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Conformal Deformations of Surfaces and Their Effects on Connections and Geodesics

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Abstract. This paper focuses on the effects of conformal changes on the first fundamental form, with particular emphasis on the transformation of Christoffel symbols, Gaussian curvature, covariant derivatives, and geodesic equations. We study how conformal scaling alters the connection and curvature properties of surfaces, influencing the intrinsic geometry without changing the underlying conformal class. Special attention is given to how the geodesic flow and curvature expressions adapt under conformal deformation, providing insight into the interplay between local geometry and the metric scaling. Classical results and examples are used to illustrate the impact of conformal metrics on these fundamental geometric quantities.

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1. Introduction

The notion of a *conformal change* arises across numerous areas of differential geometry. Such transformations are characterized by their preservation of angles between vectors,

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though they do not necessarily preserve lengths. Conformal transformations can be viewed as those mappings that preserve the local shape of infinitesimal figures. More precisely, a conformal map acts as a combination of a local rotation and an isotropic scaling, ensuring that the angle between any two intersecting curves remains unchanged. This property follows from the fact that the metric tensor is multiplied by a positive scalar function, known as the conformal factor, which scales lengths uniformly in all tangent directions. In this sense, conformal mappings preserve the local geometry up to a scale, maintaining the infinitesimal circularity of figures such as small disks or spheres. From a variational viewpoint, conformal transformations are intimately related to the theory of harmonic and minimal mappings, where they often arise as critical points of the Dirichlet energy functional. This connection provides a natural bridge between conformal geometry and variational principles in both differential geometry and mathematical physics.

In both Riemannian and Finsler geometries, two metric structures are considered conformally related if they differ by a conformal factor. M. S. Knebelman [1] was the earliest to develop a conformal framework in the context of Finsler geometry. Subsequent contributions by various geometers have expanded upon his foundational ideas (see [2, 3]).

In the geometry of surfaces, conformal transformations hold significant importance, with applications extending beyond pure mathematics into fields like engineering. The angles between intersecting curves on a surface is preserved, but may alter their arc lengths. At an infinitesimal level, a conformal map can be viewed as a local rotation combined with scaling—preserving the shape of small figures such as infinitesimal circles. Such transformations are particularly valued for maintaining certain intrinsic geometric features.

Given two surfaces Ω_1 and Ω_2, a one-to-one correspondence between their points defines a mapping from one to the other. Conformal mappings have long been utilized in applied sciences—such as electromagnetism, acoustics, elasticity, heat conduction, and fluid dynamics. Advances in computational techniques have led to widespread use of conformal geometry in domains like medical imaging, geometric modeling, computer vision, network analysis and computer graphics.

Unlike traditional studies that focus on conformal transformations of Riemannian or Finsler metrics defined on the same underlying manifold—where the metric is deformed while preserving angles and certain geometric structures—this paper investigates analogous transformations between two distinct surfaces. Specifically, we consider how a conformal change can relate the geometries of two different surfaces, rather than altering the metric within a single surface. This broader perspective allows us to study how geometric objects such as vector fields, curvature, and connections behave under conformal correspondences between separate domains, leading to new insights into the interplay between geometry and transformation theory.

Beyond their geometric significance, conformal deformations can also be interpreted within a variational framework. In this setting, the conformal factor $\sigma(u, v)$ may emerge as an extremal of an energy or curvature-related functional, such as the Dirichlet or total curvature functional. This viewpoint aligns the study of conformal geometry with classical variational principles, wherein geodesics appear as stationary curves of the energy functional associated with the conformally modified metric. Consequently, the geodesic equations derived in this paper correspond to Euler–Lagrange equations governing paths that minimize length under the new conformal scaling. Such an interpretation not only deepens the analytical understanding of conformal transformations but also connects the subject to applied disciplines—most notably mechanics and optimization—where variational methods provide a unified tool for analyzing equilibrium and deformation phenomena (see, e.g., [4]).

This paper investigates the behavior of key geometric quantities under conformal changes of surfaces. Specifically, we show how the Christoffel symbols transform under conformal rescaling. These changes directly affect the covariant derivative and, consequently, the geodesic equations. We also examine the transformation law for Gaussian curvature, a central intrinsic invariant that encapsulates the local bending of the surface.

The Paper follows as in Section 2, geometry of surfaces is highlighted, including a review of conformal surfaces, the first fundamental form, and Christoffel symbols. This foundational material sets the stage for the more advanced discussions that follow. In Section 3, we explore how Christoffel symbols transform under conformal changes of the metric, illustrating the results with concrete examples. Section 4 is devoted to the study of covariant derivatives in the context of conformal changes. We present both theoretical analysis and examples, culminating in a theorem that gives an explicit formula for the pushforward of the covariant derivative under a conformal mapping. In Section 5, we turn our attention to geodesic equations, examining their dependence on the Christoffel symbols. We derive how geodesic equations transform under conformal changes and provide examples to support our results. Importantly, we prove that geodesics are preserved under a conformal transformation if and only if the conformal factor is constant.

2. Preliminaries

In this part, we provide a concise summary of essential concepts and equations pertaining to the traditional theory of curves and surfaces within three-dimensional Euclidean space. Our approach aligns with established literature in differential geometry; readers seeking comprehensive discussions should refer to [5–12]. We denote by $\mathbb{R}^3$ the three-dimensional real vector space equipped with its conventional Euclidean framework. Locations within this space are identified using coordinates (x_1, x_2, x_3).

We define a subset Ω of $\mathbb{R}^3$ as a *regular surface* if, for every point $a \in \Omega$, one can find a neighborhood $W \subset \Omega$ which represents the image of a smooth function $\mathbf{x} : U \to \mathbb{R}^3$, where $U \subset \mathbb{R}^2$ is an open set and $\mathbf{x}$ forms a homeomorphism with its image. We call the pair $(U, \mathbf{x})$ a *local parameterization* of Ω at the point a, with $\mathbf{x}(U)$ being the parameterization domain. If $\mathbf{x}(U) = \Omega$, then we categorize the surface Ω as a *simple surface.*

With such a local parameterization available, we can articulate the surface via parametric equations, where $(u, v) \in U$:

$$\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v)).$$

In a broader context, the surface may be depicted as:

$$\mathbf{x}(u,v) = \left(f(u,v), g(u,v), h(u,v)\right),$$

with f, g, and h representing smooth functions. The conventional inner product defined on $\mathbb{R}^3$ gives rise to a Riemannian metric on the surface, characterized by the *first fundamental form*, which captures intrinsic geometric features such as distances and angles. We express this form as:

$$I = E\,du^2 + 2F\,du\,dv + G\,dv^2, \tag{1}$$

with the coefficients determined as follows:

$$\begin{aligned} E &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle = f_u^2 + g_u^2 + h_u^2, \\ F &= \langle \mathbf{x}_u, \mathbf{x}_v \rangle = f_u f_v + g_u g_v + h_u h_v, \\ G &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle = f_v^2 + g_v^2 + h_v^2. \end{aligned} \tag{2}$$

In these expressions, subscripts indicate partial derivatives with respect to the specified parameters.

