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Exploring Two New Combinatorial Identities and Their Applications to Catalan–Qi Numbers

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Abstract. In this paper, we establish two new and intriguing combinatorial identities, derived using two summation formulas for the terminating ${}_3F_2$ with unit argument, as introduced by Bailey. Notably, a combinatorial identity previously obtained by Wang and Sun is a special case of our main results. Furthermore, we present two new identities for the Catalan–Qi numbers. As applications, we explore connections to Catalan numbers, Super Catalan numbers, and the Wallis ratio. Moreover, we provide new formulas involving the Touchard's identity and Koshy's identity. Finally, we establish some integral representations involving the Catalan numbers.

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1. Introduction

Let

$$\mathbb{N} := 1, 2, 3, ..., \quad \mathbb{N}_0 := 0, 1, 2, 3, ...\} = \mathbb{N} \cup \{0\},$$

$$\mathbb{Z}_0^- := \{0, -1, -2, -3, ...\}, \quad \mathbb{Z}^- := \{-1, -2, -3, ...\} = \mathbb{Z}_0^- \setminus \{0\}.$$

Also, let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively, and let z be a complex variable. Then for real or complex parameters a and b, define the generalized binomial coefficients.

$$\binom{a}{b} = \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)} = \binom{a}{a-b} \quad (a, b \in \mathbb{C}), \tag{1}$$

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in which

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx,$$

denotes the well-known gamma function for $\mathcal{R}(z) > 0$.
For $x \in \mathbb{C}$ and $k \in \mathbb{R}$, Diaz and Pariguan [[1], p. 180] introduced the Pochhammer k-symbol as

$$(x)_{n,k} = x(x+k)(x+2k)...(x+(n-1)k), \quad n \geq 0. \tag{2}$$

In particular, we have [2–4],

$$(x)_{n,1} = (x)_n = \begin{cases} x(x+1)(x+2)...(x+n-1) & , n \geq 1, \\ 1 & , n = 0 \end{cases}$$

and

$$(-x)_n = (-1)^n(x-n+1)_n.$$

The relation (1) can be reduced to the particular case.

$$\binom{a}{n} = \frac{(-1)^n(-a)_n}{n!} \tag{3}$$

Furthermore, in terms of the Pochhammer symbol, the generalized hypergeometric series, which involves p parameters in the numerator and q parameters in the denominator, is defined in [5–8]

$${}_pF_q\left[\begin{array}{c} a_1,...,a_p \\ b_1,...,b_q \end{array}; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n...(a_p)_n}{(b_1)_n...(b_q)_n}\frac{z^n}{n!}, \tag{4}$$

where p and q are positive integers or zero (interpreting any empty product as 1), and we assume that the variable z of the series, the numerator parameters $a_1, ..., a_p$ and the denominator parameters $b_1, ..., b_q$ take complex values, provided that b_j for $j = 1, 2, .., q$ should not be zero or a negative integer. Further, assuming that none of the numerator parameters is zero or a negative integer (otherwise the question of convergence will not arise), and with the restriction that b_j $(j = 0, 1, ..., q)$ should not be zero or a negative integer, by well-known ratio tests, it can easily be seen that the generalized hypergeometric series defined in (4):
(i) convergent for all $|z| < \infty$ provided $p \leq q$,
(ii) converges for all $|z| < 1$ provided $p = q + 1$,
and **(iii)** diverges for all z, $z \neq 0$ provided $p > q + 1$.
Furthermore, if we set the following.

$$w = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j,$$

then, with the relation $p = q + 1$, the series (4) is seen to be
(i) absolutely convergent for $|z| = 1$ provided $\mathcal{R}(w) > 0$,
(ii) conditionally convergent for $|z| = 1$, $z \neq 1$, provided $-1 < \mathcal{R}(w) \leq 0$,

and **(iii)** divergent for $|z| = 1$ provided $\mathcal{R}(w) \leq -1$.
For more details on generalized hypergeometric series, we refer to the standard books [5–8].
It should be remarked here that wherever a generalized hypergeometric series reduces to the gamma function, the results are very important from the applications point of view. Thus, the classical summation theorems such as those of Gauss, Gauss second, Kummer and Bailey for the series ${}_2F_1$ (together with their respective terminating forms); The series ${}_3F_2$, along with others discussed in standard references like [5, 8], plays a crucial role in the theory of generalized hypergeometric series, as explored by Watson, Dixon, Whipple, and Saalschutz. However, in our current study, we will focus on the following classical summation theorem for the ${}_3F_2$ series, namely

- Classical Dixon summation theorem [6]

$$ {}_3F_2\left[\begin{array}{c} a, b, c \\ 1+a-b, 1+a-c \end{array}; 1\right] = \frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1-b-c+\frac{1}{2}a)}{\Gamma(1+a)\Gamma(1-b+\frac{1}{2}a)\Gamma(1-c+\frac{1}{2}a)\Gamma(1+a-b-c)} \tag{5} $$

provided $\mathcal{R}(a-2b-2c) > -2$.

