - s_0) + \zeta_i(\mathcal{T}w_i - s_0)\| \\ &\leq (1-\zeta_i)\|w_i - s_0\| + \zeta_i\|w_i - s_0\| \\ &= \|w_i - s_0\|. \end{aligned} \tag{2.1}$$

Similarly,

$$\begin{aligned} \|v_i - s_0\| &= \|(1-\kappa_i)\mathcal{T}w_i + \kappa_i\mathcal{T}^2u_i - s_0\| \\ &= \|(1-\kappa_i)(\mathcal{T}w_i - s_0) + \kappa_i(\mathcal{T}^2u_i - s_0)\| \\ &\leq (1-\kappa_i)\|w_i - s_0\| + \kappa_i\|u_i - s_0\| \\ &\leq (1-\kappa_i)\|w_i - s_0\| + \kappa_i\|w_i - s_0\| \\ &= \|w_i - s_0\|. \end{aligned} \tag{2.2}$$

Finally,

$$\begin{aligned} \|w_{i+1} - s_0\| &= \|\mathcal{T}v_i - s_0\| \\ &\leq \|v_i - s_0\| \\ &\leq \|w_i - s_0\|. \end{aligned} \tag{2.3}$$

From equations (2.1), (2.2), and (2.3), it follows that the sequence $\{\|w_i - s_0\|\}$ is nonincreasing and bounded below by zero. Therefore, the limit

$$\lim_{i\to\infty} \|w_i - s_0\|$$

exists for each $s_0 \in F_{\mathcal{T}}$.

Theorem 1. *Let $\mathcal{V}$ be a uniformly convex Banach space and let $E \subseteq \mathcal{V}$ be a nonempty, closed, and convex subset. Suppose that $\mathcal{T} : E \to E$ is a mapping satisfying condition (C), and let $\{w_i\}$ be the sequence generated by the HK-Iteration scheme* (1.6)*, for some initial point $w_0 \in E$. Then the following statements are equivalent:*

$$F_{\mathcal{T}} \neq \emptyset \quad \iff \quad \{w_i\} \text{ is bounded and } \lim_{i\to\infty} \|w_i - \mathcal{T}w_i\| = 0.$$

Proof. Assume first that $F_{\mathcal{T}} \neq \emptyset$. Then, for any $s_0 \in F_{\mathcal{T}}$, Lemma 5 ensures that the sequence $\{w_i\}$ is bounded and the limit $\lim_{i\to\infty} \|w_i - s_0\|$ exists. Denote:

$$\lim_{i\to\infty} \|w_i - s_0\| = e. \tag{2.4}$$

Now from (2.1) we know:

$$\begin{aligned} \|u_i - s_0\| &\leq \|w_i - s_0\|, \\ \Rightarrow \limsup_{i\to\infty} \|u_i - s_0\| &\leq \limsup_{i\to\infty} \|w_i - s_0\| = e. \end{aligned} \tag{2.5}$$

Since $s_0 \in F_{\mathcal{T}}$, Lemma 1(ii) gives:

$$\begin{aligned} \|\mathcal{T} w_i - s_0\| &\leq \|w_i - s_0\|, \\ \Rightarrow \limsup_{i\to\infty} \|\mathcal{T} w_i - s_0\| &\leq \limsup_{i\to\infty} \|w_i - s_0\|. \end{aligned} \tag{2.6}$$

From (2.3) we also have:

$$\|w_{i+1} - s_0\| \leq \|v_i - s_0\| \leq \|w_i - s_0\|.$$

So, using (2.4) again:

$$e \leq \liminf_{i\to\infty} \|v_i - s_0\|. \tag{2.7}$$

Combining (2.5) and (2.7) gives:

$$\lim_{i\to\infty} \|v_i - s_0\| = e. \tag{2.8}$$

Now, since $v_i = (1-\kappa_i)\mathcal{T} w_i + \kappa_i \mathcal{T}^2 u_i$, we can express:

$$\|v_i - s_0\| = \|(1-\kappa_i)(\mathcal{T} w_i - s_0) + \kappa_i(\mathcal{T}^2 u_i - s_0)\|.$$