The parameterization's regularity is ensured by the condition:

$$EG - F^2 = (f_u^2 + g_u^2 + h_u^2)(f_v^2 + g_v^2 + h_v^2) - (f_u f_v + g_u g_v + h_u h_v)^2 \neq 0,$$

ensuring that the vectors $\mathbf{x}_u$ and $\mathbf{x}_v$ maintain linear independence.

For a point $p = (u_0, v_0)$ situated on the surface, the vectors $\mathbf{x}_u(p)$ and $\mathbf{x}_v(p)$ generate the tangent space $T_{\mathbf{x}(p)}\Omega$. The unit normal vector at p is computed as:

$$\mathbf{n}(p) = \frac{\mathbf{x}_u(p) \times \mathbf{x}_v(p)}{\|\mathbf{x}_u(p) \times \mathbf{x}_v(p)\|}.$$

The *second fundamental form*, characterizing the extrinsic curvature of the surface, is defined through the quantities:

$$\begin{aligned} e &= \langle \mathbf{n}, \mathbf{x}_{uu} \rangle, \\ f &= \langle \mathbf{n}, \mathbf{x}_{uv} \rangle = \langle \mathbf{n}, \mathbf{x}_{vu} \rangle, \\ g &= \langle \mathbf{n}, \mathbf{x}_{vv} \rangle. \end{aligned}$$

These coefficients quantify the surface's bending behavior within the ambient space.

The *Gaussian curvature* K at a point is determined by the expression:

$$K = \frac{eg - f^2}{EG - F^2}. \tag{3}$$

Conformal change of first fundamental form In what follows, we present some basic definitions and results regarding the conformal change of surfaces. For more details, see [5, 13, 14].

Definition 1. *Consider two surfaces Ω and $\overline{\Omega}$ embedded in $\mathbb{R}^3$, defined via parametrizations $\mathbf{x}(u,v)$ and $\overline{\mathbf{x}}(u,v)$ over a shared domain $U \subset \mathbb{R}^2$. These surfaces are termed* conformally equivalent *when their first fundamental forms satisfy*

$$\overline{I} = \overline{E}\,du^2 + 2\overline{F}\,du\,dv + \overline{G}\,dv^2 = \sigma(u,v)\left(E\,du^2 + 2F\,du\,dv + G\,dv^2\right) = \sigma(u,v)I,$$

where $\sigma(u,v)$ denotes a smooth, strictly positive function, while I and $\overline{I}$ represent the first fundamental forms of Ω and $\overline{\Omega}$, respectively.

The transformation

$$\overline{I} = \sigma(u,v)I, \tag{4}$$

is termed the conformal change *of Ω. Through this modification, Ω experiences a conformal deformation, which indicates that there is a conformal mapping from Ω to $\overline{\Omega}$.*

Next, we examine the transformation of geometric quantities related to $\overline{\Omega}$ under the conformal change (4). In particular, we express the Gaussian curvature in terms of the coefficients of the first fundamental form.

Lemma 1. *[14] For a surface Ω parameterized by $\mathbf{x}(u,v)$, the Gaussian curvature K is given by*

$$K = \frac{1}{4(F^2-EG)^2}\begin{pmatrix} E_v^2 G + 4EF_{uv}G - 2E_uF_vG + 2E_{vv}(F^2-EG) \\ + E_uGG_u + EG_u^2 - 2EGG_{uu} \\ + 2F^2(-2F_{uv} + G_{uu}) + E(E_v - 2F_u)G_v \\ + F(4F_uF_v - 2F_uG_u - E_v(2F_v + G_u) + E_uG_v) \end{pmatrix}, \tag{5}$$

where subscripts u and v denote partial differentiation with respect to these variables.

The change of Gaussian curvature under a conformal scaling reflects how the local bending of the surface responds to uniform stretching or compression. From a physical viewpoint, K measures the intrinsic curvature or stored bending energy per unit area. When the surface metric is scaled by $\sigma(u,v)$, the curvature term transforms proportionally, showing how geometric stiffness or bending intensity redistributes across the surface under the deformation. The following theorem provides the change of the Gaussian curvature K under the change (4).

Theorem 1. *[14] Taking into account (4), the Gaussian curvature $\overline{K}$ of $\overline{\Omega}$ is given by*

$$\overline{K} = \frac{1}{\sigma}K + D_K, \tag{6}$$

where

$$\begin{aligned} D_K \;=\; & \frac{1}{4\left(F^2-EG\right)^2\sigma^3}\big((-2\,E^2G\sigma_{vv} + E^2\,G_v\,\sigma_v - E\,E_v\,G\sigma_v + 2\,EF^2\sigma_{vv} - 2\,EF\,F_v\,\sigma_v \\ & +4\,EFG\sigma_{uv} - EF\,G_u\,\sigma_v - EF\,G_v\,\sigma_u + 2\,E\,F_u\,G\sigma_v + 2\,E\,F_v\,G\sigma_u - 2\,EG^2\sigma_{uu} \\ & -EG\,G_u\,\sigma_u - E_u\,FG\sigma_v + E_u\,G^2\sigma_u + 2\,E_v\,F^2\sigma_v - E_v\,FG\sigma_u - 4\,F^3\sigma_{uv} \\ & +2\,F^2G\sigma_{uu} + 2\,F^2\,G_u\,\sigma_u - 2\,F\,F_u\,G\sigma_u)\sigma + 2\,{\sigma_v}^2E^2G - 2\,EF^2{\sigma_v}^2 \\ & -4\,GEF\sigma_u\,\sigma_v + 2\,EG^2{\sigma_u}^2 + 4\,F^3\sigma_u\,\sigma_v - 2\,F^2G{\sigma_u}^2\big). \end{aligned} \tag{7}$$

3. Conformal change of Christoffel symbols

Since $\{\mathbf{x}_u, \mathbf{x}_v, \mathbf{n}\}$ gives a basis for $\mathbb{R}^3$, there are functions Γ^1_{11}, Γ^2_{11}, $\Gamma^1_{12} = \Gamma^1_{21}$, $\Gamma^2_{12} = \Gamma^2_{21}$, Γ^1_{22}, and Γ^2_{22} so that

$$\begin{aligned} \mathbf{x}_{uu} &= \Gamma^1_{11}\mathbf{x}_u + \Gamma^2_{11}\mathbf{x}_v + \ell\mathbf{n}, \\ \mathbf{x}_{uv} &= \Gamma^1_{12}\mathbf{x}_u + \Gamma^2_{12}\mathbf{x}_v + m\mathbf{n}, \\ \mathbf{x}_{vv} &= \Gamma^1_{22}\mathbf{x}_u + \Gamma^2_{22}\mathbf{x}_v + n\mathbf{n}, \end{aligned} \tag{8}$$

these functions are called *Christoffel symbols.* Therefore, we can compute the Christoffel symbols as follows:

$$\begin{aligned} \Gamma^1_{11} &= \frac{GE_u - 2FF_u + FE_v}{2\left(EG - F^2\right)}, \quad \Gamma^1_{12} = \frac{GE_v - FG_u}{2\left(EG - F^2\right)}, \\ \Gamma^1_{22} &= \frac{2GF_v - GG_u - FG_v}{2\left(EG - F^2\right)}, \quad \Gamma^2_{11} = \frac{2EF_u - EE_v - FE_u}{2\left(EG - F^2\right)}, \\ \Gamma^2_{12} &= \frac{EG_u - FE_v}{2\left(EG - F^2\right)}, \quad \Gamma^2_{22} = \frac{EG_v - 2FF_v + FG_u}{2\left(EG - F^2\right)}. \end{aligned} \tag{9}$$

