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A Self-Adaptive Inertial Subgradient Extragradient Method for Approximating Common Solutions of Bilevel Equilibrium and Inclusion Problems in Hilbert Spaces

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Abstract. In this paper, we study bilevel equilibrium (with a pseudomonotone bifunction) and inclusion problems in the framework of Hilbert spaces. We propose an inertial subgradient extragradient algorithm with self-adaptive step size for finding common solutions to the aforementioned problems. Unlike several existing works on inclusion problem in the literature, our underlying operator is monotone and Lipschitz continuous. Also, we employ the inertial technique to accelerate the rate of strong convergence of the sequences generated by our proposed method. Moreover, the implementation of our proposed method does not require prior knowledge of the Lipschitz constant of the monotone operator. Furthermore, we give some numerical examples to illustrate the efficiency of our proposed algorithm.

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Key Words and Phrases: Bilevel equilibrium problem, pseudomonotone operator, monotone operator, inertial technique, subgradient extragradient

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1. Introduction

Let C be a nonempty, closed and convex subset of a real Hilbert space H. The Variational Inclusion Problem (VIP) is defined as finding $x \in C$ such that

$$0 \in (B + D)x, \tag{1}$$

where $D : H \to 2^H$ is a set-valued mapping and $B : H \to H$ is a single-valued mapping. We denote the solution set of (1) by $(B + D)^{-1}(0)$. Numerous problems in image processing, machine learning, and linear inverse problems can be modeled mathematically as problem (1) (for instance, see [1–5]). The forward-backward splitting method (see [6–12]) is a widely recognized approach for solving the inclusion problem (1) and it is defined as follows:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = J_r^D(x_n - rBx_n), r > 0, n \geq 1, \end{cases}$$

where $J_r^D = (I + rD)^{-1}$ is the resolvent of D and I is the identity operator on H. Alvarez and Attouch [13] employed the heavy ball method for approximating the zeros of maximal monotone operators by using the inertial proximal point algorithm which is given as follows:

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (I + r_n D)^{-1} y_n, \forall n \geq 1. \end{cases}$$

A weak convergence result was obtained under the condition that $\{r_n\}$ is nondecreasing and $\{\theta_n\} \subset [0, 1)$ with

$$\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty. \tag{2}$$

Furthermore, Moudafi and Oliny [14] introduced the following inertial forward backward algorithm for solving zero problem for the sum of two monotone operators:

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (I + r_n D)^{-1}(y_n - r_n B y_n), n \geq 1. \end{cases} \tag{3}$$

They established that the sequence generated by (3) converges weakly provided that $r_n < \frac{2}{L}$, where L is the Lipschitz constant of B and (2) is satisfied.
Moreover, the following inertial forward-backward splitting algorithm was proposed by Cholamjiak et al. [15] in 2018 for solving variational inclusion problem (VIP) in the framework of Hilbert spaces:

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = \alpha_n v + \beta_n y_n + \sigma_n J_{r_n}^D(y_n - r_n B y_n), n \geq 1, \end{cases}$$

where $B : H \to H$ and $D : H \to 2^H$ are inverse-strongly monotone and maximal monotone , respectively, $J_{r_n}^D = (I + r_n D)^{-1}, \{\alpha_n\}, \{\beta_n\}, \{\sigma_n\} \subset [0, 1]$, and $\alpha_n + \beta_n + \sigma_n = 1$. A strong convergence result was established under some suitable conditions.

On the other hand, let C be a nonempty, closed and convex subset of real Hilbert space H, and let $f : C \times C \to \mathbb{R}$ be a bifunction such that $f(x, x) = 0$ for all $x \in H$. The Equilibrium Problem associated with f and C is defined as follows: Find $x \in C$ such that

$$f(x, y) \geq 0, \forall y \in C. \tag{4}$$

We denote the set of solutions to the equilibrium problem (4) by $EP(f, C)$. It is well known that the equilibrium problem is a generalization of several other optimization problems which include variational inequality problem, fixed point problem, convex minimization problem, Nash equilibrium problem (see [16, 17]). Several methods such as projection methods, Bregman distance method, proximal method, regularization methods, etc., (see [18, 19] and references therein) have been proposed for solving EP (4) and several other related optimization problems.

The Bilevel Equilibrium Problem (BEP) introduced by Chadli et al. [20] in 2000 is defined as follows: Find $x \in EP(f, C)$ such that

$$g(x, y) \geq 0, \forall y \in EP(f, C), \tag{5}$$

$g : C \times C \to \mathbb{R}$ is a bifunction. The solution set of (5) is denoted by Γ i.e. $\Gamma = \{x \in EP(f, C) : g(x, y) \geq 0 \ \forall y \in EP(f, C)\}$. The BEP has recently attracted the interest of several authors, hence, iterative algorithms have been proposed for solving it (see [1, 18, 21] and the references there in).

In 2014, Quy [22] proposed an algorithm by combining proximal method and Halpern's method to establish a strong convergece result for approximating a common solution to bilevel equilibrium and fixed point problems in real Hilbert spaces.

In 2018, the following extragradient method for solving BEP (5) was introduced by Yuying et al. [21] :

Algorithm 1.

Step 0: *Set* $x_0 \in H, \{\alpha_n\} \subset [0, 1], 0 < \lambda < \frac{2\eta}{c^2}, \{\delta_n\}, \{\gamma_n\}$ *satisfying the following*

$$\begin{cases} \lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty, \\ 0 \leq \delta_n \leq 1 - \alpha_n, \lim_{n\to\infty} \delta_n = \delta < 1, \forall n \geq 0, \\ 0 \leq \gamma < \gamma_n < \psi < \min\left(\frac{1}{2c_1}, \frac{1}{2c_2}\right). \end{cases}$$

Set $n = 0$ *and go to Step 1.*

Step 1: *Compute*

$$s_n = \arg\min\left\{\gamma_n g(x_n, y) + \frac{1}{2}\|y - x_n\|^2 : y \in C\right\},$$
$$t_n = \arg\min\left\{\gamma_n g(s_n, y) + \frac{1}{2}\|y - x_n\|^2 : y \in C\right\},$$

Step 2: *Compute* $w_n \in \delta_2 f(t_n, t_n)$ *and*

$$x_{n+1} = \delta_n x_n + (1 - \delta_n)u_n - \alpha_n \lambda w_n,$$

*Set $n := n + 1$ and go back to **Step 1**.*

where f satisfies strong monotonicity and g satisfies pseudomonotonicity and Lipschitz-like condition. Under some appropriate conditions, the authors established that the sequence generated by the Algorithm 1 converges strongly to a solution of BEP (5). We note that the step size in the Algorithm 1 depends on the prior estimate of the Lipschitz-like constants a_1 and a_2.

Furthermore, the following subgradient extragradient method was proposed by Anh and An [23] for solving BEP (5):

$$\begin{cases} s_n = \arg\min\left\{\gamma_n g(x_n, y) + \frac{1}{2}\|y - x_n\|^2 : y \in C\right\}, \\ t_n = \arg\min\left\{\gamma_n g(y_n, y) + \frac{1}{2}\|z - x_n\|^2 : z \in D_n\right\}, \\ x_{n+1} = \arg\min\left\{\beta_n f(t_n, y) + \frac{1}{2}\|t - z_n\|^2 : t \in C\right\}, \end{cases} \tag{6}$$

where $D_n = \{v \in D : \langle x_n - \gamma_n w_n - s_n, v - s_n\rangle \leq 0\}, w_n \in \delta_2 g(x_n, s_n), \{\gamma_n\}$ and $\{\beta_n\}$ are two nonnegative sequences.

Li [24] introduced the following iterative algorithms for solving pseudomonotone equilibrium problem :

Algorithm 2.

Step 0: *Select $x_0, y_0, y_1 \in C$ and the parameters $\beta > 0$ and $\lambda > 0$, put $n = 0$.*

Step 1: *for a given x_n, solve the strongly convex problems :*

$$\begin{cases} u_n = \arg\min\left\{\beta g(x_n, y) + \frac{1}{2}\|x_n - y\|^2 : y \in C\right\}, \\ t_n = \arg\min\left\{\beta g(u_n, t) + \frac{1}{2}\|x_n - y\|^2 : t \in C_n\right\}, \end{cases} \tag{7}$$

where $C_n = \{v \in H : \langle x_n - \beta w_n - u_n, v - u_n\rangle \leq 0\}$ with $w_n \in \delta_2 g(x_n, u_n)$.

Step 2: *Solve the strongly convex problems:*

$$\begin{cases} x_{n+1} = \arg\min\left\{\lambda f(y_n, y) + \frac{1}{2}\|x_n - y\|^2 : y \in H_n\right\}, \\ y_{n+1} = \arg\min\left\{\lambda f(y_n, y) + \frac{1}{2}\|x_{n+1} - y\|^2 : y \in T_n\right\}, \end{cases} \tag{8}$$

where $H_n = \{z \in H : \langle x_n - \lambda v_n - y_n, z - y_n\rangle \leq 0\}$ with $v_n \in \delta_2 f(y_{n-1}, y_n), T_n = \{z \in H : \|z - t_n\| \leq \|z - x_n\|\}$

Step 3: *If $x_{n+1} = y_n = x_n = u_n$, then the algorithm stops, $x_n \in \Omega$; otherwise, set $n = n + 1$ and return to Step 1.*

Algorithm 3.