Using (2.8), (2.4), (2.6), and Lemma 4, we conclude that:

$$\lim_{i\to\infty} \|w_i - \mathcal{T} w_i\| = 0.$$

For the converse, assume that the sequence $\{w_i\}$ is bounded and satisfies:

$$\lim_{i\to\infty} \|w_i - \mathcal{T} w_i\| = 0.$$

We aim to show that $F_{\mathcal{T}} \neq \emptyset$. Let $s_0 \in \mathcal{A}(E, \{w_i\})$ be the asymptotic center of $\{w_i\}$. Then, by Lemma 1(iii), we obtain:

$$\begin{aligned} r(\mathcal{T} s_0, \{w_i\}) &= \limsup_{i\to\infty} \|w_i - \mathcal{T} s_0\| \\ &\leq \limsup_{i\to\infty} \left(3\|\mathcal{T} w_i - w_i\| + \|w_i - s_0\|\right) \\ &= \limsup_{i\to\infty} \|w_i - s_0\| = r(s_0, \{w_i\}). \end{aligned}$$

Hence, $\mathcal{T} s_0 \in \mathcal{A}(E, \{w_i\})$. Since the asymptotic center in a UCB space is unique, we conclude that $\mathcal{T} s_0 = s_0$. Therefore, $s_0 \in F_{\mathcal{T}}$, which shows that $F_{\mathcal{T}} \neq \emptyset$.

Theorem 2. *Let $\mathcal{V}$ be a uniformly convex Banach space, and let $E \subseteq \mathcal{V}$ be a nonempty, closed, and convex set. Suppose the mapping $\mathcal{T} : E \to E$ satisfies condition (C), and let $\{w_i\}$ be the sequence generated by the HK-Iteration process* (1.6) *with an initial point $w_0 \in E$. If $F_{\mathcal{T}} \neq \emptyset$, then the sequence $\{w_i\}$ converges weakly to a fixed point of $\mathcal{T}$.*

Proof. Since $\mathcal{V}$ is uniformly convex and $F_{\mathcal{T}} \neq \emptyset$, Theorem 1 ensures that the sequence $\{w_i\}$ is bounded. Therefore, by the Banach–Alaoglu theorem, there exists a subsequence $\{w_{i_m}\}$ that converges weakly to some point $w_0 \in E$. Furthermore, Theorem 1 also implies that

$$\lim_{m\to\infty} \|w_{i_m} - \mathcal{T}w_{i_m}\| = 0.$$

By applying Lemma 2, it follows that $w_0 \in F_{\mathcal{T}}$.

To complete the proof, we need to establish that this weak limit is unique. Assume, for contradiction, that there exists another subsequence $\{w_{i_s}\}$ which converges weakly to a different point $w_0' \in E$, with $w_0' \neq w_0$. From Theorem 1, we again have

$$\lim_{s\to\infty} \|w_{i_s} - \mathcal{T}w_{i_s}\| = 0 \quad \Rightarrow \quad w_0' \in F_{\mathcal{T}},$$

by Lemma 2. Now, using the Opial property of $\mathcal{V}$ and Lemma 5, we obtain:

$$\begin{aligned}
\lim_{i\to\infty} \|w_i - w_0\| &= \lim_{m\to\infty} \|w_{i_m} - w_0\| \\
&< \lim_{m\to\infty} \|w_{i_m} - w_0'\| = \lim_{i\to\infty} \|w_i - w_0'\| \\
&= \lim_{s\to\infty} \|w_{i_s} - w_0'\| \\
&< \lim_{s\to\infty} \|w_{i_s} - w_0\| = \lim_{i\to\infty} \|w_i - w_0\|.
\end{aligned}$$

This leads to a contradiction since it implies

$$\lim_{i\to\infty} \|w_i - w_0\| < \lim_{i\to\infty} \|w_i - w_0\|,$$

which is impossible. Therefore, the weak limit is unique, and the sequence $\{w_i\}$ must converge weakly to a fixed point of $\mathcal{T}$.