The coefficients E, F, and G are, in fact, the components of the metric tensor $\mathbf{g}$, i.e.,

$$(g_{ij}) = \begin{bmatrix} E & F \\ F & G \end{bmatrix}, \quad (g^{ij}) = \frac{1}{EG - F^2}\begin{bmatrix} G & -F \\ -F & E \end{bmatrix}.$$

In terms of the metric g_{ij}, the formulae in (9) of the Christoffel symbols Γ^i_{jk} are equivalent to the

$$\Gamma^i_{jk} = \frac{1}{2}g^{ir}(\partial_k g_{rj} + \partial_j g_{kr} - \partial_r g_{jk}). \tag{10}$$

Here, the indices $i, j, k, \ldots$ run over the values 1 and 2. Thus for conformal surfaces, we have

$$\overline{g}_{ij} = \sigma g_{ij}, \quad \overline{g}^{ij} = \frac{1}{\sigma}g^{ij}. \tag{11}$$

The modification in the metric g leads to a corresponding change in the Christoffel symbols, as described below. The transformation of the Christoffel symbols under a conformal change can be interpreted physically as a modification of the local connection or "acceleration field" on the surface. In mechanical or physical terms, these coefficients determine how tangent vectors—and hence particle trajectories constrained to the surface—adjust when the metric is scaled by the factor $\sigma(u, v)$. A larger σ locally amplifies or reduces the curvature of the coordinate grid, effectively altering the inertial guidance of motion along the surface.

Theorem 2. *For two conformal surfaces, under the conformal change* (11)*, the Christoffel symbols transform as follows:*

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + \frac{1}{2\sigma}(\delta^i_j\partial_k\sigma + \delta^i_k\partial_j\sigma - g_{jk}g^{il}\partial_l\sigma).$$

Or,

$$\begin{aligned}
\overline{\Gamma}_{11}^{1} &= \Gamma_{11}^{1} + \frac{1}{2\sigma(EG-F^2)}(EG\sigma_u - 2F^2\sigma_u + FE\sigma_v),\\
\overline{\Gamma}_{12}^{1} &= \Gamma_{12}^{1} + \frac{1}{2\sigma(EG-F^2)}(EG\sigma_v - FG\sigma_u),\\
\overline{\Gamma}_{22}^{1} &= \Gamma_{22}^{1} + \frac{1}{2\sigma(EG-F^2)}\left(GF\sigma_v - G^2\sigma_u\right),\\
\overline{\Gamma}_{11}^{2} &= \Gamma_{11}^{2} + \frac{1}{2\sigma(EG-F^2)}\left(EF\sigma_u - E^2\sigma_v\right),\\
\overline{\Gamma}_{12}^{2} = \overline{\Gamma}_{21}^{2} &= \Gamma_{12}^{2} + \frac{1}{2\sigma(EG-F^2)}\left(EG\sigma_u - FE\sigma_v\right),\\
\overline{\Gamma}_{22}^{2} &= \Gamma_{22}^{2} + \frac{1}{2\sigma(EG-F^2)}\left(EG\sigma_v - 2F^2\sigma_v + FG\sigma_u\right).
\end{aligned}$$

Proof. Starting from the Levi-Civita connection formula

$$\Gamma_{jk}^{i} = \tfrac{1}{2}g^{ir}\left(\partial_k g_{rj} + \partial_j g_{rk} - \partial_r g_{jk}\right),$$

and substituting the conformal relation $g_{ij} = \sigma g_{ij}$ with $g^{ij} = \frac{1}{\sigma}g^{ij}$, we differentiate to obtain

$$\partial_k g_{ij} = \partial_k \sigma g_{ij} + \sigma\, \partial_k g_{ij}.$$

Collecting the resulting terms yields the additional expression proportional to $\partial\sigma$, which gives the required transformation law.

Let's provide the following example.

Example 1. *The unit sphere S^2 with the parametrization (see Figure 1a):*

$$\Phi_1(u,v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{-1+u^2+v^2}{1+u^2+v^2}\right),$$

is conformally related to the Enneper surface, which is represented by the parameterization: (see Figure 1b)

$$\Phi_2(u,v) = \left(\frac{u}{3}\left(1 - \frac{u^2}{3} + v^2\right), \frac{v}{3}\left(1 + u^2 - \frac{v^2}{3}\right), \frac{1}{3}(u^2 - v^2)\right).$$

In this example, the conformal factor σ is given by

$$\sigma = \frac{1}{36}(1+u^2+v^2)^4.$$

The first fundamental form coefficients of the of the surfaces Φ_1 and Φ_2 are, respectively, given by

$$E = \frac{4}{(1+u^2+v^2)^2}, \quad F = 0, \quad G = \frac{4}{(1+u^2+v^2)^2},$$

$$\overline{E} = \sigma E = \frac{1}{9}(1+u^2+v^2)^2, \quad \overline{F} = \sigma F = 0, \quad \overline{G} = \sigma G = \frac{1}{9}(1+u^2+v^2)^2.$$

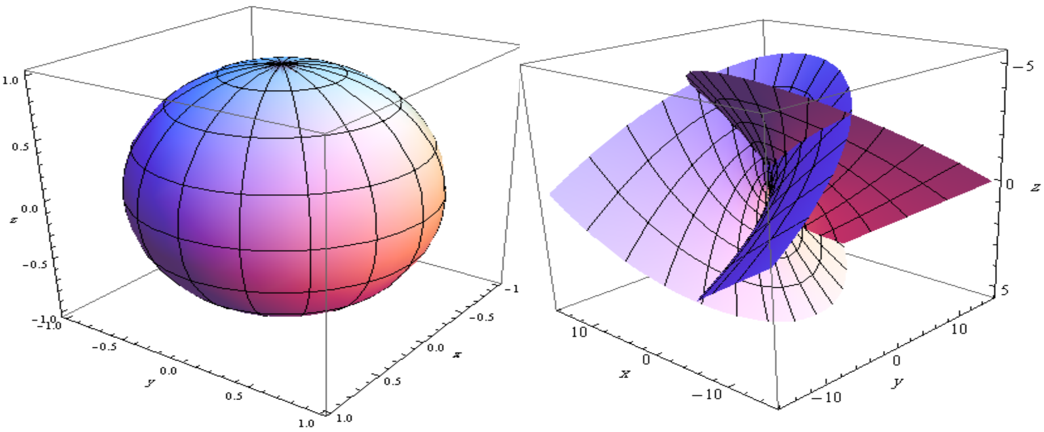
The Gaussian Curvature $\overline{K}$ *of* Φ_2 *is given by*

$$\overline{K} = \frac{K}{\sigma} + D_K = \frac{36}{(1+u^2+v^2)^4} - \frac{72}{(1+u^2+v^2)^4} = -\frac{36}{(1+u^2+v^2)^4},$$

where the curvature of the unit sphere is $K = 1$.