Step 0: *Select* $x_0, y_{-1}, y_0 \in C$ *and the parameters* $\beta > 0$ *and* $\lambda > 0$*, put* $n = 0$*.*
Step 1: *for a given* x_n*, solve the strongly convex problems :*

$$\begin{cases} u_n = \arg\min\left\{\beta g(x_n, y) + \frac{1}{2}\|x_n - y\|^2 : y \in C\right\}, \\ t_n = \arg\min\left\{\beta g(u_n, t) + \frac{1}{2}\|x_n - y\|^2 : t \in C_n\right\}, \end{cases} \tag{9}$$

where $C_n = \{v \in H : \langle x_n - \beta w_n - u_n, v - u_n\rangle \leq 0\}$ *with* $w_n \in \delta_2 g(x_n, u_n)$*.*
Step 2: *Solve the strongly convex problems:*

$$\begin{cases} x_{n+1} = \arg\min\left\{\lambda f(y_n, y) + \frac{1}{2}\|x_n - y\|^2 : y \in H_n\right\}, \\ y_{n+1} = \arg\min\left\{\lambda f(y_n, y) + \frac{1}{2}\|x_{n+1} - y\|^2 : y \in T_n\right\}, \end{cases} \tag{10}$$

where $H_n = \{z \in H : \langle x_n - \lambda v_n - y_n, z - y_n\rangle \leq 0\}$ *with* $v_n \in \delta_2 f(y_{n-1}, y_n), T_n = \{z \in H : \|z - t_n\| \leq \|z - x_n\|\}$*.*

$$\lambda_{n+1} = \begin{cases} \lambda_n, & f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n) - f(y_n, x_{n+1}) \leq 0, \\ \min\left\{\lambda, \frac{\mu(\|y_n - y_{n-1}\|^2 + \|y_n - x_{n+1}\|^2)}{f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n) - f(y_n, x_{n+1})}\right\} & \textit{otherwise.} \end{cases} \tag{11}$$

Step 3: *Modify* β_{n+1} *by the following formula:*

$$\beta_{n+1} = \begin{cases} \beta_n, & g(x_n, t_n) - g(x_n, u_n) - g(u_n, t_n) \leq 0, \\ \min\left\{\beta_n, \frac{\mu(\|u_n - x_n\|^2 + \|t_n - x_n\|^2)}{g(x_n, t_n) - g(x_n, u_n) - g(u_n, t_n)}\right\} & \textit{otherwise} \end{cases} \tag{12}$$

Step 4: *If* $x_{n+1} = y_{n+1} = x_n = y_n$*, then the algorithm stops,* $x_n \in \Omega$*; otherwise, set* $n = n + 1$ *and return to Step 1.*

Using the above methods, they proved that the sequence of iterates converges weakly when the associated operators f and g are pseudomonotone bifunctions.

The following Algorithm 4 proposed by [21] is an improvement on Agorithm 1, although the two algorithms necessitate calculating two projections onto the feasible set C for each iteration, which can be computationally demanding if the set C is complex. In addition, the line search procedure in [21] involves an inner iteration that requires extra computation and execution time.

Algorithm 4.

Initialization: *Select* $x_0 \in H, 0 < \mu < \frac{2\eta}{c^2}$*, the sequences* $\{\alpha_n\} \subset (0, 1), \{\delta_n\}$ *and* $\{\gamma_n\}$ *are such that*

$$\begin{cases} \lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=0}^{\infty} \alpha_n = \infty, \\ 0 \leq \delta_n \leq 1 - \alpha_n, \forall n \geq 0, \qquad \lim_{n\to\infty} \delta_n = \delta < 1, \\ 0 <\leq \gamma \leq \gamma_n \leq \bar{\gamma} < \min\left(\frac{1}{2a_1}, \frac{1}{2a_2}\right). \end{cases}$$

Set $n = 0$ and go to Step 1.

Step 1. *Compute*

$$\begin{cases} y_n = \arg\min\left\{\gamma_n g(x_n, y) + \frac{1}{2}\|y - x_n\|^2 : y \in C\right\}, \\ z_n = \arg\min\left\{\gamma_n g(y_n, y) + \frac{1}{2}\|y - x_n\|^2 : y \in C_n\right\}, \end{cases}$$

Step2. *Compute $u_n \in \partial_2 f(z_n, z_n)$ and*

$$x_{n+1} = \delta_n x_n + (1 - \delta_n) z_n - \alpha_n \mu u_n.$$

Set $n = n + 1$ and go back to Step 1.

In light of the above, this paper presents a novel inertial self-adaptive subgradient extragradient method for solving a common solution of BEP and inclusion problems in real Hilbert spaces. Additionally, our algorithm uses only one strongly convex optimization problem within the feasible set, while the second level optimization problem is solved over a constructible half-space that can be easily computed using established techniques in convex optimization. Moreover, the algorithm's stepsize is determined by a step-adaptive process, eliminating the need for prior estimates of the Lipschitz-like constants. We establish a strong convergence theorem for approximating the solution of BEP and inclusion problem in real Hilbert spaces. Furthermore, we present numerical experiments to demonstrate the efficiency of the proposed method and the results outlined in our article build upon and broaden several related findings in the existing literature.

We emphasize the features provided by our proposed algorithm as follows:

(i) We employ the inertial technique to accelerate the rate of convergence of our proposed algorithm.

(ii) Our convergence analysis is not established under two-cases approach which is commonly used.

(iii) Our cost operator of bilevel equilibrium problem is pseudomonotone which is more general than monotone.

(iv) We solve a common solution of BEP and inclusion problem.

(v) Our algorithm focuses solely on one strongly convex optimization problem within the feasible set, while the second optimization problem is addressed over a constructible half-space, which can be readily computed using well-established techniques in convex optimization.

(vi) We obtain a strong convergence result unlike in [24] where a weak convergence result was obtained.

2. Preliminaries

In this section, we present some useful definitions and lemmas required to establish our result. Let $x_n \to x$ denotes strong convergence and $x_n \rightharpoonup x$ denotes weak convergence. Let C be a nonempty closed convex subset of a real Hilbert space H.

Definition 1. *Let $f : C \times C \to \mathbb{R}$ be a bifunction. Then f is said to be*

(i) monotone on C if

$$f(x, y) + f(y, x) \leq 0, \forall x, y \in C;$$

(ii) pseudomonotone on C if

$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0, \forall x, y \in H.$$

Definition 2. *Let C be a nonempty subset of a real Hilbert space H. A function $f : H \to \mathbb{R}$ is said to be convex if*

$$f(cx + (1 - c)y) \leq cf(x) + (1 - c)f(y),\ \forall\ c \in (0, 1), x, y \in C,$$

Definition 3. *A subdifferential ∂f of $f : H \to \mathbb{R}$ is defined by*

$$\partial f(x) := \{v \in H : f(y) \geq f(x) + \langle v, y - x\rangle, \quad x, y \in H\}$$

Definition 4. *Let $f : C \to \mathbb{R} \cup \{+\infty\}$ be a function.*

(i) The effective domain of f, denoted by $domf$ is defined by

$$domf := \{x \in H : f(x) < +\infty\};$$

(ii) f is lower semicontinuous at $x_0 \in domf$ if and only if

$$f(x_0) \leq \liminf_{x \to x_0} f(x);$$

(iii) f is upper semicontinuous at $x_0 \in domf$ if and only if

$$f(x_0) \geq \liminf_{x \to x_0} f(x).$$

Definition 5. *A mapping $A : H \to H$ is said to be*

(i) monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall \ \ x, y \in H;$$

(ii) $\beta-$strongly monotone if for all $\beta > 0$

$$\langle Ax - Ay, x - y\rangle \geq \beta\|x - y\|^2, \ \ \forall x, y \in H;$$

(iii) *α-inverse strongly monotone if there exists a constant $\alpha > 0$ such that*

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in H;$$

(iv) *nonexpansive if*

$$\|Ax - Ay\| \leq \|x - y\|, \forall x, y \in H;$$

(v) *firmly nonexpansive if*

$$\langle Ax - Ay, x - y \rangle \geq \|Ax - Ay\|^2, \quad \forall \ \ x, y \in H;$$

(vi) *Lipschitz continuous if for all $L > 0$*

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in H;$$

if $0 \leq L < 1$, then A is a contraction mapping.

It is known that nonexpansive mappings are 1-Lipschitz continuous mappings. Moreover, inverse strongly monotone mappings are monotone and Lipschitz continuous. However, the converse does not hold.

Definition 6. *Let $B : H \to 2^H$ be a multivalued operator on H. Then*

(i) *The effective domain of B denoted by $D(B)$ is given as $D(B) = \{x \in H : Bx \neq \emptyset\}$.*

(ii) *the graph $G(B)$ is defined by*

$$G(B) := \{(x, u) \in H \times H : u \in B(x)\};$$

(iii) *the operator B is said to be monotone if $\langle x - y, u - v \rangle \geq 0$ for all $x, y \in D(B), u \in Bx$, and $v \in By$;*

(iv) *a monotone operator B on H is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on H;*

(v) *for a maximal monotone multivalued mapping B on H and $r > 0$, the operator*

$$J_r^B := (I + rB)^{-1} : H \to D(B)$$

is called the resolvent of B, where I is the identity operator on H. It is known that for any $r > 0$, the resolvent mapping J_r^B is single-valued and firmly nonexpansive.

For all $x, y \in H$, the following inequalities hold:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \tag{13}$$

and

$$\|x + y\|^2 = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2. \tag{14}$$

Let $N_C(x)$ be the normal cone at $x \in C$. Then,

$$N_C(x) = \{w \in H : \langle w, x - y \rangle \geq 0, \forall y \in C\}. \tag{15}$$

Lemma 1. *[25] Let $B : H \to 2^H$ be a maximal monotone mapping and $A : H \to H$ be a Lipschitz continuous and monotone mapping. Then the mapping $A + B$ is maximal monotone.*

Lemma 2. *[26] Let C be a nonempty closed convex subset of a real Hilbert space H and $\phi : C \to \mathbb{R}$ be a convex and subdifferentiable function on C. Then x^* is a solution to the convex problem : minimize$\{\phi(x) : x \in C\}$ if and only if $0 \in \partial\phi(x^*) + N_C(x^*)$, where $\partial\phi(x)$ denotes the subdifferential of ϕ and $N_C(x^*)$ is the normal cone of C at x^*.*

Lemma 3. *[27] Let $\{c_n\}$ be a sequence of non-negative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0, 1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\{b_n\}$ be a sequence of real numbers. Suppose that*

$$c_{n+1} \leq (1 - \alpha_n)c_n + \alpha_n b_n, n \geq 1.$$

If $\limsup_{k\to\infty} b_{n_k} \leq 0$ for every subsequence $\{c_{n_k}\}$ of $\{c_n\}$ satisfying the condition $\liminf_{k\to\infty}(c_{n_k+1} - c_{n_k}) \geq 0$, then $\lim_{n\to\infty} c_n = 0$.