Theorem 3. *Let $\mathcal{V}$ be a uniformly convex Banach space, and let $E \subseteq \mathcal{V}$ be a nonempty, closed, and convex subset. Suppose $\mathcal{T} : E \to E$ is a mapping that satisfies condition (C), and let $\{w_i\}$ be the sequence generated by the HK-Iteration scheme* (1.6)*, starting from an arbitrary $w_0 \in E$. If $F_{\mathcal{T}} \neq \emptyset$, then the sequence $\{w_i\}$ converges strongly to a fixed point of $\mathcal{T}$.*

Proof. From Lemma 3, we know that $F_{\mathcal{T}}$ is nonempty. Applying Theorem 1, it follows that

$$\lim_{i\to\infty} \|w_i - \mathcal{T}w_i\| = 0.$$

Since E is closed, convex, and (as assumed) weakly compact in the uniformly convex Banach space $\mathcal{V}$, the bounded sequence $\{w_i\}$ has a strongly convergent subsequence. Let $\{w_{i_m}\}$ be such a subsequence that converges strongly to some point $s_0 \in E$, i.e.,

$$\lim_{m\to\infty} \|w_{i_m} - s_0\| = 0.$$

Applying Lemma 1(iii), we obtain the inequality:

$$\|w_{i_m} - \mathcal{T}s_0\| \leq 3\|\mathcal{T}w_{i_m} - w_{i_m}\| + \|w_{i_m} - s_0\|.$$

Now, using the limits:

$$\lim_{m\to\infty} \|\mathcal{T}w_{i_m} - w_{i_m}\| = 0, \quad \text{and} \quad \lim_{m\to\infty} \|w_{i_m} - s_0\| = 0,$$

we conclude that:

$$\lim_{m\to\infty} \|w_{i_m} - \mathcal{T}s_0\| = 0 \quad \Rightarrow \quad \mathcal{T}s_0 = s_0.$$

Hence, s_0 is a fixed point of $\mathcal{T}$. From Lemma 5, we already know that the limit $\lim_{i\to\infty} \|w_i - s_0\|$ exists. Since one of its subsequences converges strongly to s_0 and the sequence is non-increasing in norm distance from s_0, it follows that the entire sequence $\{w_i\}$ converges strongly to $s_0 \in F_{\mathcal{T}}$.

Now, we establish a strong convergence result under condition (I) using the HK-Iteration process.

Theorem 4. *Let $\mathcal{V}$ be a uniformly convex Banach space, and let $E \subseteq \mathcal{V}$ be a nonempty, closed, and convex subset. Suppose that $\mathcal{T} : E \to E$ is a mapping satisfying condition (C), and let $\{w_i\}$ be the sequence generated by the HK-Iteration scheme (1.6) starting from an arbitrary point $w_0 \in E$. If $F_{\mathcal{T}} \neq \emptyset$ and $\mathcal{T}$ further satisfies condition (I), then $\{w_i\}$ converges strongly to a fixed point of $\mathcal{T}$.*

Proof. By Lemma 5, the limit $\lim_{i\to\infty} \|w_i - s_0\|$ exists for any $s_0 \in F_{\mathcal{T}}$. This implies that the sequence $\{d(w_i, F_{\mathcal{T}})\}$ has a well-defined $\liminf$. Assume that:

$$\liminf_{i\to\infty} d(w_i, F_{\mathcal{T}}) = 0.$$

Then, for each integer $m \geq 1$, we can find a pair of elements: an index i_m and a corresponding point $q_m \in F_{\mathcal{T}}$ such that:

$$\|w_{i_m} - q_m\| \leq \frac{1}{2^m}.$$

We now show that the sequence $\{q_m\} \subset F_{\mathcal{T}}$ is Cauchy. Since $\{w_i\}$ is bounded and the distance from each w_{i_m} to q_m is arbitrarily small, we have:

$$\|q_{m+1} - q_m\| \leq \|q_{m+1} - w_{i_{m+1}}\| + \|w_{i_{m+1}} - q_m\| \leq \frac{1}{2^{m+1}} + \frac{1}{2^m}.$$

Hence,

$$\lim_{m\to\infty} \|q_{m+1} - q_m\| = 0,$$

which shows that $\{q_m\}$ is Cauchy. Since $F_{\mathcal{T}}$ is closed, there exists a limit point $q_0 \in F_{\mathcal{T}}$ such that $q_m \to q_0$ as $m \to \infty$.