The Christoffel symbols of the Enneper surface can be calculated by using Theorem 2 as follows:

$$\begin{aligned}
\overline{\Gamma}^1_{11} &= -\frac{2u}{u^2+v^2+1} + \frac{4u}{u^2+v^2+1} = \frac{2u}{u^2+v^2+1} \\
\overline{\Gamma}^1_{12} &= -\frac{2v}{u^2+v^2+1} + \frac{4v}{u^2+v^2+1} = \frac{2v}{u^2+v^2+1} \\
\overline{\Gamma}^1_{22} &= \frac{2u}{u^2+v^2+1} - \frac{4u}{u^2+v^2+1}) = -\frac{2u}{u^2+v^2+1} \\
\overline{\Gamma}^2_{11} &= \frac{2v}{u^2+v^2+1} + \frac{4v}{u^2+v^2+1} = -\frac{2v}{u^2+v^2+1} \\
\overline{\Gamma}^2_{12} = \overline{\Gamma}^2_{21} &= -\frac{2u}{u^2+v^2+1} + \frac{4u}{u^2+v^2+1} = \frac{2u}{u^2+v^2+1} \\
\overline{\Gamma}^2_{22} &= -\frac{2v}{u^2+v^2+1} + \frac{4v}{u^2+v^2+1} = \frac{2v}{u^2+v^2+1}.
\end{aligned}$$



(a) (b)

Figure 1: (a) The unit sphere Φ_1, (b) The Enneper surface Φ_2.

4. Covariant derivative

Consider two surfaces Ω and $\overline{\Omega}$ in $\mathbb{R}^3$, given by the local parameterizations $\mathbf{x}$ and $\overline{\mathbf{x}}$ on the same domain $U \subset \mathbb{R}^2$. If the surfaces are conformally equivalent—i.e., $\exists$ a conformal diffeomorphism $f : \Omega \to \overline{\Omega}$—then we have

$$f_* : T_p\Omega \to T_{f(p)}\overline{\Omega}.$$

So f_* (or df) pushforwards tangent vectors on the surface Ω to tangent vectors on $\overline{\Omega}$, see the diagram bellow.

$$\begin{tikzcd} T_p\Omega \arrow[r, "f_*"] \arrow[d, "\pi"] & T_{f(p)}\overline{\Omega} \arrow[d, "\pi'"] \\ p \in \Omega \arrow[r, "f"] & f(p) \in \overline{\Omega} \end{tikzcd}$$

Let $X = X^i\,\partial_i$ be a vector field on Ω, then

$$f_*X = X^i \frac{\partial f^k}{\partial x^i} \frac{\partial}{\partial y^k},$$

where y^i are the coordinates on $\overline{\Omega}$ and $\left\{\frac{\partial}{\partial y^k}\right\}$ are basis of tangent spaces on $\overline{\Omega}$. That is, the components of f_*X in the $\left\{\frac{\partial}{\partial y^k}\right\}$ basis are:

$$(f_*X)^k = \frac{\partial f^k}{\partial x^i} X^i.$$

This gives a vector field on the surface $\overline{\Omega}$. We can provide the following example.

Example 2. *Consider a tangent vector field* $\mathbf{V}$ *on* S^2*, namely, the longitude rotation vector field, which is tangent to the lines of longitude. That is,* $\mathbf{V}$ *is defined by*

$$\mathbf{V}(x, y, z) = (-y, x, 0).$$

This vector field rotates around the z-axis, see Figure 2a. We verify tangency using the position vector $\mathbf{n} = (x, y, z)$*:*

$$\mathbf{n} \cdot \mathbf{V} = x(-y) + y(x) + z(0) = -xy + xy = 0$$

which implies $\mathbf{V} \perp \mathbf{n}$*. That is,* $\mathbf{V} \in T_pS^2$ *at every point.*

The stereographic projection $\pi : S^2 \setminus \{(0, 0, 1)\} \to \mathbb{R}^2$ *is given by:*

$$(u, v) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right).$$

This transformation assigns to each point on the sphere (excluding the north pole) a corresponding point in the plane $z = 0$. The stereographic projection functions as a conformal correspondence between the spherical surface and the planar surface.

The spherical coordinates (x, y, z) can be formulated using the parameters (u, v) as follows:

$$x = \frac{2u}{u^2 + v^2 + 1}, \quad y = \frac{2v}{u^2 + v^2 + 1}, \quad z = \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}$$

To push forward a vector field, we compute the Jacobian matrix π_* of the stereographic projection. Using the facts that

$$u = \frac{x}{1 - z}, \quad v = \frac{y}{1 - z},$$

we have the following partial derivatives:

$$\frac{\partial u}{\partial x} = \frac{1}{1 - z}, \quad \frac{\partial u}{\partial z} = \frac{x}{(1 - z)^2}, \quad \frac{\partial v}{\partial y} = \frac{1}{1 - z}, \quad \frac{\partial v}{\partial z} = \frac{y}{(1 - z)^2}.$$

So the Jacobian (differential) π_* is given by

$$\pi_* = \begin{bmatrix} \frac{1}{1-z} & 0 & \frac{x}{(1-z)^2} \\ 0 & \frac{1}{1-z} & \frac{y}{(1-z)^2} \end{bmatrix}.$$

Then, the pushforward $\overline{\mathbf{V}}$ of $\mathbf{V}$ is obtained as follows:

$$\overline{\mathbf{V}} = \pi_*(\mathbf{V}) = \begin{bmatrix} \frac{1}{1-z} & 0 & \frac{x}{(1-z)^2} \\ 0 & \frac{1}{1-z} & \frac{y}{(1-z)^2} \end{bmatrix} \begin{bmatrix} -y \\ x \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{-y}{1-z} \\ \frac{x}{1-z} \end{bmatrix}$$

So the pushforward of the vector field on the sphere to the plane is:

$$\overline{\mathbf{V}}(u, v) = \left(\frac{-y}{1 - z}, \frac{x}{1 - z} \right) = \frac{u^2 + v^2 + 1}{2}(-v, u).$$

This is a rotational vector field in the plane, see Figure 2b, scaled by a factor depending on the distance from the origin. The direction is preserved (angle preserved), but the length is rescaled by a conformal factor.

In differential geometry, understanding how vectors change as they move along a surface is crucial. This is formalized through the concepts of the covariant derivative and parallel transport, both of which can be expressed using the Christoffel symbols derived from the first fundamental form of the surface.