Lemma 4. *[21] Let $f : H \times H \to \mathbb{R}$ be an η-strongly monotone bifunction and satisfies c-Lipschitz continuous condition i.e. for each $x, y \in H, u \in \partial_2 f(x, \cdot)(x)$ and $v \in \partial_2 f(y, \cdot)(y)$. Suppose $0 < \alpha \leq 1, 0 \leq \partial \leq 1 - \alpha$ and $0 < \lambda < \frac{2\eta}{c^2}$, then*

$$\|(1 - \delta)x - \alpha\lambda u - [(1 - \delta)y - \alpha\lambda v]\| \leq (1 - \delta - \alpha\tau)\|x - y\|,$$

where $\tau = 1 - \sqrt{1 - \lambda(2\eta - \lambda c^2)} \in (0, 1]$.

Lemma 5. *[28] Let $\{a_n\}, \{c_n\} \subset \mathbb{R}_+, \{\sigma_n\} \subset (0, 1)$ and $\{b_n\} \subset \mathbb{R}$ be sequences such that*

$$a_{n+1} \leq (1 - \sigma_n)a_n + b_n + c_n, \forall n \geq 0.$$

Assume $\sum_{n=0}^{\infty} |c_n| < \infty$. Then the following results hold:

(i) If $b_n \leq \beta\sigma_n$ for some $\beta \geq 0$, then $\{a_n\}$ is a bounded sequence,

(ii) If we have

$$\sum_{n=0}^{\infty} \sigma_n = \infty \quad \text{and} \quad \limsup_{n\to\infty} \frac{b_n}{\sigma_n} \leq 0,$$

then $\lim_{n\to\infty} a_n = 0$.

3. Proposed Method

In this section, we highlight some assumptions required to establish our strong convergence result, and then we present our proposed method.

Assumption 1. *Let $f : H \times H \to \mathbb{R}$ be a bifunction that satisfies the following:*

(i) $f(\cdot, y)$ is weakly upper semicontinuous on H for every fixed $y \in C$;

(ii) *f is β- strongly monotone on H;*

(iii) *$f(x,\cdot)$ is convex, weakly lower semicontinuous and subdifferentiable on H for every fixed $x \in C$;*

(iv) *f is c-Lipschitz continuous, i.e. for every $x, y \in H$, there exists a constant $c > 0$ such that*

$$\digamma(\delta_2 f(x,x), \delta_2 f(y,y)) \leq c\|x-y\|^2,$$

where the Hausdorff metric $\digamma$ on H is defined by

$$\digamma(A,B) := \max\{\sup d(x,B), \sup d(x,A)\},$$

$\forall x \in A$, $\forall y \in B$ and $\forall A, B \in CB(H)$ ($CB(H)$ is the family of nonempty closed bounded subsets of H).

Assumption 2. *Let $g : H \times H \to \mathbb{R}$ be a bifunction such that the following are satisfied:*

(i) *g is convex, weakly lower semicontinuous, and subdifferentiable on H for every fixed $x \in H$;*

(ii) *g is pseudomonotone on C with respect to $Ep(f,C)$ i.e.*

$$g(x,q) \leq 0, \forall x \in C, q \in Ep(f,C);$$

(iii) *$g(\cdot, y)$ is weakly upper semicontinuous on H for every fixed $y \in C$;*

(iv) *g is Lipschitz-type continuous i.e. there exists two positive constants a_1 and a_2 such that*

$$g(x,y) + g(y,z) \geq g(x,z) - a_1\|x-y\|^2 - a_2\|y-z\|^2, \forall x,y,z \in H.$$

It is worthy to note that the solution set $Sol(g,C)$ is closed and convex when the bifunction g satisfies Assumption 2(i), (ii) and (iii).

Assumption 3.

(i) *Let $\{\epsilon_n\}$ be a positive sequence such that $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$, where $\{\alpha_n\}, \{\epsilon_n\} \subset (0,1)$;*

(ii) $\lim_{n\to\infty}\alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $0 < \delta_n \leq 1 - \alpha_n, \lim_{n\to\infty} \delta_n = \delta < 1, \{\delta_n\} \subset (0,1), \forall n \geq 0$;

(iv) *The solution set $\Omega = \Gamma \cap (B+D)^{-1}(0) \neq \emptyset$*

Next, we present our proposed method as follows:

Algorithm 5.

Step 0: *Set* $x_0, x_1 \in H, \gamma_1, r_1 > 0, \mu, \theta, \rho \in (0,1), 0 < \lambda < \frac{2\eta}{c^2}$.
Set $n = 1$ *and go to Step 1.*
Step 1: *Given* $(n-1)th$ *and* nth *iterates, compute*

$$w_n = x_n + \theta_n(x_n - x_{n-1}),$$

where

$$\theta_n = \begin{cases} \min\left\{\frac{\epsilon_n}{\|x_n - x_{n-1}\|}, \theta\right\}, & \text{if } x_n \neq x_{n-1} \\ \theta & \text{otherwise} \end{cases} \tag{16}$$

Step 2: *Compute*

$$s_n = \arg\min\left\{\gamma_n g(w_n, y) + \frac{1}{2}\|w_n - y\|^2 : y \in C\right\},$$
$$t_n = \arg\min\left\{\gamma_n g(s_n, y) + \frac{1}{2}\|w_n - y\|^2 : y \in D_n\right\},$$

where $D_n = \{x \in H : \langle w_n - \gamma_n \zeta_n - s_n, x - s_n\rangle \leq 0\}$ *and* $\zeta_n \in \partial_2 g(w_n, s_n)$ *is chosen such that*
$w_n - \gamma_n \zeta_n - s_n \in N_C(s_n)$.
Step 3: *Compute*

$$v_n = (I + r_n D)^{-1}(I - r_n B)t_n$$

and

$$u_n = v_n - r_n(Bv_n - Bt_n).$$

Step 4: *Select* $\sigma_n \in \partial_2 f(u_n, \cdot)(u_n)$ *and compute*

$$x_{n+1} = \delta_n w_n + (1 - \delta_n)u_n - \alpha_n \lambda \sigma_n,$$

where

$$\gamma_{n+1} = \begin{cases} \min\left\{\gamma_n, \frac{\mu(\|w_n - s_n\|^2) + \|t_n - s_n\|^2}{2(g(w_n,t_n) + g(w_n,s_n) - g(s_n,t_n))}\right\}, & \text{if } g(w_n, t_n) + g(w_n, s_n) - g(s_n, t_n) > 0, \\ \gamma_n & \text{otherwise} \end{cases} \tag{17}$$

and

$$r_{n+1} = \begin{cases} \min\left\{\frac{\rho\|t_n - v_n\|}{\|Bt_n - Bv_n\|}, r_n\right\}, & \text{if } Bt_n \neq Bv_n, \\ r_n & \text{otherwise.} \end{cases} \tag{18}$$

Set $n := n + 1$ *and go back to* ***Step 1***.

Remark 1.
From (16)*, we have*

$$\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq \frac{\epsilon_n}{\alpha_n},$$

hence, from Assumption 3(i), we obtain

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq \lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0.$$

Hence, we have

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0.$$

4. Convergence Analysis

Lemma 6. *Let* $\{r_n\}$ *be a sequence generated by* (18)*. Then* $\{r_n\}$ *is a non-increasing sequence. Hence*

$$\lim_{n\to\infty} r_n = r \geq \min\left\{r_1, \frac{\rho}{L}\right\}$$

Proof.
It is clear that $\{r_n\}$ is non-increasing. Also, if $Bt_n - Bv_n \neq 0$, then we have

$$\frac{\rho\|t_n - v_n\|}{\|Bt_n - Bv_n\|} \geq \frac{\rho}{L}.$$

Therefore, $\{r_n\}$ has the lower bound $\min\left\{r_1, \frac{\rho}{L}\right\}$.

Lemma 7. *The sequence* $\{\gamma_n\}$ *generated by* (17) *is monotonically non-increasing and*

$$\lim_{n\to\infty} \gamma_n = \gamma \geq \frac{\mu}{2\max\{a_1, a_2\}}$$

Proof.
It is obvious that $\{\gamma_n\}$ is non-increasing and g satisfies Assumption 2(iv), we have

$$\begin{aligned}\frac{\mu(\|w_n - s_n\|^2 + \|t_n - s_n\|^2)}{2(g(w_n, t_n) - g(w_n, s_n) - g(s_n, t_n))} &\geq \frac{\mu(\|w_n - s_n\|^2 + \|t_n - s_n\|^2)}{2(a_1\|w_n - s_n\|^2 + a_2\|s_n - t_n\|^2)} \\ &\geq \frac{\mu}{2\max\{a_1, a_2\}}.\end{aligned}$$

Therefore, $\{\gamma_n\}$ is bounded below by $\frac{\mu}{2\max\{a_1,a_2\}}$. Thus, we have

$$\lim_{n\to\infty} \gamma_n = \gamma \geq \frac{\mu}{2\max\{a_1, a_2\}}.$$

Lemma 8. *Let $\{x_n\}$ be a sequence generated by Algorithm 5, then*

$$\|u_n-q\|^2 \leq \|w_n-q\|^2-\left(1-\mu\frac{\gamma_n}{\gamma_{n+1}}\right)\left[\|w_n-s_n\|^2+\|s_n-t_n\|^2\right]-\left(1-\rho^2\frac{r_n^2}{r_{n+1}^2}\right)\|t_n-v_n\|^2, q \in \Omega.$$

Proof.
From the definition of $\{r_n\}$, we obtain

$$\|Bt_n - Bv_n\| \leq \frac{\rho}{r_{n+1}}\|t_n - v_n\|, \forall n \in \mathbb{N}. \tag{19}$$