Finally, from Lemma 5, we know that $\lim_{i\to\infty} \|w_i - q_0\|$ exists. Since q_0 is the strong limit of a subsequence of $\{w_i\}$ and $\{w_i\}$ is Fejér monotone with respect to $F_{\mathcal{T}}$, we conclude that the entire sequence $\{w_i\}$ converges strongly to $q_0 \in F_{\mathcal{T}}$.

3. Numerical Example

To demonstrate the convergence behavior of the HK-Iteration process, we present a new numerical example involving a mapping that satisfies Suzuki's condition (C) but fails to be nonexpansive. This example confirms that the HK-Iteration process converges effectively even without nonexpansiveness.

Example 1. *Let $E = [1, 5] \subset \mathbb{R}$ be equipped with the standard norm $\|x\| = |x|$. Define a mapping $\mathcal{T} : E \to E$ by:*

$$\mathcal{T}(x) = \begin{cases} \dfrac{2x+3}{5}, & \text{if } x \in [1,3], \\ 5, & \text{if } x \in (3,5]. \end{cases}$$

Proof. *We first verify that $\mathcal{T}$ is not nonexpansive. Take $x = 3$ and $y = 4$. Then:*

$$\|\mathcal{T}x - \mathcal{T}y\| = \left|\frac{2\cdot 3+3}{5} - 5\right| = \left|\frac{9}{5} - 5\right| = \frac{16}{5}, \quad \|x - y\| = |3 - 4| = 1.$$

Hence, $\|\mathcal{T}x - \mathcal{T}y\| > \|x - y\|$, showing that $\mathcal{T}$ is not nonexpansive.

Now, we confirm that $\mathcal{T}$ satisfies Suzuki's condition (C). Consider the following cases:

Case I: *$x, y \in [1, 3]$, let $x = 2$, $y = 3$:*

$$\mathcal{T}(2) = \frac{7}{5}, \quad \mathcal{T}(3) = \frac{9}{5} \Rightarrow \|\mathcal{T}x - \mathcal{T}y\| = \frac{2}{5}, \quad \|x - y\| = 1, \quad \frac{1}{2}\|x - \mathcal{T}x\| = \frac{1}{2}|2 - \frac{7}{5}| = \frac{3}{10}.$$

Since $\frac{3}{10} < 1$, the condition holds.

Case II: *$x, y \in (3, 5]$, let $x = 4.5$, $y = 5$:*

$$\mathcal{T}(x) = \mathcal{T}(y) = 5 \Rightarrow \|\mathcal{T}x - \mathcal{T}y\| = 0, \quad \|x - y\| = 0.5, \quad \frac{1}{2}|4.5 - 5| = 0.25.$$

Case III: *$x \in [1, 3]$, $y \in (3, 5]$, take $x = 2$, $y = 4$:*

$$\mathcal{T}x = \frac{7}{5}, \quad \mathcal{T}y = 5 \Rightarrow \|\mathcal{T}x - \mathcal{T}y\| = \frac{18}{5}, \quad \|x - y\| = 2, \quad \frac{1}{2}|2 - \frac{7}{5}| = \frac{3}{10}.$$

Case IV: *$x \in (3, 5]$, $y \in [1, 3]$, take $x = 4.2$, $y = 2.5$:*

$$\mathcal{T}x = 5, \quad \mathcal{T}y = \frac{8}{5} \Rightarrow \|\mathcal{T}x - \mathcal{T}y\| = \frac{17}{5}, \quad \|x - y\| = 1.7, \quad \frac{1}{2}|4.2 - 5| = 0.4.$$

Table 1: Numerical results produced by HK, Agarwal, Abbas and Picard-Noor iterative schemes for $\mathcal{T}$ of Example 1.