- The Classical Watson Summation Theorem [6]

$$ {}_3F_2\left[\begin{array}{c} a, b, c \\ \frac{1}{2}(a+b+1), 2c \end{array}; 1\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(a+b+1)\right)\Gamma\left(c-\frac{1}{2}(a+b-1)\right)}{\Gamma\left(\frac{1}{2}(a+1)\right)\Gamma\left(\frac{1}{2}(b+1)\right)\Gamma\left(c-\frac{1}{2}(a-1)\right)\Gamma\left(c-\frac{1}{2}(b-1)\right)}, \tag{6} $$

provided $\mathcal{R}(2c-a-b) > -1$.

- Classical Whipple summation theorem [6]

$$ {}_3F_2\left[\begin{array}{c} a, 1-a, c \\ e, 2c-e+1 \end{array}; 1\right] = \frac{\pi 2^{1-2c}\Gamma(e)\Gamma(2c-e+1)}{\Gamma\left(\frac{1}{2}(a+e)\right)\Gamma\left(\frac{1}{2}(1-a+e)\right)\Gamma\left(c+1-\frac{1}{2}(a+e)\right)\Gamma\left(c+\frac{1}{2}(a-e+1)\right)}, \tag{7} $$

provided $\mathcal{R}(c) > 0$.

Remark 1. *It is not out of place to mention here that the above-mentioned classical summation theorems for the series ${}_2F_1$ and ${}_3F_2$ have been generalized and extended. For this, we refer to interesting research papers by Qi, et al. [9–11], Lavoie, et al. [12–14], Rakha and Rathie [15] and Kim et al. [16].*

In addition to this, by employing the classical hypergeometric summation theorem (6) in the following well-known transformation formula [17], i.e.

$$ {}_3F_2\left[\begin{array}{c} a, b, -n \\ e, f \end{array}; 1\right] = \frac{(f-b)_n}{(f)_n}{}_3F_2\left[\begin{array}{c} e-a, b, -n \\ e, 1+b-f-n \end{array}; 1\right] \tag{8} $$

by setting
(i) $e = \frac{1}{2}(a+b+1)$, $f = -2n$

and
(ii) $b = 1 - a$, $f = 1 - 2n - e$
respectively, we get the following two elegant terminating identities for the series ${}_3F_2$ with unit argument, due to Bailey [17], that is.

$$ {}_3F_2\left[\begin{matrix} a, b, -n \\ \frac{1}{2}(a+b+1), -2n \end{matrix} ; 1\right] = \frac{\left(\frac{1}{2}a+\frac{1}{2}\right)_n\left(\frac{1}{2}b+\frac{1}{2}\right)_n}{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}(a+b+1)\right)_n} \tag{9} $$

and

$$ {}_3F_2\left[\begin{matrix} a, 1-a, -n \\ e, 1-2n-e \end{matrix} ; 1\right] = \frac{2^{2n}\left(\frac{1}{2}+\frac{1}{2}e-\frac{1}{2}a\right)_n\left(\frac{1}{2}e+\frac{1}{2}a\right)_n}{(e)_{2n}}. \tag{10} $$

We would like to mention here that only a few hypergeometric identities for the terminating series are available in the literature.

On the other hand, in combinatorial mathematics, Catalan numbers form a sequence of natural numbers that appear in various counting problems, often involving recursively defined objects. The nth Catalan number can be expressed in terms of the central binomial coefficients

$$ C_n = \frac{1}{n+1}\binom{2n}{n}, \quad for \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}. \tag{11} $$

In terms of the hypergeometric function, it is represented by

$$ {}_2F_1\left[\begin{matrix} -n, -n+1 \\ 2 \end{matrix} ; 1\right], \quad for \quad n \in \mathbb{N}_0. \tag{12} $$

Recently, an alternative and analytical generalization of the Catalan numbers C_n, known in the literature as the Catalan–Qi numbers $C(a, b; n)$ was introduced and is defined by

$$ C(a, b; n) = \left(\frac{b}{a}\right)^n \frac{(a)_n}{(b)_n}, \tag{13} $$

for $\mathcal{R}(a) > 0$, $\mathcal{R}(b) > 0$ and for $n \in \mathbb{N}_0$.