The covariant derivative allows us to differentiate vector fields along the surface in a manner that accounts for the surface's curvature. For a vector field $\mathbf{V} = V^i \mathbf{e}_i$, where $\mathbf{e}_i$ are the basis vectors, the covariant derivative along the coordinate direction x^j is:

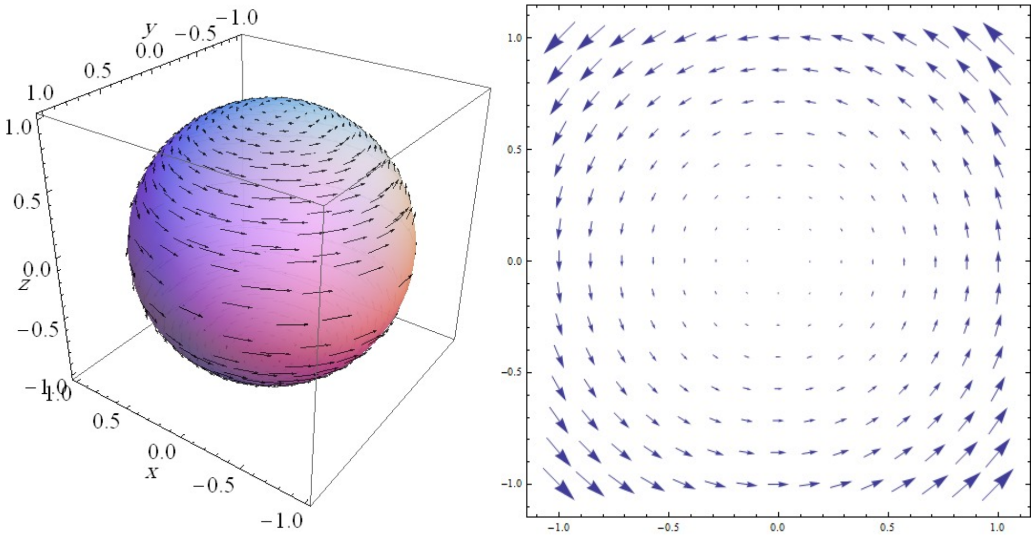


Figure 2: (a) The vector field $\mathbf{V}$ on S^2 (b) The vector field $\overline{\mathbf{V}}$ on the plane

$$\nabla_j V^i = \frac{\partial V^i}{\partial x^j} + \Gamma^i_{jk} V^k. \tag{12}$$

This operation ensures that the derivative of the vector field remains within the tangent space of the surface.

We want to compute:

$$\overline{\nabla}_{f_* Y}(f_* X)$$

for some vector fields X and Y on Ω. Such a relation is given by the following theorem where we use the notation ∂_i to denote the basis vector $\frac{\partial}{\partial x^i}$.

The conformal modification of the covariant derivative illustrates how vector directions evolve when the underlying metric changes scale. In continuum mechanics or general relativity, this corresponds to adjusting parallel transport rules so that vectors maintain their orientation relative to the new scaled geometry. Thus, the additional term involving σ represents the compensating effect of the varying scale factor on the rate of change of tangent vectors along the surface.

Theorem 3. *Let $f : \Omega \to \overline{\Omega}$ be a conformal diffeomorphism between two surfaces, and $X = X^i \partial_i$ and $Y = Y^i \partial_i$ are vector fields on Ω, then we have*

$$\overline{\nabla}_{f_* Y}(f_* X) = f_* \left(\nabla_Y X\right) + \frac{1}{2\sigma} \left[\delta^k_i(\partial_j \sigma) + \delta^k_j(\partial_i \sigma) - g_{ij} g^{kl}(\partial_l \sigma)\right] X^j Y^i \, f_*(\partial_k).$$

Proof. If $f : \Omega \to \overline{\Omega}$ is a conformal diffeomorphism between two surfaces, $X = X^i \partial_i$ and $Y = Y^i \partial_i$ are vector fields on Ω, then our goal is to prove that

$$\overline{\nabla}_{f_* Y}(f_* X) = f_*(\nabla_Y X) + T(X, Y),$$

where $T(X,Y)$ is the difference term due to the conformal factor. For the surface Ω and $\overline{\Omega}$, we have $\overline{g}_{ij} = \sigma g_{ij}$. Hence, by Theorem 2, we have the following difference

$$\overline{\Gamma}^i_{jk} - \Gamma^i_{jk} = \frac{1}{2\sigma}(\delta^i_j \partial_k \sigma + \delta^i_k \partial_j \sigma - g_{jk} g^{il} \partial_l \sigma).$$

In local coordinates, by (12), we the following formulae:

$$\nabla_Y X = Y^j \left(\partial_j X^k + \Gamma^k_{jl} X^l \right) \partial_k,$$

$$f_*(\nabla_Y X) = Y^j \left(\partial_j X^k + \Gamma^k_{jl} X^l \right) f_*(\partial_k),$$

$$\overline{\nabla}_{f_* Y}(f_* X) = Y^j (f_*(\partial_j)(X^k \circ f^{-1}) + \overline{\Gamma}^k_{jl} X^l) f_*(\partial_k) = Y^j (\partial_j X^k + \overline{\Gamma}^k_{jl} X^l) f_*(\partial_k).$$

where we used the fact that $f_*\left(\partial_j\right)(h) = \partial_j(h \circ f)$ for any smooth function h on Ω. That is, we get

$$\overline{\nabla}_{f_* Y}(f_* X) - f_*(\nabla_Y X) = Y^j X^l \left(\overline{\Gamma}^k_{jl} - \Gamma^k_{jl} \right) f_*(\partial_k).$$

Substitute the difference of Christoffel symbols, then we get

$$\overline{\Gamma}^k_{jl} - \Gamma^k_{jl} = \frac{1}{2\sigma}(\delta^k_j \partial_l \sigma + \delta^k_l \partial_j \sigma - g_{jl} g^{km} \partial_m \sigma),$$

which yields

$$T(X,Y) = \frac{1}{2\sigma} Y^j X^l \left(\delta^k_j \partial_l \sigma + \delta^k_l \partial_j \sigma - g_{jl} g^{km} \partial_m \sigma \right) f_*(\partial_k).$$

We conclude this section with the following remark.

Remark 1. *Parallel transport involves moving a vector along a curve on the surface such that it remains parallel with respect to the surface's connection. Given a curve $\gamma(t)$ on the surface, a vector field $\mathbf{V}(t)$ is said to be parallel transported along γ if its covariant derivative along the curve vanishes:*

$$\frac{D\mathbf{V}}{dt} = \nabla_{\dot{\gamma}(t)} \mathbf{V}(t) = \frac{dV^i}{dt} + \Gamma^i_{jk} \frac{dx^j}{dt} V^k = 0.$$

Solving this differential equation provides the components $V^i(t)$ of the vector field along the curve $\gamma(t)$. Understanding these concepts is fundamental in differential geometry, as they provide the tools to study how vectors and tensors behave on curved surfaces.