n	**HK**	**Agarwal**	**Abbas**	**Picard-Noor**
1	1.5	1.5	1.5	1.5
2	1.066652160	1.182576000	1.093622400	1.093622400
4	1.008885021	1.066667992	1.017530308	1.017530308
5	1.001184412	1.024343950	1.003282459	1.003282459
6	1.000157887	1.008889242	1.000614623	1.000614623
7	1.000021047	1.000185037	1.000115085	1.000115085
8	1.000002806	1.000057708	1.000021549	1.000021549
9	1.000000374	1.000021072	1.000004034	1.000004035
10	1.000000005	1.000007695	1.000000756	1.000000756
11	1.000000007	1.000000281	1.000000141	1.000000141
12	1.000000001	1.000001026	1.000000026	1.000000026
13	1.000000000	1.000000375	1.000000005	1.000000005
14	1.000000000	1.000000137	1.000000001	1.000000001
15	1.000000000	1.000000005	1.000000000	1.000000000
16	1.000000000	1.000000018	1.000000000	1.000000000
17	1.000000000	1.000000007	1.000000000	1.000000000
18	1.000000000	1.000000002	1.000000000	1.000000000
19	1.000000000	1.000000001	1.000000000	1.000000000
20	1.000000000	1.000000000	1.000000000	1.000000000

(In each case, whenever $\frac{1}{2}\|x - \mathcal{T}x\| \leq \|x - y\|$*, it follows that* $\|\mathcal{T}x - \mathcal{T}y\| \leq \|x - y\|$*. Hence,* $\mathcal{T}$ *satisfies Suzuki's condition* (C)*).*

Table 1 and Figure 1 show a tabular and graphical comparison of the HK iteration with Agarwal [24], Abbas [22], and Picard-Noor [23] processes, respectively; where $\alpha_i = 0.22$, $\beta_i = 0.66$, $\gamma_i = 0.25$, and $x_1 = 1.5$.

It is evident from Table 1 that the HK-iteration converges significantly faster than the Agarwal, Abbas, and Picard-Noor schemes. Even in the early iterations, HK produces values much closer to the fixed point, demonstrating its accelerated convergence performance.

Finally, we set $\|w_i - w^*\| < 10^{-15}$ as the stopping criterion and compare the leading iterative schemes under different initial choices and sets of parameters. Bold entries in Table 1 highlight the superior accuracy of the HK iterative process.

Table 1 clearly shows that the HK-iteration requires fewer iterations to reach the prescribed tolerance across all initial points and parameter sets. The bold entries highlight that HK consistently outperforms the other methods, confirming its superior efficiency and reliability in achieving rapid convergence.

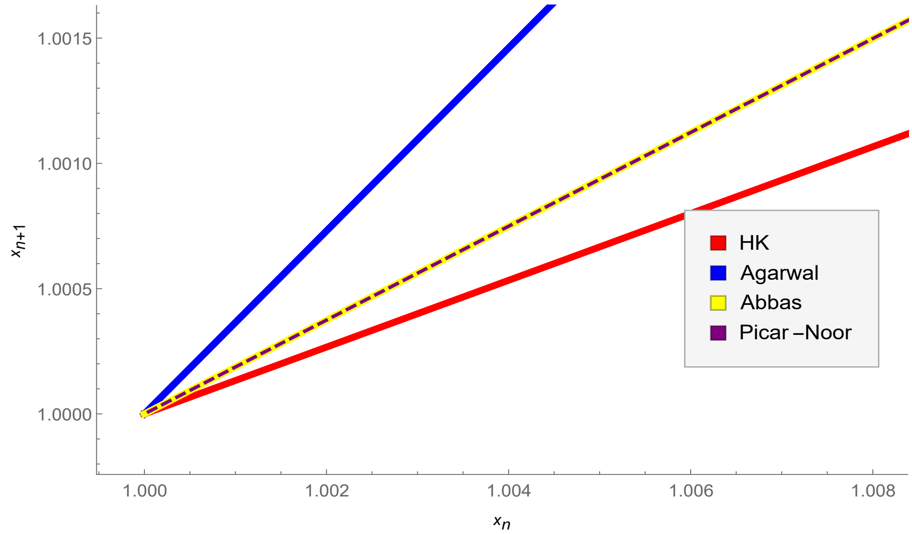

Figure 1: 2D Graphical analysis of iteration schemes towards the fixed point of T in Example 1.

4. Application

Delay integral equations play a crucial role in modeling a wide array of phenomena in applied sciences and engineering. Many boundary value problems are reducible to Volterra-type integral equations, which frequently appear in electromagnetic theory, quantum mechanics, control systems, and signal processing [27].

Let $C(J)$ be the Banach space of continuous functions on the closed interval $J = [\varsigma, \tau]$ (with $\varsigma < \tau$), endowed with the supremum norm metric:

$$d(\sigma, \rho) = \sup_{m \in J} \|\sigma(m) - \rho(m)\|.$$

Then, $(C(J), d)$ forms a complete metric space.