It is not difficult to verify that

$$ C\left(\frac{1}{2}, 2; n\right) = C_n. \tag{14} $$

For a comprehensive study, we refer to Koshy's standard text [18] and related research papers [2, 19–29].
It is interesting to mention here that the generalization of the Catalan numbers known as the Super Catalan numbers is defined in [30–33] as follows:

$$ S_{m,n} = \frac{(2m)!(2n)!}{m!n!(m+n)!}, \qquad m, n \geq 0. \tag{15} $$

In particular, we have

$$\begin{aligned} S_{0,n} = &\ \frac{(2n)!}{(n!)^2}, \quad n \geq 0, \\ S_{1,n} = &\ 2\frac{(2n)!}{n!(n+1)!}, \quad n \geq 0, \\ S_{n,n} = &\ \frac{(2n)!}{(n!)^2}, \quad n \geq 0, \\ S_{m,n} = &\ S_{n,m}, \quad m, n \geq 0. \end{aligned}$$

It is clear that

$$S_{1,n} = 2C_n, \qquad n \geq 0. \tag{16}$$

It is well known that the Wallis ratio is defined by

$$W_n = \frac{(2n-1)!!}{(2n)!!} = \frac{(2n)!}{2^{2n}(n!)^2} = \frac{1}{\sqrt{\pi}}\frac{\Gamma(n+1/2)}{\Gamma(n+1)}, \ n \geq 0. \tag{17}$$

Hence, it is easy to see that

$$C_n = \frac{4^n}{n+1}W_n, \ n \geq 0. \tag{18}$$

The Wallis ratio has been extensively studied and applied by numerous mathematicians, as seen in [34], and the related literature.
The goal of this paper is to present two novel and general combinatorial identities, utilizing the identities (9) and (10). An intriguing combinatorial identity derived by Wang and Sun appears as a special case of our main results, which will be discussed in Section 2. Additionally, we introduce two new identities for Catalan–Qi numbers and, as an application, derive new identities for Catalan numbers using the relation (14). In a similar manner, we also present new findings for the super-Catalan numbers and the Wallis ratio. Moreover, we provide new formulas involving the Touchard's identity and Koshy's identity. Finally, we establish some integral representations involving the Catalan numbers.

2. Two new and general combinatorial identities

In this section, we establish two new and general combinatorial identities asserted in the following theorem.

Theorem 1. *The identities*

$$\sum_{k=0}^{n} \frac{2^k (a)_k (b)_k}{k!(a+b+1)_{k,2}} \binom{2n-k}{n} = \frac{(n+1)_n}{n!} \frac{(a+1)_{n,2}(b+1)_{n,2}}{(1)_{n,2}(a+b+1)_{n,2}} \tag{19}$$

and

$$\sum_{k=0}^{n} \frac{(a)_k (1-a)_k}{k!(e)_k} \binom{2n+e-1-k}{n+e-1} = \frac{(n+e)_n}{n!} \frac{(1+e-a)_{n,2}(e+a)_{n,2}}{(e)_{2n}}, \tag{20}$$

hold true

Proof. Denoting the left hand side of (19) by S, we have

$$S = \sum_{k=0}^{n} \frac{2^k (a)_k (b)_k}{k!(a+b+1)_{k,2}} \binom{2n-k}{n}.$$

Using the definition (3), it is not difficult to see that

$$\binom{2n-k}{n} = \frac{\Gamma(2n+1)}{\Gamma^2(n+1)} \frac{(-n)_k}{(-2n)_k}.$$

Thus, we have

$$S = \frac{\Gamma(2n+1)}{\Gamma^2(n+1)} \sum_{k=0}^{n} \frac{2^k (a)_k (b)_k (-n)_k}{k!(a+b+1)_{k,2}(-2n)_k}.$$

then,

$$S = \frac{\Gamma(2n+1)}{\Gamma^2(n+1)} \sum_{k=0}^{n} \frac{(a)_k (b)_k (-n)_k}{k!\left(\frac{1}{2}(a+b+1)\right)_k (-2n)_k}.$$

Summing up the series, we finally have

$$S = \frac{(n+1)_n}{n!} {}_3F_2 \left[\begin{matrix} a, b, -n \\ \frac{1}{2}(a+b+1), -2n \end{matrix} ; 1 \right].$$