5. Conformal change of geodesics

The geodesic equations on a surface describe the paths that locally minimize distance, and they can be expressed in terms of Christoffel symbols derived from the surface's first fundamental form (metric tensor). If a surface is parametrized by coordinates (u^1, u^2), typically denoted as (u, v), the geodesic equations are:

$$\frac{d^2u^k}{dt^2} + \Gamma^k_{ij}\frac{du^i}{dt}\frac{du^j}{dt} = 0,$$

or more concretely (writing explicitly for $u^1 = u$ and $u^2 = v$):

$$\begin{aligned}
&\frac{d^2u}{dt^2} + \Gamma^1_{11}\left(\frac{du}{dt}\right)^2 + 2\Gamma^1_{12}\frac{du}{dt}\frac{dv}{dt} + \Gamma^1_{22}\left(\frac{dv}{dt}\right)^2 = 0,\\
&\frac{d^2v}{dt^2} + \Gamma^2_{11}\left(\frac{du}{dt}\right)^2 + 2\Gamma^2_{12}\frac{du}{dt}\frac{dv}{dt} + \Gamma^2_{22}\left(\frac{dv}{dt}\right)^2 = 0.
\end{aligned}$$

Now to study these equations under the transformation we set

$$\begin{aligned}
\gamma_1 &:= \frac{d^2u}{dt^2} + \Gamma^1_{11}\left(\frac{du}{dt}\right)^2 + 2\Gamma^1_{12}\frac{du}{dt}\frac{dv}{dt} + \Gamma^1_{22}\left(\frac{dv}{dt}\right)^2,\\
\gamma_2 &:= \frac{d^2v}{dt^2} + \Gamma^2_{11}\left(\frac{du}{dt}\right)^2 + 2\Gamma^2_{12}\frac{du}{dt}\frac{dv}{dt} + \Gamma^2_{22}\left(\frac{dv}{dt}\right)^2.
\end{aligned}$$

Now, we have

$$\begin{aligned}
\overline{\gamma}_1 &= \frac{d^2u}{dt^2} + \overline{\Gamma}^1_{11}\left(\frac{du}{dt}\right)^2 + 2\overline{\Gamma}^1_{12}\frac{du}{dt}\frac{dv}{dt} + \overline{\Gamma}^1_{22}\left(\frac{dv}{dt}\right)^2\\
&= \gamma_1 + \frac{1}{2\sigma(EG-F^2)}\Bigg((EG\sigma_u - 2F^2\sigma_u + FE\sigma_v)\left(\frac{du}{dt}\right)^2 + 2(EG\sigma_v - FG\sigma_u)\frac{du}{dt}\frac{dv}{dt}\\
&+ (GF\sigma_v - G^2\sigma_u)\left(\frac{dv}{dt}\right)^2\Bigg), \qquad (13)
\end{aligned}$$

$$\begin{aligned}
\overline{\gamma}_2 &= \frac{d^2v}{dt^2} + \overline{\Gamma}^2_{11}\left(\frac{du}{dt}\right)^2 + 2\overline{\Gamma}^2_{12}\frac{du}{dt}\frac{dv}{dt} + \overline{\Gamma}^2_{22}\left(\frac{dv}{dt}\right)^2\\
&= \gamma_2 + \frac{1}{2\sigma(EG-F^2)}\Bigg((EF\sigma_u - E^2\sigma_v)\left(\frac{du}{dt}\right)^2 + 2(EG\sigma_u - FE\sigma_v)\frac{du}{dt}\frac{dv}{dt}\\
&+ (EG\sigma_v - 2F^2\sigma_v + FG\sigma_u)\left(\frac{dv}{dt}\right)^2\Bigg). \qquad (14)
\end{aligned}$$

The following theorem has peen proven in [14], but here we provide an alternative proof based on the change of the Christoffel symbols under the conforma change.

Theorem 4. *Suppose $\gamma(t)$ is a geodesic on the surface Ω. Then the curve $\overline{\gamma}(t) = f(\gamma(t))$, where f is a conformal mapping, will trace a geodesic on $\overline{\Omega}$ if and only if the conformal factor σ is constant.*

Proof. If γ is a geodesic on Ω and the conformal factor σ is constant, then by (13) and (14), $\overline{\gamma}$ is a geodesic on $\overline{\Omega}$.

Now, assume that γ is a geodesic on Ω and $\overline{\gamma}$ is a geodesic on $\overline{\Omega}$, then by (13) we have the following system:

$$(EG\sigma_u - 2F^2\sigma_u + FE\sigma_v)\left(\frac{du}{dt}\right)^2 + 2(EG\sigma_v - FG\sigma_u)\frac{du}{dt}\frac{dv}{dt} + (GF\sigma_v - G^2\sigma_u)\left(\frac{dv}{dt}\right)^2 = 0$$
$$(EF\sigma_u - E^2\sigma_v)\left(\frac{du}{dt}\right)^2 + 2(EG\sigma_u - FE\sigma_v)\frac{du}{dt}\frac{dv}{dt} + (EG\sigma_v - 2F^2\sigma_v + FG\sigma_u)\left(\frac{dv}{dt}\right)^2 = 0.$$

This system can be considered as algebraic system at every point on the surface Ω. Moreover, it can be seen as a system of two polynomials in $\frac{du}{dt}$ and $\frac{dv}{dt}$. That is, we can get the following equations:

$$\begin{aligned} EG\sigma_u - 2F^2\sigma_u + FE\sigma_v &= 0 \\ 2(EG\sigma_v - FG\sigma_u) &= 0 \\ GF\sigma_v - G^2\sigma_u &= 0. \end{aligned}$$

The last two equations in the above system has non trivial solutions for σ_u and σ_v if and only if

$$G^2(EG - F^2) = 0.$$

Which is a contradiction since $G \neq 0$ and $EG - F^2 \neq 0$. That is, we must have $\sigma_u = 0$ and $\sigma_v = 0$ i.e. σ is a constant.

Example 3. *Once again, consider the surfaces in Example 2. That is, the sphere with the parametrization (see Figure 3a):*

$$\Phi_1(u,v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{-1+u^2+v^2}{1+u^2+v^2}\right).$$

Using the Christoffel symbols, the geodesic equations in $u(t)$, $v(t)$ are given by:

$$\frac{d^2u}{dt^2} + \frac{2u}{1+u^2+v^2}\left[\left(\frac{du}{dt}\right)^2 - \left(\frac{dv}{dt}\right)^2\right] + \frac{4v}{1+u^2+v^2}\frac{du}{dt}\frac{dv}{dt} = 0$$
$$\frac{d^2v}{dt^2} + \frac{2v}{1+u^2+v^2}\left[\left(\frac{dv}{dt}\right)^2 - \left(\frac{du}{dt}\right)^2\right] + \frac{4u}{1+u^2+v^2}\frac{du}{dt}\frac{dv}{dt} = 0.$$

That is, the geodesics on the sphere are the great circles parametrized by

$$\gamma(t) = (\cos t, \sin t, 0).$$

Now, for the Enpper surface with the parameterization (see Figure 3b):

$$\Phi_2(u,v) = \left(\frac{u}{3}\left(1-\frac{u^2}{3}+v^2\right), \frac{v}{3}\left(1+u^2-\frac{v^2}{3}\right), \frac{1}{3}(u^2-v^2)\right),$$

the geodesic equations are given by:

$$\frac{d^2u}{dt^2} + \frac{2u}{u^2+v^2+1}\left(\left(\frac{du}{dt}\right)^2 - \left(\frac{dv}{dt}\right)^2\right) + \frac{4v}{u^2+v^2+1}\frac{du}{dt}\frac{dv}{dt} = 0$$
$$\frac{d^2v}{dt^2} + \frac{2v}{u^2+v^2+1}\left(\left(\frac{dv}{dt}\right)^2 - \left(\frac{du}{dt}\right)^2\right) + \frac{4u}{u^2+v^2+1}\frac{du}{dt}\frac{dv}{dt} = 0$$