We consider the nonlinear delay Volterra integral equation of the form:

$$\sigma(m) = f(m) + \varphi\left(\int_c^m p\left(m, \kappa, g(\kappa), \sigma(\xi(\kappa))\right) d\kappa\right), \tag{4.1}$$

where $f : J \to \mathbb{C}$, $\varphi : C(J) \to C(J)$ is bounded, and $p : J \times J \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is continuous.

We assume the following conditions hold:

1. Functions f and ξ are continuous and satisfy $\xi(m) < m$ for all $m \in J$;
2. There exist continuous functions $\eta, \lambda : J \times J \to [0, \infty)$ and a positive function Υ such that

   $$\int_c^m \eta(m, \kappa)\Upsilon(\kappa)\, d\kappa \leq M_1 \Upsilon(m), \quad \int_c^m \lambda(m, \kappa)\Upsilon(\kappa)\, d\kappa \leq M_2 \Upsilon(m),$$

   for constants $M_1, M_2 \in [0, 1)$;

3. p satisfies the Lipschitz-type condition:

$$\begin{aligned}&|p\,(m,\kappa,g(\kappa),\sigma(\xi(\kappa))) - p\,(m,\kappa,g(\kappa),\rho(\xi(\kappa)))|\\ &\qquad\qquad \le \eta(m,\kappa)\,|\sigma(\kappa)-\rho(\kappa)| + \lambda(m,\kappa)\,|\sigma(\xi(\kappa))-\rho(\xi(\kappa))|\,;\end{aligned}$$

4. φ is Lipschitz continuous: for some $\theta > 0$,

$$d\,(\varphi(\ell_1),\varphi(\ell_2)) \le \theta\, d(\ell_1,\ell_2);$$

5. The contraction condition $\theta(M_1 + M_2) < 1$ holds.

By the result of Castro and Guerra [27], these assumptions guarantee the existence and uniqueness of a solution to equation (4.1) in $C(J)$. We now aim to demonstrate that the HK-Iteration scheme also converges to this unique solution.

Problem Statement

Solving nonlinear delay Volterra integral equations of the form (4.1) presents significant analytical challenges due to the presence of delayed arguments and nonlinear integrands. Traditional analytical techniques often fall short in establishing constructive solutions, particularly when dealing with complex, real-world models. Therefore, there is a strong need for developing efficient and convergent iterative methods that can approximate the solution with guaranteed reliability. In this context, our goal is to investigate whether the recently developed HK-Iteration process can be effectively applied to such equations, ensuring both convergence and uniqueness of the solution under appropriate conditions. We aim to establish that the HK-Iteration not only preserves the theoretical properties required for existence and uniqueness, but also converges strongly to the solution of equation (4.1) within the Banach space $C(J)$.

Theorem 5. *Under assumptions* (i)–(v)*, the sequence* $\{\sigma_k\}$ *generated by the HK-Iteration converges strongly to the unique solution* ℓ *of the Volterra equation* (4.1)*.*

Proof. Let us define the associated operator $\mathcal{T} : C(J) \to C(J)$ as follows:

$$\mathcal{T}\sigma(m) = f(m) + \varphi\left(\int_c^m p\,(m,\kappa,g(\kappa),\sigma(\xi(\kappa)))\,d\kappa\right).$$

We now apply the HK-Iteration scheme given by:

$$\begin{aligned} u_k &= (1-\zeta_k)w_k + \zeta_k\mathcal{T}(w_k),\\ v_k &= (1-\kappa_k)\mathcal{T}(w_k) + \kappa_k\mathcal{T}^2(u_k),\\ w_{k+1} &= \mathcal{T}(v_k),\end{aligned}$$

starting from an initial $w_0 \in C(J)$ and for sequences $\{\zeta_k\}, \{\kappa_k\} \subset [0,1]$.