Moreso, if $Bt_n = Bv_n$, hence (19) holds. Otherwise, we get

$$r_{n+1} = \min\left\{\frac{\rho\|t_n - v_n\|}{\|Bt_n - Bv_n\|}, r_n\right\} \leq \frac{\mu\|t_n - v_n\|}{\|Bt_n - Bv_n\|},$$

from which we have

$$\|Bt_n - Bv_n\| \leq \frac{\rho}{r_{n+1}}\|t_n - v_n\|, \forall n \in \mathbb{N}. \tag{20}$$

Hence, (19) holds when $Bt_n = Bv_n$ and $Bt_n \neq Bv_n$.
From the definition of s_n and Lemma 2, we obtain

$$0 \in \partial_2\left(\gamma_n g(w_n, \cdot) + \frac{1}{2}\|w_n - y\|^2\right)(s_n) + N_C(s_n), \forall y \in C.$$

This implies that $\zeta_n' \in \partial_2 g(w_n, .)(s_n)$ and $\varkappa \in N_C(s_n)$ exist, such that

$$\gamma_n\zeta_n' + s_n - w_n + \varkappa = 0.$$

Since $\varkappa \in N_C(s_n)$, therefore $\langle \varkappa, y - s_n\rangle \leq 0, \forall y \in C$. Hence

$$\langle w_n - \gamma_n\zeta_n' - s_n, y - s_n\rangle \leq 0, \forall y \in C.$$

Thus, $C \subset D_n$. Also, from the definition of t_n and Lemma 2, we have that

$$0 \in \partial_2\left(\gamma_n g(y_n, y) + \frac{1}{2}\|w_n - y\|^2\right)(t_n) + N_{D_n}(t_n), \forall y \in D_n.$$

Therefore, $\bar{\zeta}_n \in \partial_2 g(s_n, \cdot)(t_n)$ and $\bar{\varkappa} \in N_{D_n}(t_n)$ exist, such that

$$\gamma_n\bar{\zeta}_n + t_n - w_n + \bar{\varkappa} = 0,$$

and for all $y \in D_n$. Since $\bar{\varkappa} \in N_{D_n}(t_n)$, so $\langle \bar{\varkappa}, y - t_n\rangle \leq 0$. Thus

$$\gamma_n\langle\bar{\zeta}_n, y - t_n\rangle \geq \langle w_n - t_n, y - t_n\rangle, \forall y \in D_n. \tag{21}$$

Also, from $\bar{\zeta}_n \in \partial_2 g(s_n, t_n)$, we have

$$g(s_n, y) - g(s_n, t_n) \geq \langle\bar{\zeta}_n, y - t_n\rangle, \forall y \in H. \tag{22}$$

By substituting $y = q$ in (21), then we obtain

$$\gamma_n \langle \bar{\zeta}_n, q - t_n \rangle \geq \langle w_n - t_n, q - t_n \rangle. \tag{23}$$

Since $\gamma_n > 0$, therefore it follows from (22) and (23) that

$$\gamma_n \big(g(s_n, q) - g(s_n, t_n) \big) \geq \langle w_n - t_n, q - t_n \rangle.$$

Since $q \in sol(f, C)$ together with Assumption 2(ii) which is the pseudomonotonicity of g, then we have $g(s_n, q) \leq 0$. Thus, we obtain

$$-\gamma_n g(s_n, t_n) \geq \langle w_n - t_n, q - t_n \rangle. \tag{24}$$

Also, since $t_n \in D_n$, we have

$$\langle w_n - \gamma_n \zeta_n - s_n, t_n - s_n \rangle \leq 0.$$

Thus,

$$\langle w_n - s_n, t_n - s_n \rangle \leq \gamma_n \langle \zeta_n, t_n - s_n \rangle.$$

Also, $\zeta_n \in \partial_2 g(w_n, \cdot)(s_n)$, hence

$$g(w_n, y) - g(w_n, s_n) \geq \langle \zeta_n, y - s_n \rangle, \forall y \in H.$$

Therefore,

$$\gamma_n (g(w_n, t_n) - g(w_n, s_n)) \geq \gamma_n \langle \zeta_n, t_n - s_n \rangle \geq \langle w_n - s_n, t_n - s_n \rangle. \tag{25}$$

By adding (25) and (24), we get

$$\begin{aligned} 2\gamma_n (g(w_n, t_n) - g(w_n, s_n) - g(s_n, t_n)) &\geq 2\langle w_n - s_n, t_n - s_n \rangle + 2\langle w_n - t_n, q - t_n \rangle \\ &= \|w_n - s_n\|^2 + \|t_n - s_n\|^2 - \|w_n - t_n\|^2 \\ &+ \|w_n - t_n\|^2 + \|t_n - q\|^2 - \|w_n - q\|^2, \end{aligned}$$

which implies that

$$\begin{aligned} \|t_n - q\|^2 \leq \|w_n - q\|^2 - \|w_n - s_n\|^2 - \|t_n - s_n\|^2 \\ + 2\gamma_n (g(w_n, t_n) - g(w_n, s_n) - g(s_n, t_n)). \end{aligned}$$

Using the update γ_{n+1}, we have

$$\begin{aligned} \|t_n - q\|^2 &\leq \|w_n - q\|^2 - \|w_n - s_n\|^2 - \|t_n - s_n\|^2 + 2\gamma_n (g(w_n, t_n) - g(w_n, s_n) - g(s_n, t_n)) \\ &\leq \|w_n - q\|^2 - \|w_n - s_n\|^2 - \|t_n - s_n\|^2 + \frac{\gamma_n}{\gamma_{n+1}} \mu (\|w_n - s_n\|^2 + \|t_n - s_n\|^2) \\ &= \|w_n - q\|^2 - \left(1 - \mu \frac{\gamma_n}{\gamma_{n+1}}\right) \|w_n - s_n\|^2 - \left(1 - \mu \frac{\gamma_n}{\gamma_{n+1}}\right) \|s_n - t_n\|^2 \end{aligned}$$

$$= \|w_n - q\|^2 - \left(1 - \mu\frac{\gamma_n}{\gamma_{n+1}}\right)\left[\|w_n - s_n\|^2 + \|s_n - t_n\|^2\right]. \tag{26}$$

By combining (20) and (26), we have

$$\begin{aligned}
\|u_n - q\|^2 &= \|v_n - r_n(Bv_n - Bt_n) - q\|^2 \\
&= \|v_n - q\|^2 + r_n^2\|Bv_n - Bt_n\|^2 - 2r_n\langle v_n - q, Bv_n - Bt_n\rangle \\
&= \|t_n - q\|^2 + \|v_n - t_n\|^2 + 2\langle v_n - t_n, t_n - q\rangle + r_n^2\|Bv_n - Bt_n\|^2 - 2r_n\langle v_n - q, Bv_n - Bt_n\rangle \\
&= \|t_n - q\|^2 + \|v_n - t_n\|^2 - 2\|v_n - t_n\|^2 + 2\langle v_n - t_n, v_n - q\rangle + r_n^2\|Bv_n - Bt_n\|^2 \\
&\quad - 2r_n\langle v_n - q, Bv_n - Bt_n\rangle \\
&= \|t_n - q\|^2 - \|v_n - t_n\|^2 + 2\langle v_n - t_n, v_n - q\rangle + r_n^2\|Bv_n - Bt_n\|^2 - 2r_n\langle v_n - q, Bv_n - Bt_n\rangle \\
&= \|t_n - q\|^2 - \|v_n - t_n\|^2 - 2\langle t_n - v_n - r_n(Bt_n - Bv_n), v_n - q\rangle + r_n^2\|Bv_n - Bt_n\|^2 \\
&\leq \|w_n - q\|^2 - \left(1 - \mu\frac{\gamma_n}{\gamma_{n+1}}\right)\left[\|w_n - s_n\|^2 + \|s_n - t_n\|^2\right] - \left(1 - \rho^2\frac{r_n^2}{r_{n+1}^2}\right)\|t_n - v_n\|^2 \\
&\quad - 2\langle t_n - v_n - r_n(Bt_n - Bv_n), v_n - q\rangle. \qquad (27)
\end{aligned}$$

From the definition of v_n, we get $(I - r_nB)t_n \in (I + r_nD)v_n$. By the maximal monotonocity of D, we have $h_n \in Dv_n$ such that $(I - r_nB)t_n = v_n + r_nh_n$, which implies that

$$h_n = \frac{1}{r_n}(t_n - v_n - r_nBt_n). \tag{28}$$

Furthermore, $0 \in (B + D)q$ and $Bv_n + h_n \in (B + D)v_n$. Since $B + D$ is maximal monotone, we get

$$\langle Bv_n + h_n, v_n - q\rangle \geq 0, \tag{29}$$

by substituting (28) in (29), we obtain

$$\frac{1}{r_n}\langle t_n - v_n - r_nBt_n + r_nBv_n, v_n - q\rangle \geq 0,$$

from which we have

$$\langle t_n - v_n - r_n(Bt_n - Bv_n), v_n - q\rangle \geq 0. \tag{30}$$

By applying (30) into (27), we get

$$\|u_n - q\|^2 \leq \|w_n - q\|^2 - \left(1 - \mu\frac{\gamma_n}{\gamma_{n+1}}\right)\left[\|w_n - s_n\|^2 + \|s_n - t_n\|^2\right] - \left(1 - \rho^2\frac{r_n^2}{r_{n+1}^2}\right)\|t_n - v_n\|^2. \tag{31}$$

Lemma 9. *Let $\{x_n\}$ be the sequence generated by Algorithm 5 under Assumption 3. Then $\{x_n\}$ is bounded.*