We now note that the ${}_3F_2$ series can be evaluated using Bailey's result (9), which allows us to easily derive the right-hand side of (19). This concludes the proof of the result (19) stated in the theorem.
Denoting the left hand side of (20) by S', we have

$$S' = \sum_{k=0}^{n} \frac{(a)_k (1-a)_k}{k!(e)_k} \binom{2n+e-1-k}{n+e-1}.$$

Using the definition (3), it is not difficult to see that

$$\binom{2n+e-1-k}{n+e-1} = \frac{(n+e)_n}{n!} \frac{(-n)_k}{(1-2n-e)_k}.$$

Thus, we have

$$S' = \frac{(n+e)_n}{n!} \sum_{k=0}^{n} \frac{(a)_k (1-a)_k (-n)_k}{k!(e)_k (1-2n-e)_k}.$$

Summing up the series, we finally have

$$S' = \frac{(n+e)_n}{n!} {}_3F_2 \left[\begin{matrix} a, 1-a, -n \\ e, 1-2n-e \end{matrix} ; 1 \right].$$

We now note that the ${}_3F_2$ series can be evaluated using Bailey's result (10), which allows us to easily derive the right-hand side of (20). This concludes the proof of the result (20) stated in the theorem. The proof of this Theorem is now complete.

Corollary 1. *If we set $a=-n$ and $b=n+1$, in the result (19) or if we set $a=-n$ and $e=1$, in the result (20), then we have*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n-k}{n}\binom{n+k}{k}=\frac{(n+1)_n}{2^n(n!)^2}\frac{(1-n)_{n,2}(n+2)_{n,2}}{(1)_{n,2}} \tag{21}$$

which upon considering the even and odd cases, reduce the following interesting result

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n-k}{n}\binom{n+k}{k}=\begin{cases}(-1)^m\dfrac{(3m)!}{(m!)^3} & ,\textit{for } n=2m \textit{ (even)},\\ 0 & ,\textit{for } n=2m+1 \textit{ (odd)}.\end{cases} \tag{22}$$

3. Two new identities for the Catalan–Qi–numbers

In this section, we establish two new identities for the Catalan–Qi–numbers.

Theorem 2. *For a and b are complex numbers, and for $n\geq 0$, the identities for the Catalan–Qi numbers.*

$$\begin{aligned}\sum_{k=0}^{n}\binom{2n-k}{n}\Big(\frac{2ab}{a+b+1}\Big)^k C\Big(b,\frac{a+b+1}{2};k\Big)C(a,1;k)&\\ =\frac{(n+1)_n}{n!}\Big(1+\frac{ab}{a+b+1}\Big)^n C\Big(\frac{b+1}{2},\frac{a+b+1}{2};n\Big)C\Big(\frac{a+1}{2},\frac{1}{2};n\Big)&\end{aligned} \tag{23}$$

are valid for $\mathcal{R}(a),\mathcal{R}(b)>0$
and

$$\begin{aligned}\sum_{k=0}^{n}\binom{2n+e-1-k}{n+e-1}\Big(\frac{a(1-a)}{e}\Big)^k C(1-a,1;k)C(a,e;k)&\\ =\frac{(n+e)_n}{n!}\Big(\frac{(1+e-a)(e+a)}{e(e+1)}\Big)^n C\Big(\frac{e-a+1}{2},\frac{e+1}{2};n\Big)C\Big(\frac{e+a}{2},\frac{e}{2};n\Big)&\end{aligned} \tag{24}$$

are valid for $0<\mathcal{R}(a)<1$ and $\mathcal{R}(e)>0$.

Proof. Applying the relation (13) to (19), we obtain the result (23).
Using the identity

$$(a)_{2n}=2^{2n}\Big(\frac{1}{2}a\Big)_n\Big(\frac{1}{2}a+\frac{1}{2}\Big)_n$$

and relation (13) in (20), we have the result (24). For simplicity, we prefer to omit the details. The proof of Theorem 2 is now complete.

Corollary 2. *The results for the Catalan numbers*

$$\sum_{k=0}^{n} \binom{2n-k}{n} \frac{(5)_{k,2}}{8^k k!} C_k = \frac{(7)_{n,4}(3)_{n,4}}{8^n n!(1)_{n,2}} C_n \tag{25}$$

and

$$\sum_{k=0}^{n} \binom{2n+1-k}{n+1} \frac{k+1}{16^k} C_k^2 = \frac{(5)_{n,4}^2(2n+1)}{8^n n!(3)_{n,2}} C_n, \tag{26}$$

holds for $n \geq 0$.