Given a point $(x,y,z) \in S^2 \setminus \{(0,0,1)\}$, the stereographic projection is:

$$u = \frac{x}{1-z}, \quad v = \frac{y}{1-z}$$

Using the (u,v) coordinates from above, plug into the Enneper surface:

$$\Phi(u,v) = \left(\frac{u}{3}\left(1-\frac{u^2}{3}+v^2\right),\ \frac{v}{3}\left(1+u^2-\frac{v^2}{3}\right),\ \frac{1}{3}(u^2-v^2)\right)$$

So, the explicit conformal mapping $\Psi : S^2 \setminus \{(0,0,1)\} \to \mathbb{R}^3$ is:

$$f(x,y,z) = \left(\frac{x(x^2-3y^2-3z^2+6z-3)}{9(z-1)^3}, -\frac{y(3x^2-y^2+3z^2-6z+3)}{9(z-1)^3}, \frac{(x^2-y^2)}{3(z-1)^2}\right)$$

This gives the conformal map from the unit sphere (minus the north pole) to the Enneper surface.

Therefore, the transformed curves are

$$\overline{\gamma} = f\circ\gamma = \left(-\frac{2}{9}(2\cos^2 t-3)\cos t, \frac{2}{9}(2\cos^2 t+1)\sin t, -\frac{1}{3}(1-2\cos^2 t)\right),$$

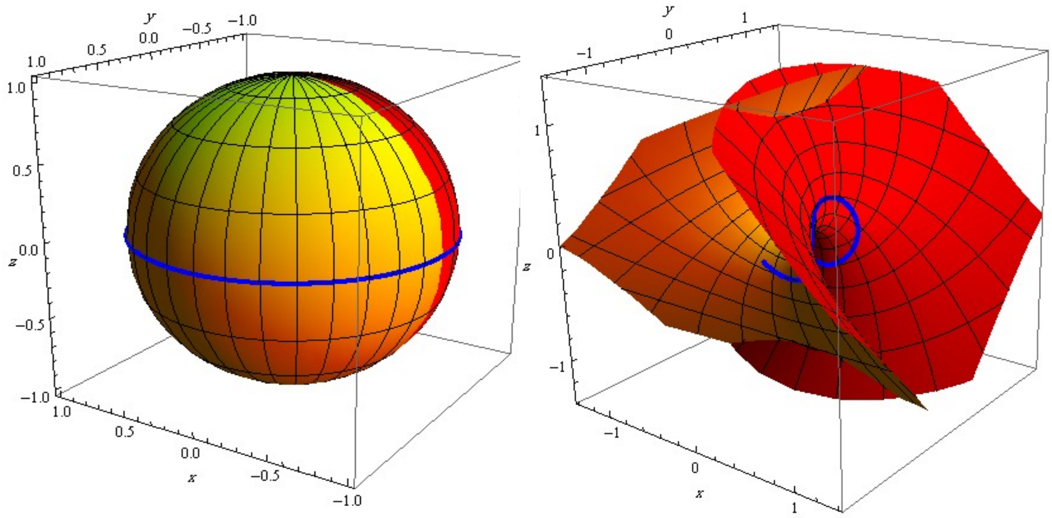
which are not geodesics on the Enneper.

Example 4. *Consider the regular surfaces Ω and $\overline{\Omega}$, defined by the following parameterizations:*

$$\Phi_1(u,v) = (\cos(u),\, \sin(u),\, v)\,, \quad \Phi_2(u,v) = (\cosh(u)\cos(v),\, \cosh(u)\sin(v),\, u)\,.$$

A direct computation shows that the curve

$$\gamma(t) = (\cos(t),\, \sin(t),\, 0)$$



(a) (b)

Figure 3: (a) The geodesic $\gamma(t)$ on the sphere $\Phi_1(u, v)$, (b) The curve $\overline{\gamma}$ which is not a geodesic on the Enneper surface $\Phi_2(u, v)$.

lies entirely on Ω, and its image under an appropriate conformal transformation also lies on $\overline{\Omega}$, where it corresponds to a geodesic.

The first fundamental form coefficients for the surfaces are computed as follows: for the cylinder Ω,

$$E = 1, \quad F = 0, \quad G = 1, \quad K = -\operatorname{sech}^4(u),$$

while for the surface $\overline{\Omega}$, which is conformally related to Ω, we obtain

$$\overline{E} = \sigma E = \cosh^2(u), \quad \overline{F} = \sigma F = 0, \quad \overline{G} = \sigma G = \cosh^2(u), \quad \overline{K} = 0.$$

The conformal map $f : \Omega \to \overline{\Omega}$ is defined by

$$f(x, y, z) = \left(\cosh(\cos^{-1}(x)) \cos z, \, \cosh(\sin^{-1}(y)) \sin z, \, \cos^{-1}(x)\right).$$

On the cylinder Ω, the Christoffel symbols vanish, i.e., $\Gamma^i_{jk} = 0$, and thus the geodesic equations reduce to

$$\frac{d^2u}{dt^2} = 0, \quad \frac{d^2v}{dt^2} = 0.$$

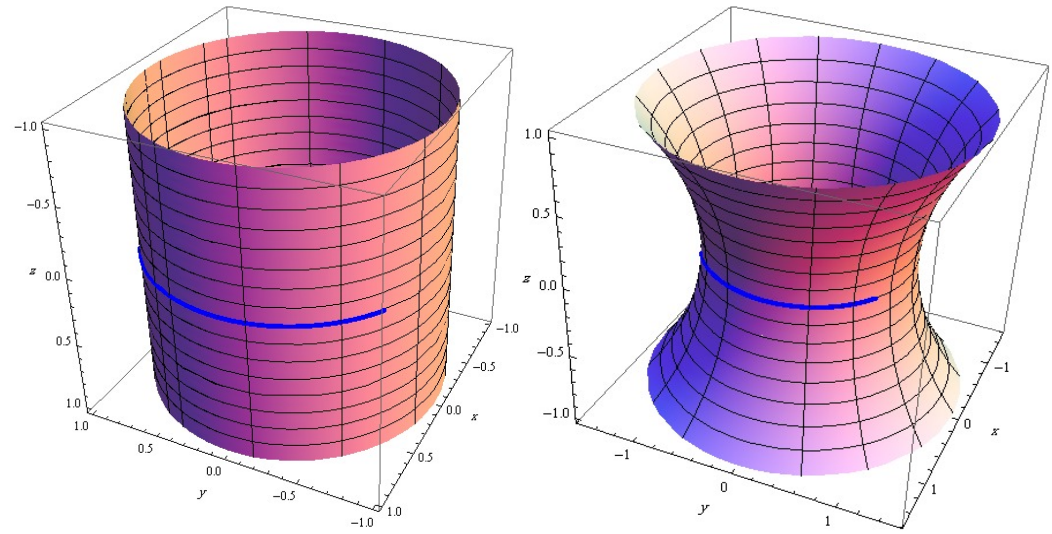
This implies that geodesics on the cylinder are straight lines in the parameter domain, and are given explicitly by

$$\gamma(t) = (\cos(a + bt), \, \sin(a + bt), \, c + dt),$$

where $u = a + bt$, $v = c + dt$, and $a, b, c, d \in \mathbb{R}$ are constants.