Proof.
Since $\lim_{n\to\infty}\gamma_n=\gamma$ exists, therefore we have $\lim_{n\to\infty}\left(1-\mu\frac{\gamma_n}{\gamma_{n+1}}\right)=1-\mu>0,\forall n\geq 1$. Also, since $\lim_{n\to\infty}\left(1-\rho^2\frac{r_n^2}{r_{n+1}^2}\right)=1-\rho^2>0$, then there exists $n_0\in\mathbb{N}$ such that $\left(1-\rho^2\frac{r_n^2}{r_{n+1}^2}\right)>0,\forall n\geq 1$. Therefore from (31), we get

$$\|u_n-q\|^2\leq\|w_n-q\|^2. \tag{32}$$

From the definition of w_n, we have

$$\begin{aligned}\|w_n-q\|&=\|x_n+\theta_n(x_n-x_{n-1})-q\|\\&\leq\|x_n-q\|+\theta_n\|x_n-x_{n-1}\|\\&\leq\|x_n-q\|+\alpha_n\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|.\end{aligned} \tag{33}$$

From Remark 1, we have $\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|=0$. Then, it follows that there exists a constant M_1 such that $\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\leq M_1$. By applying this in (33), we have

$$\|w_n-q\|\leq\|x_n-q\|+\alpha_nM_1. \tag{34}$$

From (34), Assumption 1(ii), (iv), and Lemma 4, we obtain

$$\begin{aligned}\|x_{n+1}-q\|&=\|\delta_nw_n+(1-\delta_n)u_n-\alpha_n\lambda\sigma_n-q\|\\&=\|\delta_nw_n+(1-\delta_n)u_n-\alpha_n\lambda\sigma_n-\delta_nq-(1-\delta_n)q+\alpha_n\lambda v-\alpha_n\lambda v\|\\&\leq\|(1-\delta_n)u_n-\alpha_n\lambda\sigma_n-\big[(1-\delta_n)q-\alpha_n\lambda q\big]\|+\delta_n\|w_n-q\|+\alpha_n\lambda\|v\|\\&\leq(1-\delta_n-\alpha_n\bar{\tau})\|u_n-q\|+\delta_n\|w_n-q\|+\alpha_n\lambda\|v\|\\&\leq(1-\delta_n-\alpha_n\bar{\tau})\|w_n-q\|+\delta_n\|w_n-q\|+\alpha_n\lambda\|v\|\\&\leq(1-\alpha_n\bar{\tau})\big[\|x_n-q\|+\alpha_nM_1\big]+\alpha_n\lambda\|v\|\\&\leq(1-\alpha_n\bar{\tau})\|x_n-q\|+\alpha_n\big[M_1+\lambda\|v\|\big]\\&\leq(1-\alpha_n\bar{\tau})\|x_n-q\|+\alpha_n\bar{\tau}\left(\frac{M_1}{\bar{\tau}}+\frac{\lambda\|v\|}{\bar{\tau}}\right),\end{aligned} \tag{35}$$

where $\bar{\tau}=1-\sqrt{1-\lambda(2\eta-\lambda c^2)}$. Now, let $M_2:=\sup_{n\in\mathbb{N}}\left\{\frac{M_1}{\bar{\tau}}+\frac{\lambda\|v\|}{\bar{\tau}}\right\}$. From (35), we have

$$\|x_{n+1}-q\|\leq(1-\alpha_n\bar{\tau})\|x_n-q\|+\alpha_n\bar{\tau}M_2.$$

Setting $a_n:=\|x_n-q\|$, $\sigma_n:=\alpha_n\bar{\tau}$, $b_n:=\alpha_n\bar{\tau}M_2$ and $c_n:=0$. Then by Lemma 5(i) and assumptions on the control parameters, we can conclude that $\{\|x_n-q\|\}$ is bounded and this implies that $\{x_n\}$ is bounded. Consequently, $\{u_n\},\{t_n\},\{s_n\},v_n$ and $\{w_n\}$ are also bounded.

Lemma 10. *Let $\{v_n\}$ and $\{t_n\}$ be sequences generated by Algorithm 5. If Assumption 3 holds such that $\lim_{k\to\infty}\|v_{n_k}-t_{n_k}\|=0$ for some subsequences $\{v_{n_k}\}$ of $\{v_n\}$ and $\{t_{n_k}\}$ of $\{t_n\}$, and $\{t_{n_k}\}$ converges weakly to $z\in H$, then $z\in(B+D)^{-1}(0)$.*

Proof.
Let $(u,v)\in G(B+D)$, that is $v-Bu\in Du$. Since $v_{n_k}=(I+r_{n_k}D)^{-1}(I-r_{n_k}B)t_{n_k}$, we obtain $(I-r_{n_k}B)t_{n_k}\in(I+r_{n_k}D)v_{n_k}$ which implies that

$$\frac{1}{r_{n_k}}(t_{n_k}-v_{n_k}-r_{n_k}Bt_{n_k})\in Dv_{n_k}.$$

Since D is maximal monotone, we have

$$\left\langle u-v_{n_k},v-Bu-\frac{1}{r_{n_k}}(t_{n_k}-v_{n_k}-r_{n_k}Bt_{n_k})\right\rangle\geq 0.$$

This implies that

$$\langle u-v_{n_k},v\rangle-\left\langle u-v_{n_k},Bu+\frac{1}{r_{n_k}}(t_{n_k}-v_{n_k}-r_{n_k}Bt_{n_k})\right\rangle\geq 0.$$

Thus, we have

$$\begin{aligned}\langle u-v_{n_k},v\rangle&\geq\left\langle u-v_{n_k},Bu+\frac{1}{r_{n_k}}(t_{n_k}-v_{n_k}-r_{n_k}Bt_{n_k})\right\rangle\\&=\langle u-v_{n_k},Bu-Bt_{n_k}\rangle+\left\langle u-v_{n_k},\frac{1}{r_{n_k}}(t_{n_k}-v_{n_k})\right\rangle\\&=\langle u-v_{n_k},Bu-Bv_{n_k}\rangle+\langle u-v_{n_k},Bv_{n_k}-Bt_{n_k}\rangle+\left\langle u-v_{n_k},\frac{1}{r_{n_k}}(t_{n_k}-v_{n_k})\right\rangle\\&\geq\langle u-v_{n_k},Bv_{n_k}-Bt_{n_k}\rangle+\left\langle u-v_{n_k},\frac{1}{r_{n_k}}(t_{n_k}-v_{n_k})\right\rangle.\end{aligned}$$

Since $\lim_{k\to\infty}\|t_{n_k}-v_{n_k}\|=0$ and B is Lipschitz continuous, then we have $\lim_{k\to\infty}\|Bt_{n_k}-Bv_{n_k}\|=0$. Also, since $\lim_{n\to\infty}r_n=r\geq\min\left\{r_0,\frac{\rho}{L}\right\}$, we obtain

$$\langle u-z,v\rangle=\lim_{k\to\infty}\langle u-v_{n_k},u\rangle\geq 0.$$

By the maximal monotonicity of $B+D$, it implies that $0\in(B+D)z$, that is $z\in(B+D)^{-1}(0)$.

Lemma 11. *The following inequality holds for all $q\in\Omega$:*

$$\|x_{n+1}-q\|^2\leq(1-\alpha_n\bar{\tau})\|x_n-q\|^2+\alpha_n\bar{\tau}\left(2\langle v,q-x_{n+1}\rangle+\frac{3M_3\theta_n\|x_n-x_{n-1}\|}{\alpha_n\bar{\tau}}\right),\forall n,M_3>0.$$

Proof.
Let $q \in \Omega$. Then by Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}\|w_n - q\|^2 &= \|x_n + \theta_n(x_n - x_{n-1})\|^2\\ &= \|x_n - q\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 + 2\theta_n\langle x_n - q, x_n - x_{n-1}\rangle\\ &\le \|x_n - q\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 + 2\theta_n\|x_n - q\|\|x_n - x_{n-1}\|\\ &= \|x_n - q\|^2 + \theta_n\|x_n - x_{n-1}\|\big[\theta_n\|x_n - x_{n-1}\| + 2\|x_n - q\|\big]\\ &= \|x_n - q\|^2 + 3M_3\alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|, \end{aligned} \tag{36}$$

where $M_3 := \sup_{n\in\mathbb{N}}\{\|x_n - q\|, \theta_n\|x_n - x_{n-1}\|\} > 0$. By Lemma 5, (36) and (13), we obtain

$$\begin{aligned}\|x_{n+1} - q\|^2 =&\|\delta_n w_n + (1-\delta_n)u_n - \alpha_n\lambda\sigma_n - q\|^2\\ &= \|(1-\delta_n)u_n - \alpha_n\lambda\sigma_n - \big[(1-\delta_n)q - \alpha_n\lambda v\big] + \delta_n(w_n - q) - \alpha_n\lambda v\|^2\\ &\le \|(1-\delta_n)u_n - \alpha_n\lambda\sigma_n - \big[(1-\delta_n)q - \alpha_n\lambda v\big] + \delta_n(w_n - q)\|^2 + 2\alpha_n\lambda\langle v, q - x_{n+1}\rangle.\\ &\le \|(1-\delta_n)u_n - \alpha_n\lambda\sigma_n - \big[(1-\delta_n)q - \alpha_n\lambda v\big]\|^2 + \|\delta_n(w_n - q)\|^2\\ &+ 2\delta_n\|(1-\delta_n)u_n - \alpha_n\lambda\sigma_n - \big[(1-\delta_n)q - \alpha_n\lambda v\big]\|\|w_n - q\| + 2\alpha_n\lambda\langle v, q - x_{n+1}\rangle\\ &\le (1-\delta_n - \alpha_n\bar\tau)^2\|u_n - q\|^2 + \delta_n^2\|w_n - q\|^2 + 2(1-\delta_n-\alpha_n\bar\tau)\delta_n\|u_n - q\|\|w_n - q\|\\ &+ 2\alpha_n\bar\tau\langle v, q - x_{n+1}\rangle\\ &\le (1-\delta_n-\alpha_n\bar\tau)^2\|u_n - q\|^2 + \delta_n^2\|w_n - q\|^2\\ &+ (1-\delta_n-\alpha_n\bar\tau)\delta_n(\|u_n - q\|^2 + \|w_n - q\|^2) + 2\alpha_n\bar\tau\langle v, q - x_{n+1}\rangle\\ &= (1-\delta_n-\alpha_n\bar\tau)(1-\alpha_n\bar\tau)\|u_n - q\|^2 + \delta_n(1-\alpha_n\bar\tau)\|w_n - q\|^2 + 2\alpha_n\bar\tau\langle v, q - x_{n+1}\rangle \quad (37)\\ &\le (1-\alpha_n\bar\tau)\|w_n - q\|^2 + 2\alpha_n\bar\tau\langle v, q - x_{n+1}\rangle\\ &\le (1-\alpha_n\bar\tau)\left[\|x_n - q\|^2 + 3M_3\alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|\right] + 2\alpha_n\bar\tau\langle v, q - x_{n+1}\rangle\\ &\le (1-\alpha_n\bar\tau)\|x_n - q\|^2 + (1-\alpha\bar\tau)3M_3\alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + 2\alpha_n\bar\tau\langle v, q - x_{n+1}\rangle\\ &\le (1-\alpha_n\bar\tau))\|x_n - q\|^2 + \alpha_n\bar\tau\left(2\langle v, q - x_{n+1}\rangle + \frac{3M_3\theta_n\|x_n - x_{n-1}\|}{\alpha_n\bar\tau}\right).\end{aligned}$$

