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Some New Results and Application of Caputo-Fractional-Order Ostrowski's-Type Inequality

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Abstract. This article is about a study of a new concept of fractional order Ostrowski's type inequality for interval-valued functions using Caputo generalized Hukuhara (gH) differentiability. As fractional calculus gains traction in mathematical modeling, particularly in contexts involving uncertainty, our focus lies on extending Ostrowski's type inequality—a fundamental result in numerical analysis—within this framework. We begin by defining the Caputo gH-derivative and exploring its properties as applied to interval-valued functions. Through rigorous analysis, we derive a general form of Ostrowski's type inequality that incorporates fractional order derivatives, demonstrating how this approach accommodates the inherent variability of interval-valued data. We also give a comparison of fractional order Ostrowski's type inequality with the classical order Ostrowski's type inequality. Existence and uniqueness of Caputo integral inequity is verified with Lipschitz property and also stability analysis is derived with Lyapunov function for Ostrowski's type inequality. The results not only enhance the theoretical landscape of fractional calculus but also have significant implications for practical applications across various scientific fields, including engineering and economics, where interval data frequently arises. To validate our findings, we provide illustrative examples that showcase the applicability of the generalized Ostrowski's type inequality, highlighting its potential to manage complex phenomena in interval-valued functions. Our results are new, novel and generalization of the classical Ostrowski's type inequality. Ultimately, this work aims to bridge the gap between fractional calculus and interval analysis, paving the way for future research in this promising area and expanding the toolkit available for mathematicians and practitioners alike.

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1. Introduction

Fractional calculus is about the derivatives and integrals of arbitrary order and very famous concept in mathematical theory. Over three centuries ago, it was initially a purely theoretical concept, but during recent years it spread widely due to its applications like in viscoelasticity [1], diffusion processes [2], electrochemistry [3] and many others. The most recent work in this manner can be seen in [4], [5], [6], [7], [8], [9], [10], and [11]. Integral inequalities are mathematical concepts that provide bounds or estimates for the values of integrals. They are used to establish relationships between functions, allowing for comparison and control over the behavior of integrals without requiring explicit solutions. Examples are Jensen's inequality which is crucial in Probability and statistics and other is Hölder's inequality which is foundation in L^p spaces and functional analysis. These inequalities not only help in theoretical proofs but also support applications in differential equations, optimization, and various scientific models [12].

In general, the behavior of inequalities refers to their behavior or about maintaing their properties under various mathematical operations and transformations. Key behaviors of inequalities are monotonicity, convexity/concavity and stability. Ostrowski's type inequality exhibit similar behaviors to general inequalities, including monotonicity preservation, predictable scaling, and stability under mathematical operations. However, unlike inequalities dependent on convexity, Ostrowski's type inequality focuses on estimating the deviation of a function's value at a specific point from its average value over an interval. This inequality is also sensitive to the integration domain, much like Holder's inequality. In essence, Ostrowski's type inequality is tailored for estimating bounds and errors in approximations. In [13] Chalco-Cano et al. obtained Ostrowski-type inequalities for interval-valued functions by using Hukuhara derivative for interval-valued functions. Some improvements and further generalizations on the Ostrowski-Gruss inequalities involving bounded once and twice differentiable mappings has been found in [14], also in [15], B.G.Pachpatte, solved the Ostrowski type inequality with two independent variables.

The significant inequality, known as Ostrowski's type inequality was established by Ostrowski in 1938 defined as

$$\left|\frac{1}{b-a}\int_a^b f(y)dy - f(x)\right| \leq \left(\frac{1}{4}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right)(b-a)\left\|f'\right\|_\infty, \tag{1}$$

where $f \in C^1[a,b]$, $x \in [a,b]$ see [16].

The inequality defined in (1) has been the subject of extensive research in recent times and several scholars are considering its variations, expansions, and generalizations ([17]-[18]). While estimating the integral average $\frac{1}{(b-a)}\int_a^b f(y)dy$ at a point $x \in [a,b]$, this inequality provides an upper bound for the deviation of the function $f(x)$ from its integral average (see [19]).