Proof.
The result follows from (23) by noting that when $a = \frac{5}{2}$ and $b = \frac{1}{2}$, we have by (14) the relation (25). The result follows from (24) by noting that when $a = \frac{1}{2}$ and $e = 2$, we have by (14) the relation (26).

Corollary 3. *The results for the Super Catalan numbers*

$$\sum_{k=0}^{n} \binom{2n-k}{n} \frac{(5)_{k,2}}{8^k k!} S_{1,k} = \frac{(7)_{n,4}(3)_{n,4}}{8^n n!(1)_{n,2}} S_{1,n} \tag{27}$$

and

$$\sum_{k=0}^{n} \binom{2n+1-k}{n+1} \frac{k+1}{16^k} S_{1,k}^2 = \frac{2(5)_{n,4}^2(2n+1)}{8^n n!(3)_{n,2}} S_{1,n}, \tag{28}$$

holds for $n \geq 0$.

Proof.
Using the relation (16) to equation (25) we obtain the result (27). Using the relation (16) to equation (26) we obtain the result (28). The required proof is now complete.

Corollary 4. *The results of the Wallis ratio*

$$\sum_{k=0}^{n} \binom{2n-k}{n} \frac{(5)_{k,2}}{2^k (k+1)!} W_k = \frac{(7)_{n,4}(3)_{n,4}}{2^n (n+1)!(1)_{n,2}} W_n \tag{29}$$

and

$$\sum_{k=0}^{n} \binom{2n+1-k}{n+1} \frac{1}{k+1} W_k^2 = \frac{(2n+1)(5)_{n,4}^2}{2^n (n+1)!(3)_{n,2}} W_n, \tag{30}$$

holds true for $n \geq 0$.

Proof.
Using the relation (18) to equation (25) we obtain the result (27). Using the relation (18) to equation (26) we obtain the result (28). The proof of Corollary 4 is now complete.

Now, we will present two important mathematical identities: Touchard's identity and Koshy's identity [18]. Touchard's identity is a powerful tool in analyzing algebraic structures, enabling precise expression of coefficients. On the other hand, Koshy's identity is used in coefficient calculations and analysis of sets. Touchard's identity can be expressed as

$$C_{n+1} = \sum_{r=0}^{n} 2^{n-2r} \binom{n}{2r} C_r, \tag{31}$$

Koshy's identity expresses the following relationship:

$$C_n = \sum_{r=1}^{n} (-1)^{r-1} \binom{n-r+1}{r} C_{n-r}. \tag{32}$$

These two identities are considered cornerstones in the mathematical literature because of their broad applications in many areas of mathematics.

Theorem 3. *The result for the Touchard's identity*

$$\sum_{k=0}^{n} \binom{2n-k}{n} \frac{(5)_{k,2}}{8^k k!} C_k = \frac{(7)_{n,4}(3)_{n,4}}{8^n n!(1)_{n,2}} \frac{(n+2)}{(2n+1)} \sum_{r=0}^{n} 2^{n-2r-1} \binom{n}{2r} C_r \tag{33}$$

and

$$\sum_{k=0}^{n} \binom{2n+1-k}{n+1} \frac{k+1}{16^k} C_k^2 = \frac{(5)_{n,4}^2 (n+2)}{8^n n!(3)_{n,2}} \sum_{r=0}^{n} 2^{n-2r-1} \binom{n}{2r} C_r, \tag{34}$$

holds for $n \geq 0$.

Proof. Using the following recurrence relation for the Catalan numbers:

$$C_0 = 1 \quad and \quad C_{n+1} = \frac{2(2n+1)}{n+2} C_n, \quad n \geq 0,$$

in right-hand side of equations (25) and (26), we obtain equations (33) and (34).

Theorem 4. *The result for the Koshy's identity*

$$\sum_{k=0}^{n} \binom{2n-k}{n} \frac{(5)_{k,2}}{8^k k!} C_k = \frac{(7)_{n,4}(3)_{n,4}}{8^n n!(1)_{n,2}} \sum_{r=1}^{n} (-1)^{r-1} \binom{n-r+1}{r} C_{n-r} \tag{35}$$

and

$$\sum_{k=0}^{n} \binom{2n+1-k}{n+1} \frac{k+1}{16^k} C_k^2 = \frac{(5)_{n,4}^2 (2n+1)}{8^n n!(3)_{n,2}} \sum_{r=1}^{n} (-1)^{r-1} \binom{n-r+1}{r} C_{n-r}, \tag{36}$$

holds true for $n \geq 0$.