Theorem 1. *Let $\{x_n\}$ be a sequence generated by Algorithm 5 satisfying Assumptions 1, 2 and 3. Then $\{x_n\}$ converges strongly to $\bar x = P_\Omega x_0$.*

Proof.
Let $\bar x \in \Omega$, then from Lemma 11, we get

$$\|x_{n+1} - \bar x\|^2 \le (1-\alpha_n\bar\tau))\|x_n - \bar x\|^2 + \alpha_n\bar\tau\left(2\langle v, \bar x - x_{n+1}\rangle + \frac{3M_3\theta_n\|x_n - x_{n-1}\|}{\alpha_n\bar\tau}\right)$$

$$= (1 - \alpha_n \bar{\tau}))\|x_n - \bar{x}\|^2 + \alpha_n \bar{\tau} b_n, \tag{38}$$

where $b_n = 2\langle v, \bar{x} - x_{n+1}\rangle + \frac{3M_3\theta_n\|x_n - x_{n-1}\|}{\alpha_n\bar{\tau}}$.
Now, we claim that $\{\|x_n - \bar{x}\|\}$ converges to zero. In order to establish this, by applying Lemma 3, we have to show that $\limsup_{k\to\infty} b_{n_k} \leq 0$ for every subsequence $\{\|x_{n_k} - \bar{x}\|\}$ of $\{\|x_n - \bar{x}\|\}$ such that

$$\liminf_{k\to\infty}\left(\|x_{n_k+1} - \bar{x}\| - \|x_{n_k} - \bar{x}\|\right) \geq 0. \tag{39}$$

From (31), (36) and (37), we have

$$\begin{aligned}
\|x_{n+1} - \bar{x}\|^2 &= \|\delta_n w_n + (1-\delta_n)u_n - \alpha_n\lambda\sigma_n - \bar{x}\|^2 \\
&\leq (1-\delta_n - \alpha_n\bar{\tau})(1-\alpha_n\bar{\tau})\|u_n - \bar{x}\|^2 + \delta_n(1-\alpha_n\bar{\tau})\|w_n - \bar{x}\|^2 + 2\alpha_n\bar{\tau}\langle v, \bar{x} - x_{n+1}\rangle \\
&\leq (1-\delta_n - \alpha_n\bar{\tau})(1-\alpha_n\bar{\tau})\Bigg[\|w_n - \bar{x}\|^2 - \left(1 - \mu\frac{\gamma_n}{\gamma_{n+1}}\right)\left[\|w_n - s_n\|^2 + \|s_n - t_n\|^2\right] \\
&- \left(1 - \rho^2\frac{r_n^2}{r_{n+1}^2}\right)\|t_n - v_n\|^2\Bigg] + \delta_n(1-\alpha_n\bar{\tau})\|w_n - \bar{x}\|^2 + 2\alpha_n\bar{\tau}\langle v, \bar{x} - x_{n+1}\rangle \\
&\leq (1-\alpha_n\bar{\tau})\|w_n - \bar{x}\|^2 - (1-\delta_n - \alpha_n\bar{\tau})(1-\alpha_n\bar{\tau})\Bigg[\left(1 - \mu\frac{\gamma_n}{\gamma_{n+1}}\right)\big[\|w_n - s_n\|^2 \\
&+ \|s_n - t_n\|^2\big] - \left(1 - \rho^2\frac{r_n^2}{r_{n+1}^2}\right)\|t_n - v_n\|^2\Bigg] + 2\alpha_n\bar{\tau}\langle v, \bar{x} - x_{n+1}\rangle \\
&\leq (1-\alpha_n\bar{\tau})\|x_n - \bar{x}\|^2 + 3M_3\alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \\
&- (1-\delta_n - \alpha_n\bar{\tau})(1-\alpha_n\bar{\tau})\Bigg[\left(1 - \mu\frac{\gamma_n}{\gamma_{n+1}}\right)\left[\|w_n - s_n\|^2 + \|s_n - t_n\|^2\right] \\
&- \left(1 - \rho^2\frac{r_n^2}{r_{n+1}^2}\right)\|t_n - v_n\|^2\Bigg] + 2\alpha_n\bar{\tau}\langle v, \bar{x} - x_{n+1}\rangle.
\end{aligned} \tag{40}$$

Suppose that $\{\|x_{n_k} - \bar{x}\|\}$ is a subsequence of $\{\|x_n - \bar{x}\|\}$ such that (39) holds. From (40), we obtain

$$\begin{aligned}
&(1-\delta_{n_k} - \alpha_{n_k}\bar{\tau})(1-\alpha_{n_k}\bar{\tau})\Bigg[\left(1 - \mu\frac{\gamma_{n_k}}{\gamma_{n_k+1}}\right)\left[\|w_{n_k} - s_{n_k}\|^2 + \|s_{n_k} - t_{n_k}\|^2\right] \\
&- \left(1 - \rho^2\frac{r_{n_k}^2}{r_{n_k+1}^2}\right)\|t_{n_k} - v_{n_k}\|^2\Bigg] \\
&\leq (1-\alpha_{n_k}\bar{x})\|x_{n_k} - \bar{x}\|^2 - \|x_{n_k+1} - \bar{x}\|^2 + \alpha_{n_k}\left(\frac{3M_3\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k} - x_{n_k-1}\| + 2\bar{\tau}\langle v, \bar{x} - x_{n_k+1}\rangle\right).
\end{aligned}$$

By applying (39) together with the fact that $\lim_{k\to\infty}\alpha_{n_k} = 0$, and that $\lim_{k\to\infty}\Big(1 -$

$\mu\frac{r_{n_k}}{r_{n_k+1}}\Big) = 1-\mu > 0$, we get

$$\lim_{k\to\infty}(1-\delta_{n_k}-\alpha_{n_k}\bar{\tau})(1-\alpha_{n_k}\bar{\tau})\left(1-\mu\frac{\gamma_{n_k}}{\gamma_{n_k+1}}\right)\|w_{n_k}-s_{n_k}\| = 0,$$
$$\lim_{k\to\infty}(1-\delta_{n_k}-\alpha_{n_k}\bar{\tau})(1-\alpha_{n_k}\bar{\tau})\left(1-\mu\frac{\gamma_{n_k}}{r_{n_k+1}}\right)\|s_{n_k}-t_{n_k}\| = 0,$$

and

$$\lim_{k\to\infty}(1-\delta_{n_k}-\alpha_{n_k}\bar{\tau})(1-\alpha_{n_k}\bar{\tau})\left(1-\rho^2\frac{r^2_{n_k}}{r^2_{n_k+1}}\right)\|t_{n_k}-v_{n_k}\|^2 = 0.$$

By the conditions on the control parameters, we have

$$\lim_{k\to\infty}\|w_{n_k}-s_{n_k}\| = 0, \tag{41}$$
$$\lim_{k\to\infty}\|s_{n_k}-t_{n_k}\| = 0, \tag{42}$$

and

$$\lim_{k\to\infty}\|t_{n_k}-v_{n_k}\| = 0. \tag{43}$$

Indeed, we have

$$\|u_{n_k}-v_{n_k}\| = r_{n_k}\|Bv_{n_k}-Bt_{n_k}\|. \tag{44}$$

Since $\lim_{k\to\infty}\|v_{n_k}-t_{n_k}\| = 0$ and B is Lipschitz continuous, we have $\|Bv_{n_k}-Bt_{n_k}\| \to 0$ as $k\to\infty$. By applying this in (44), we obtain

$$\lim_{k\to\infty}\|u_{n_k}-v_{n_k}\| = 0. \tag{45}$$

Also, from (41) and (42), we get

$$\begin{aligned}\|w_{n_k}-z_{n_k}\| &= \|w_{n_k}-y_{n_k}+y_{n_k}-z_{n_k}\| \\ &\le \|w_{n_k}-y_{n_k}\|+\|y_{n_k}-z_{n_k}\| \to 0.\end{aligned}$$

Thus

$$\lim_{k\to\infty}\|w_{n_k}-z_{n_k}\| = 0. \tag{46}$$

Similarly, we have

$$\lim_{k\to\infty}\|z_{n_k}-x_{n_k}\| = 0,$$
$$\lim_{k\to\infty}\|x_{n_k}-y_{n_k}\| = 0,$$

and

$$\lim_{k\to\infty}\|v_{n_k}-x_{n_k}\| = 0. \tag{47}$$

From (45) and (47), we get

$$\lim_{k\to\infty} \|u_{n_k} - x_{n_k}\| = 0. \tag{48}$$

By Remark 1, we have

$$\|w_{n_k} - x_{n_k}\| = \alpha_{n_k} \frac{\theta_{n_k}}{\alpha_{n_k}} \|x_{n_k} - x_{n_k-1}\| \to 0 \quad \text{as} \quad n \to \infty. \tag{49}$$

From (48) and (49), we have

$$\lim_{k\to\infty} \|w_{n_k} - u_{n_k}\| = 0. \tag{50}$$

It is clear that

$$\begin{aligned}\|x_{n_k+1} - u_{n_k}\| &= \|\delta_{n_k} w_{n_k} + (1-\delta_{n_k})u_{n_k} - \alpha_{n_k}\lambda\sigma_{n_k} - u_{n_k}\| \\ &= \|\delta_{n_k}(w_{n_k} - u_{n_k}) - \alpha_{n_k}\lambda\sigma_{n_k}\| \\ &\leq \delta_{n_k}\|w_{n_k} - u_{n_k}\| + \alpha_{n_k}\|\lambda\sigma_{n_k}\|. \end{aligned} \tag{51}$$

By applying (50) and the condition on α_{n_k} in (51), we get

$$\lim_{k\to\infty} \|x_{n_k+1} - u_{n_k}\| = 0. \tag{52}$$

From (48) and (52), we obtain

$$\lim_{k\to\infty} \|x_{n_k+1} - x_{n_k}\| = 0. \tag{53}$$

Since $\{x_n\}$ is bounded, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_{k_j}} \rightharpoonup z$. From (43) and (47), we have $t_{n_k} \rightharpoonup z$. Moreover, it follows from (43) and Lemma 10 that $z \in (B+D)^{-1}(0)$.