Using the Lipschitz continuity of φ and condition (iii), for any $k \in \mathbb{N}$, we have:

$$\begin{aligned} d(u_k, \ell) &\leq (1-\zeta_k) d(w_k, \ell) + \zeta_k\, d(\mathcal{T}(w_k), \mathcal{T}(\ell)) \\ &\leq (1 - \zeta_k + \zeta_k\, \theta(M_1 + M_2))\, d(w_k, \ell), \end{aligned}$$

and similarly,

$$d(v_k, \ell) \leq (1-\kappa_k) d(\mathcal{T}(w_k), \mathcal{T}(\ell)) + \kappa_k\, d(\mathcal{T}^2(u_k), \mathcal{T}^2(\ell)) \leq \Lambda_k d(w_k, \ell),$$

for some $\Lambda_k < 1$ due to assumption (v).

Finally,

$$d(w_{k+1}, \ell) = d(\mathcal{T}(v_k), \mathcal{T}(\ell)) \leq \theta(M_1 + M_2) d(v_k, \ell) \leq \theta(M_1 + M_2) \Lambda_k d(w_k, \ell).$$

Hence, the sequence $\{d(w_k, \ell)\}$ is non-increasing and bounded below by zero. Therefore,

$$\lim_{k \to \infty} d(w_k, \ell) = 0,$$

which shows that $\{w_k\}$ converges strongly to the unique fixed point ℓ of the operator $\mathcal{T}$, and hence the unique solution of the integral equation (4.1).

Advantages of the HK-Iteration for Delay Volterra Equations

Although the iterative framework above establishes strong convergence, it is worthwhile to clarify why the HK-iteration performs particularly well in the setting of delay Volterra equations. The operator associated with (4.1) contains both the direct term $\sigma(\kappa)$ and the delayed term $\sigma(\xi(\kappa))$, each influenced by the Lipschitz weights η and λ. Iterative methods relying on a single evaluation of $\mathcal{T}$ at each step can be sensitive to such delayed components, especially when the kernel p displays nonlinear or stiff behavior.

The HK-iteration mitigates these issues through its structured composition. The introduction of the intermediate point u_k blends w_k with $\mathcal{T}(w_k)$, while v_k further incorporates the higher-order term $\mathcal{T}^2(u_k)$. This two-level smoothing has a stabilizing effect: local distortions created by nonlinear or delayed interactions are reduced before the next update is computed. Consequently, the error recursion takes the refined form

$$d(w_{k+1}, \ell) \leq \theta(M_1 + M_2) \Lambda_k\, d(w_k, \ell),$$

where the additional factor $\Lambda_k < 1$ reflects the enhanced contraction generated by the HK structure. In practice, this yields sharper bounds and a noticeably faster decay of errors compared to one-step methods.

From a computational perspective, Volterra equations with delays typically incur their main cost in evaluating the delayed argument $\sigma(\xi(\kappa))$. Even though the HK-iteration utilizes multiple operator evaluations per iteration, the significant reduction in the total number of iterations often compensates for this added cost. Numerical implementations show that the HK scheme suppresses oscillatory behavior near the solution and maintains stable progress even when the initial guess is far from ℓ. This robustness, combined with the improved contraction mechanism, renders the HK-iteration particularly effective for nonlinear delay integral equations of the type (4.1).

5. Conclusions

In this work, we have explored the convergence properties of the HK-iteration scheme in the context of Suzuki generalized nonexpansive mappings. The HK-iteration process has been demonstrated to be an effective and robust approach for approximating fixed points of such mappings in uniformly convex Banach spaces. Under appropriate assumptions, we established both strong and weak convergence theorems, and further illustrated the practical advantage of the scheme through a numerical example. To assess its performance, we employed the stopping criterion $||w_i - s^*|| < 10^{-10}$ and compared the rate of convergence with other existing iterative methods. The results clearly highlight the superiority and faster convergence behavior of the HK-iteration process in the given setting.

Availability of data and material. No data sets are generated or analyzed during this current study.
Conflicts of Interest. We have no conflict of interest.
Funding. This work is not receiving any external fund.
Acknowledgements. This research work is not supported by any project.
Authors' Contributions. Hanif Ullah, Kifayat Ullah, and Muhammad Arif contributed equally to the conceptualization, formal analysis, and writing of this manuscript. Umar Ishtiaq, Ioan-Lucian Popa, and Saif ur Rehman assisted in reviewing and improving the final draft. All authors have read and approved the final version of the paper.

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