Proof. By applying Koshy's identity (32), to the right-hand side of equations (25) and (26), we have the results (35) and (36).

4. Integral representations involving the Catalan numbers

In this section, we establish some integral representations involving Catalan numbers.

Theorem 5 ([35, Theorem 1.4]). *The Catalan numbers C_n for $n \geq 0$, can be represented by:*

$$C_n = \frac{1}{\pi} \int_0^\infty \frac{\sqrt{x}}{(x+\frac{1}{4})^{n+2}} dx = \frac{2}{\pi} \int_0^\infty \frac{x^2}{(x^2+\frac{1}{4})^{n+2}} dx. \tag{37}$$

Corollary 5. *The following results hold.*

$$\sum_{k=0}^{n} \binom{2n-k}{n} \frac{(5)_{k,2}}{8^k k!} C_k = \frac{(7)_{n,4}(3)_{n,4}}{8^n n!(1)_{n,2}} \frac{1}{\pi} \int_0^\infty \frac{\sqrt{x}}{(x+\frac{1}{4})^{n+2}} dx = \frac{(7)_{n,4}(3)_{n,4}}{8^n n!(1)_{n,2}} \frac{2}{\pi} \int_0^\infty \frac{x^2}{(x^2+\frac{1}{4})^{n+2}} dx \tag{38}$$

and

$$\sum_{k=0}^{n} \binom{2n+1-k}{n+1} \frac{k+1}{16^k} C_k^2 = \frac{(5)_{n,4}^2(2n+1)}{8^n n!(3)_{n,2}} \frac{1}{\pi} \int_0^\infty \frac{\sqrt{x}}{(x+\frac{1}{4})^{n+2}} dx = \frac{(5)_{n,4}^2(2n+1)}{8^n n!(3)_{n,2}} \frac{2}{\pi} \int_0^\infty \frac{x^2}{(x^2+\frac{1}{4})^{n+2}} dx, \tag{39}$$

holds true for $n \geq 0$.

Proof. Applying the integral representations for the Catalan numbers of Theorem 5, on the left-hand side of (25) and (26), we obtain the results (38) and (39).

Theorem 6 ([36, Proposition 4.1]). *For $n \geq 0$, the Catalan numbers C_n can be represented by:*

$$C_n = \frac{2^{2n+5}}{\pi} \int_0^1 \frac{x^2(1-x^2)^{2n}}{(1+x^2)^{2n+3}} dx. \tag{40}$$

Corollary 6. *The following results hold.*

$$\sum_{k=0}^{n} \binom{2n-k}{n} \frac{(5)_{k,2}}{8^k k!} C_k = \frac{(7)_{n,4}(3)_{n,4}}{n!(1)_{n,2}} \frac{2^{5-n}}{\pi} \int_0^1 \frac{x^2(1-x^2)^{2n}}{(1+x^2)^{2n+3}} dx \tag{41}$$

and

$$\sum_{k=0}^{n} \binom{2n+1-k}{n+1} \frac{k+1}{16^k} C_k^2 = \frac{(5)_{n,4}^2(2n+1)}{n!(3)_{n,2}} \frac{2^{5-n}}{\pi} \int_0^1 \frac{x^2(1-x^2)^{2n}}{(1+x^2)^{2n+3}} dx, \tag{42}$$

holds for $n \geq 0$.

Proof. Using the integral representations for the Catalan numbers in Theorem 6, on the left-hand side of (25) and (26), we obtain the results (41) and (42).

Concluding Remark: In this paper, we establish combinatorial identities by applying a hypergeometric series transformation attributed to Bailey. This result not only highlights the deep interplay between hypergeometric series and combinatorial structures but also provides a novel perspective on summation techniques and special function identities. Additionally, we derive new combinatorial and analytical identities for the Catalan numbers, which play a crucial role in various branches of discrete mathematics. As an application, we extend our findings to obtain new expressions for the Super Catalan numbers, which generalize the classical Catalan numbers and frequently arise in enumerative combinatorics and representation theory. Furthermore, we explore their connection to the Wallis ratio. Moreover, we establish novel algebraic and combinatorial formulations involving Touchard's identity and Koshy's identity. Finally, we derive integral representations associated with the combinatorial identities.

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