Next, we show that $z \in Sol(g,C)$. Let $q \in Sol(g,C)$, then $f(q,y) \geq 0$ for all $y \in Sol(g,C)$. Therefore q is a minimum of the convex function $f(q,y)$ over $Sol(g,C)$. Thus, by Lemma 4, we obtain

$$0 \in \delta_2 f(q,\cdot)(q) + N_{Sol(g,C)}(q).$$

Thus, there exists $v \in \delta_2 f(q,\cdot)(q)$ such that

$$\langle v, z - \bar{x}\rangle \geq 0, \forall z \in Sol(g,C). \tag{54}$$

From Lemma 4, and by the definition of $\{s_n\}$, we have

$$0 \in \delta_2\left(\lambda_n g(w_n, y) + \frac{1}{2}\|w_n - y\|^2\right)(s_n) + N_C(s_n).$$

Therefore, there exists $\bar{\omega} \in N_C(s_n)$ and $\zeta_n \in \delta_2 g(w_n,\cdot)(s_n)$ such that

$$\gamma_n \zeta_n + s_n - w_n + \bar{\omega} = 0. \tag{55}$$

Since $\bar{\omega} \in N_C(s_n)$ and $\langle \bar{\omega}, y - s_n \rangle \leq 0$. Then, we obtain from (55) that

$$\gamma_n \langle \omega, y - s_n \rangle \geq \langle w_n - s_n, y - s_n \rangle, \forall y \in C. \tag{56}$$

Furthermore, it is known that $\zeta_n \in \delta_2 g(w_n, \cdot)(s_n)$. Thus, we have

$$g(w_n, y) - g(w_n, s_n) \geq \langle \bar{\omega}, y - s_n \rangle, \forall y \in H. \tag{57}$$

From (56) and (57), we get

$$\gamma_n \Bigg(g(w_n, y) - g(w_n, s_n) \Bigg) \geq \langle w_n - s_n, y - s_n \rangle, \forall y \in C. \tag{58}$$

Let $n = n_k$ in (58), we have

$$\gamma_{n_k} \Bigg(g(w_{n_k}, y) - g(w_{n_k}, s_{n_k}) \Bigg) \geq \langle w_{n_k} - s_{n_k}, y - s_{n_k} \rangle, \forall y \in C. \tag{59}$$

Taking $k \to \infty$ and by applying Assumption 2 (i), 2(ii) together with the fact that $\|w_{n_k} - y_{n_k}\| \to 0$ as $n \to \infty$, we get

$$g(z, y) \geq 0, \forall y \in C.$$

Thus $z \in Sol(g, C)$. Therefore, $z \in \Omega = Sol(g, C) \cap (B + D)^{-1}(0)$. We now show that $\{x_n\}$ converges strongly to $\bar{x}$. From (54), we get

$$\limsup_{k \to \infty} \langle v, \bar{x} - x_{n_k+1} \rangle = \limsup_{k \to \infty} \langle v, \bar{x} - x_{n_k} \rangle = \limsup_{k \to \infty} \langle v, \bar{x} - x_{n_{k_j}} \rangle = \langle v, \bar{x} - z \rangle \leq 0,$$

which implies

$$\limsup_{k \to \infty} \langle v, \bar{x} - x_{n_k+1} \rangle \leq 0. \tag{60}$$

Since $\lim_{n \to \infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0$ and from (60), we obtain

$$\limsup_{k \to \infty} b_{n_k} \leq 0, \tag{61}$$

which together with (60), (38) and Lemma 3, we get $\lim_{n \to \infty} \|x_n - \bar{x}\| = 0$. Hence, $\{x_n\}$ converges strongly to $\bar{x}$.

5. Numerical Examples

In this section, we present some numerical examples to illustrate the efficacy the efficiency of our proposed algorithm. All simulations were carried on a personal Dell laptop with memory size 8/256 using the MATLAB version 2025a. In Example 2, we made a comparison of Algorithm 3.2 with its non-accelerated version, i.e $\theta_n = 0$. For both examples, we use the following parameters except where a different value was used and indicated in the example. We let $\theta = 1/4$, $\rho = 0.25$ $\epsilon_n = 1/(n+1)^2$, $\lambda = 1/7$, $\mu = 0.91$, $\gamma_1 = 2.7$, $\delta = \frac{n}{n+5}$ and $\alpha_n = \frac{1}{n+2}$. The stopping criterion for both experiment is $\|x_{n+1} - x_n\| \leq b$ where $b = 10^{-6}$.

Example 1. *Let* $H = L^2([0,1])$, *with* $\|x\| = \left(\int_0^1 \|x(t)\|^2 dt\right)^{\frac{1}{2}}$ *and inner product* $\langle x, y\rangle = \int_0^1 x(t)y(t)dt$ *for all* $x, y \in L^2([0,1])$. *Suppose* $C = \{x \in L^2([0,1]) : \int_0^1 \frac{t}{2}x(t)dt = 1\}$, *then the projection* P_C *of* x *onto* C *is defined by*

$$P_C x(t) = x(t) - \frac{\int_0^1 \frac{t}{2}dt - 1}{\int_0^1 \frac{t^2}{2}dt}, \forall x \in L^2([0,1]), t \in [0,1].$$

Let $g : L^2([0,1]) \times L^2([0,1]) \to \mathbb{R}$ *be a bifunction defined by*

$$g(x,y) = \langle T(x), y - x\rangle, \forall x, y \in L^2([0,1]),$$

where $T(x) = \int_0^1 \frac{x(t)}{2}dt$, *where* $x \in L^2([0,1])$. *Clearly,* g *which is monotone and hence pseudomonotone satisfies Lipschitz-type continuous with* $L_1 = L_2 = \frac{1}{\pi}$. *Also, let* $f : L^2[(0,1)] \times L^2[(0,1)] \to \mathbb{R}$ *be a mapping such that* $f(x,y) = \langle S(x), y - x\rangle$ *where* $S : L^2[(0,1)] \to L^2[(0,1)]$ *is given by* $Sx(t) = x(t) - x_0$, *hence,* S *is* 1*-strongly monotone and* 1*-Lipschitz continuous. Let* $D : L^2[(0,1)] \to L^2[(0,1)]$ *be defined by* $Dx = 3x$, *and* $Bx = 5x + 3$. *For* $s > 0$,

$$\begin{aligned} J_s^D(x - sBx) &= (I + sD)^{-1}(x - sBx) \\ &= \frac{x}{1+3s} - \frac{s(5x+3)}{1+3s} \end{aligned}$$

Let $\|x_{n+1} - x_n\| \leq 10^{-4}$ *be the stopping criterion. We consider the following cases:*
Case I : $x_0 = 1 + t^2, x_1 = \frac{1}{5}exp(5t)$,
Case II : $x_0 = t^2 - 1, x_1 = texp(-3t)$,
Case III : $x_0 = \frac{t^3}{3}, x_1 = \sin 3t$,
Case IV : $x_0 = exp(3t), x_1 = \frac{1-t}{5}$.