For the catenary, the Christofell symbols are

$$\overline{\Gamma}^1_{11} = \tanh u, \quad \overline{\Gamma}^1_{22} = -\tanh u, \quad \overline{\Gamma}^2_{12} = \overline{\Gamma}^2_{21} = \tanh u.$$



(a) (b)

Figure 4: (a) The geodesic curve γ on the surface $X(u,v)$, (b) The geodesic curve $\overline{\gamma}$ on the surface $Y(u,v)$.

Hence, the geodesic equations on the catenary are:

$$\frac{d^2u}{dt^2} + \tanh u \left(\left(\frac{du}{dt}\right)^2 - \left(\frac{dv}{dt}\right)^2 \right) = 0$$

$$\frac{d^2v}{dt^2} + 2\tanh u \frac{du}{dt}\frac{dv}{dt} = 0$$

Substituting by $u = a + bt$ and $v = c + dt$ into the above equations, we get

$$(b^2 - d^2)\tanh(a + bt) = 0, \quad bd\tanh(a + bt) = 0.$$

Thus, applying the conformal mapping f to the original curve $\gamma(t)$ yields the transformed curve

$$\overline{\gamma}(t) = (\cosh(a + bt)\cos(c + dt),\, \cosh(a + bt)\sin(c + dt),\, a + bt)\,.$$

We can have the following two cases:

Case I: (Both curves γ and $\overline{\gamma}$ are geodesics)

It can be deduced that the curve $\overline{\gamma}(t)$ lies on the catenary as a geodesic if and only if the conditions

$$b^2 - d^2 = 0, \quad bd = 0$$

are satisfied. These conditions are equivalent to $b = d = 0$. Under this constraint, the resulting curve reduces to

$$\overline{\gamma}(t) = (\cosh(a)\cos(c),\, \cosh(a)\sin(c),\, a)\,,$$

which represents a geodesic on the catenary with constant speed $\cosh^2(a)$*, see Figures 4a and 4b.*

Case II: (The curve γ is a geodesic while $\overline{\gamma}$ is not)

In this case, the equations

$$b^2 - d^2 = 0, \quad bd = 0$$

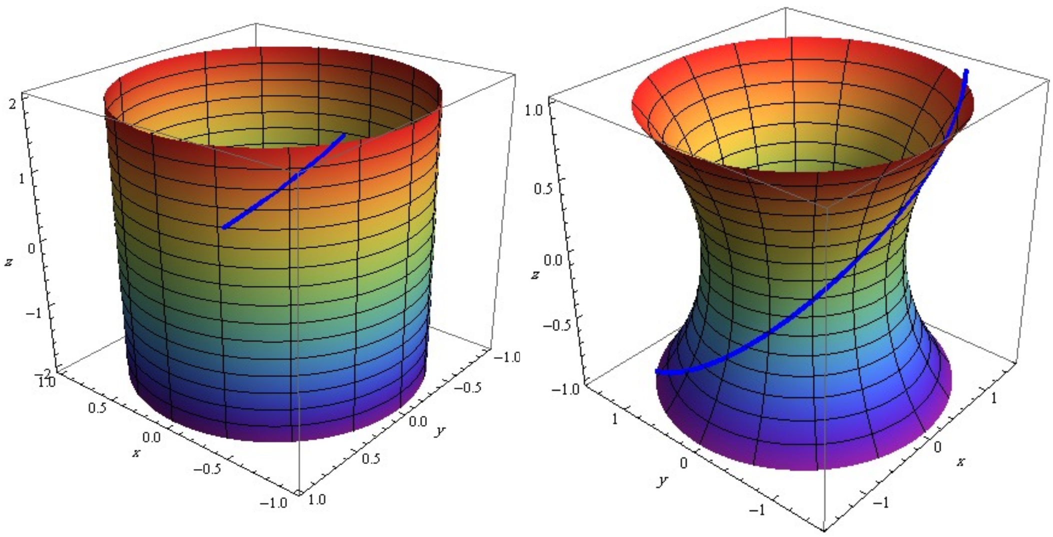
are not satisfied. So the curve The geodesics on the unit cylinder (see Figure 5a) are the curves parametrized by

$$\gamma(t) = (\cos(a + bt), \sin(a + bt), c + dt),$$

is a geodesic on the cylinder while the curve

$$\overline{\gamma} = f \circ \gamma = (\cosh(a + bt)\cos(c + dt), \cosh(a + bt)\sin(c + dt), a + bt).$$

is not a geodesic on the catenary (see Figure 5b) for which $b^2 - d^2 \neq 0$, ***or*** $bd \neq 0$*, see Figures 5a and 5b.*



(a) (b)

Figure 5: The curve γ is a geodesic on Ω, (b) The curve $\overline{\gamma}$ is not a geodesic on $\overline{\Omega}$.

6. Conclusion

In this work, we have explored the geometric implications of conformal changes to the first fundamental form of a surface. By showing the transformations of Christoffel symbols, covariant derivatives, Gaussian curvature, and geodesic equations under conformal scaling, we demonstrated how such changes preserve the conformal structure while modifying the intrinsic geometry. The behavior of geodesic curves and curvature under these transformations highlights the impact of the conformal deformation in the geometric properties. Through both theoretical development and illustrative examples, we have shown how conformal metrics influence foundational geometric constructs, offering a deeper understanding of surface geometry under metric scaling. These insights not only reinforce classical differential geometry results but also pave the way for further investigations into geometric analysis and mathematical physics where conformal structures play a central role.

The present analysis, though confined to smooth two-dimensional surfaces with non-vanishing $EG - F^2$, suggests promising directions for extending conformal geometry to irregular or fractal surfaces. Fractal structures often arise in natural and engineered systems—such as biological tissues, porous media, and microstructured interfaces—where differentiability fails globally. In such cases, conformal transformations can be investigated through *two-scale fractal dimensions*, which distinguish between macroscale and microscale roughness characteristics. Introducing a scale-dependent conformal factor $\sigma(u, v, \varepsilon)$, where ε represents the observation scale, would allow curvature, metric, and geodesic transformations to account for microscale irregularities, thereby linking conformal geometry with fractal and multiscale surface analysis.

Beyond theoretical significance, the proposed framework has notable applied relevance in several disciplines. In medical imaging, conformal deformation methods can improve anatomical surface mapping while preserving local geometric accuracy; in computer graphics and geometric modeling, understanding geodesic and curvature behavior under conformal scaling aids in texture mapping and deformation simulation. Similarly, in materials science, the study of curvature and metric variations under scaling provides insight into how surface roughness influences mechanical and functional properties (see, e.g., [4]). Collectively, these perspectives establish a bridge between classical conformal geometry and the emerging study of multiscale and fractal phenomena.

Author contributions

All authors wrote the whole paper and approved all the results.

Conflict of interest

The authors declare no conflict of interest.

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