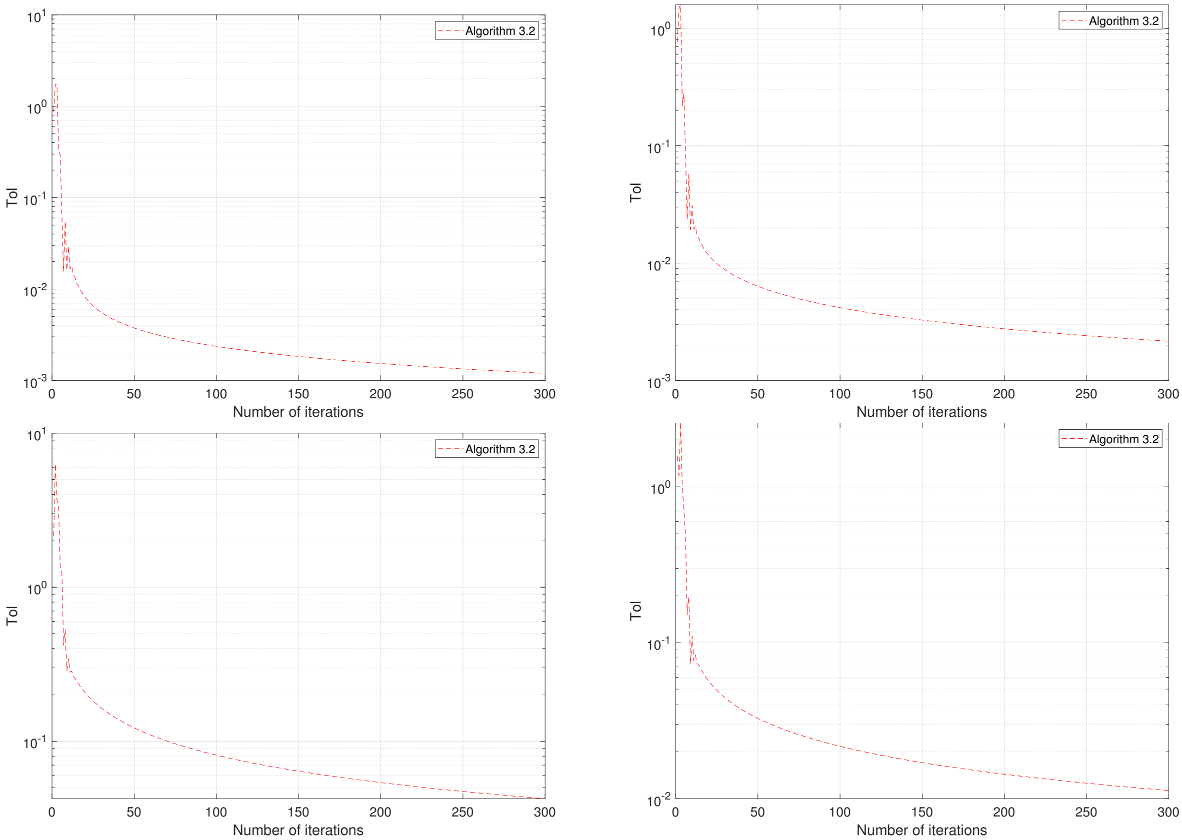

Figure 1: Example 1. Top left: Case 1, Top right: Case 2, Bottom left: Case 3, Bottom right: Case 4.

Example 2. *Let* $H = \mathbb{R}^n$ *and* $C = \{x \in \mathbb{R}^n : -5 \leq x_i \leq 5, \forall i = 1, 2, \cdots, n\}$*. Let* $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *be a bifunctional that satisfies Assumption 2, and it is defined by*

$$g(x, y) = \langle Px + Qy, y - x \rangle, \forall x, y \in \mathbb{R}^n$$

where P *and* Q *are randomly symmetric positive semidefinite matrices such that* $P - Q$ *is positive definite. It is clear that* g *is pseudomonotone and Lipschitz-type continuous with* $L_1 = L_2 = \frac{1}{2}\|P - Q\|$*. Moreover, let* $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *be defined by*

$$f(x, y) = \langle Sx + Ty, y - x \rangle, \forall x, y \in \mathbb{R}^n,$$

where S *and* T *are defined by*

$$S = N^T N + nI_n$$

and

$$T = S + M^T M + nI_n,$$

where I_n *is the identity matrix,* M *and* N *are* $n \times n$ *matrices. Now, define the operator* $D : \mathbb{R}^n \to \mathbb{R}^n$ *by* $Dx = 3x$ *and* $B : \mathbb{R}^n \to \mathbb{R}^n$ *by* $Bx = 5x + 3$*. Then following Example 5.1, we have that*

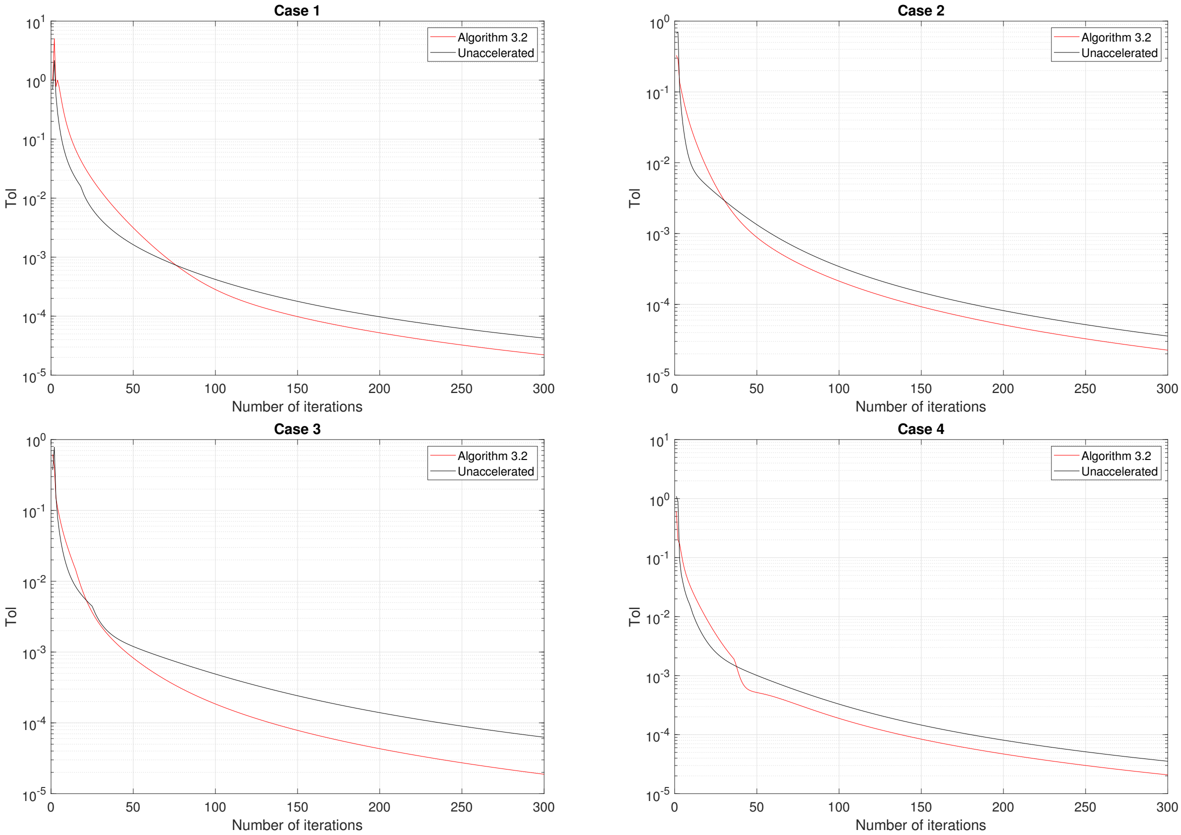

Figure 2: Example 2. Top left: Case 1, Top right: Case 2, Bottom left: Case 3, Bottom right: Case 4.

$$\begin{aligned} J_s^D(x - sBx) &= (I + sD)^{-1}(x - sBx) \\ &= \frac{x}{1+3s} - \frac{s(5x+3)}{1+3s}, \forall x \in \mathbb{R}^n. \end{aligned}$$

Let $\|x_{n+1} - x_n\| \leq 10^{-4}$ *be the stopping criterion. For this example, we vary the values of* γ_1 *as follows:*

Case 1: $\gamma_1 = 2.7$;

Case 2: $\gamma_1 = 1.5$;

Case 3: $\gamma_1 = 0.9$;

Case 4: $\gamma_1 = 0.01$.

The report of this experiment are given in Figure 2.

6. Conclusion

In this paper, we studied a problem of finding common solutions of bilevel equilibrium and inclusion problems in the framework of Hilbert spaces by using a self-adaptive inertial subgradient extragradient method. The algorithm is constructed to ensure convergence without needing a prior estimate of the Lipschitz-like constant. Furthermore, we demonstrated that the sequence produced by our proposed method converges strongly to the common solution of the aforementioned problems, bypassing the conventional two-case approach commonly employed in many studies. Also, we established a strong convergence result under mild conditions and presented numerical experiments to demonstrate the efficiency and accuracy of the proposed method.

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Appendix 1. *Algorithm of Jolaoso et al. [18]*

Step 0: *Set* $x_0, x_1 \in H, \alpha \geq 3, \gamma_1 > 0, \mu, \theta \in (0,1), 0 < \lambda < \frac{2\eta}{c^2}, \{\alpha_n\}, \{\delta_n\}, \{\epsilon_n\} \subset (0,1)$.
Step 1: *Given* $(n-1)th$ *and* nth *iterates, compute*

$$w_n = x_n + \theta_n(x_n - x_{n-1}),$$

where

$$\theta_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\max\{\|x_n - x_{n-1}\|^2, \|x_n - x_{n-1}\|\}}\right\}, & \text{if } x_n \neq x_{n-1} \\ \theta & \text{otherwise} \end{cases}$$

Step 2: *Compute*

$$s_n = \arg\min\left\{\gamma_n g(w_n, y) + \frac{1}{2}\|w_n - y\|^2 : y \in C\right\},$$
$$t_n = \arg\min\left\{\gamma_n g(s_n, y) + \frac{1}{2}\|w_n - y\|^2 : y \in D_n\right\},$$

where $D_n = \{x \in H : \langle w_n - \gamma_n \zeta_n - s_n, x - s_n \rangle\}$ *and* $\zeta_n \in \partial_2 g(w_n, s_n)$ *is chosen such that* $w_n - \gamma_n \zeta_n - s_n \in N_C(s_n)$.
Step 4: *Compute*

$$x_{n+1} = \delta_n w_n + (1 - \delta_n)t_n - \alpha_n \lambda \sigma_n,$$

where

$$\gamma_{n+1} = \begin{cases} \min\left\{\gamma_n, \frac{\mu(\|w_n - s_n\|^2) + \|t_n - s_n\|^2}{2(g(w_n, t_n) + g(w_n, s_n) - g(s_n, t_n))}\right\}, & \text{if } g(w_n, t_n) + g(w_n, s_n) - g(s_n, t_n) > 0, \\ \gamma_n & \text{otherwise} \end{cases}$$

Set $n := n + 1$ *and go back to* ***Step 1****.*

where $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty, 0 < \delta_n < 1 - \alpha_n, \lim_{n\to\infty} \delta_n = \delta < 1, \forall n \geq 0$, and $\epsilon_n = o(\alpha_n)$ which means $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$.