Fractional calculus, which extends differentiation and integration to non-integer orders, provides a realistic framework for modeling phenomena with long-term memory and fractal behaviors, unlike classical calculus. Fractional calculus supports advancements in analyzing complex and chaotic systems, fixed-point theory, Lagrange's polynomial schemes [20]. These methods, reinforced by adaptive sliding mode control [21], find applications in physics, financial analysis, signal processing, secure computing, advanced circuit design, and industrial processes, leveraging their non-linear dynamics of a piecewise modified Atangana-Baleanue Caputo derivative [22] and fractal-fractional based modeling [21]. Inspiration of fractional calculus is almost every where. In [23] V. Luplescue worked on the fractional calculus for interval-valued functions. That is why mathematicians are working to generalize the inequalities using various types of fractional order derivatives. A very recent advancement is done by Milovanovic and Pecaric in which they have introduced the first generalization of Ostrowski's type inequality see [24]. Another development of fractional derivatives is for interval-valued functions, which is done through the application of either the H-derivative or G-derivative, that are the generalized derivative concepts extended to handle interval-valued data according to the classical derivative concept. Banks and Jacobs [25], Hukuhara [26], Puri and Ralescu [27] are among the authors of the well-known and ancient Hukuhara derivative notion. In contrast, Bede and Gal [28] are the authors of the more contemporary G-derived concept. The drawback of the "classical" Hukuhara derivative, as introduced by Hukuhara, is its restrictive definition, which requires the subtraction of intervals to be well-defined. This limitation often leads to solutions with increasing interval widths over time, making it unsuitable for accurately modeling certain dynamic systems. Consequently, the longer the elapsed time, the greater will be the uncertainty of the solution [29]. The generalized Hukuhara derivative does not suffer from this drawback since it avoids, any time-dependent imprecision of solutions [30]. Recently, Stefanini and Bede ([31], [32], [30], [33]) researched on interval-valued differential equations and they introduced a new version of the *Hukuhara* derivative for interval-valued mappings. Their approach has solutions where values have non-increasing length. They introduced the concept of generalized Hukuhara-difference ($g_H - difference$). For an interval-valued functions, we have read some concepts of derivatives in [34] and [35]. These include the G-derivative [28], g_H-derivative [31], [36], [30] and π-derivative [25], [37], [38]. Particularly, according to [34] and [30], it is known that for an interval-valued functions, the idea of G-differentiation, g_H-differentiation and π-differentiation to be equal if there are on finite number of switching points.

Motivated by this detailed literature survey this paper is dedicated to studying Ostrowski's-type inequality using Caputo g_H-differentiable interval-valued functions. These results are studied for interval valued functions to attract researchers and mathematicians from different fields.

2. Preliminaries

This section focuses on some well-known fundamental definitions related to interval-valued functions and the principles of fractional calculus.

Definition 1. *Let $\mathbb{R}$ be the one-dimensional Euclidean space. From [39], let K_C denotes the family of all non-empty compact convex subset of $\mathbb{R}$, that is,*

$$K_C(\mathbb{R}) = \{[\mathrm{a},\mathrm{b}] \mid \mathrm{a},\mathrm{b} \in \mathbb{R} \text{ and } \mathrm{a} \leq \mathrm{b}\}. \tag{2}$$

The Hausdorff metric $\mathscr{H}$ on $K_{\mathrm{C}}(\mathbb{R})$ is defined by

$$\mathscr{H}(\mathrm{A},\mathrm{B}) = max\{d(\mathrm{A},\mathrm{B}), d(\mathrm{B},\mathrm{A})\}, \tag{3}$$

where

$$d(\mathrm{A},\mathrm{B}) = max_{\mathrm{a}\in A} d(\mathrm{a},\mathrm{B}), \quad d(\mathrm{a},\mathrm{B}) = min_{\mathrm{b}\in\mathrm{B}}, \quad d(\mathrm{a},\mathrm{b}) = min_{\mathrm{b}\in\mathrm{B}}|\mathrm{a}-\mathrm{b}|. \tag{4}$$

It is well known that $(K_C(\mathbb{R}), \mathscr{H})$ is a complete metric space. The space $K_C(\mathbb{R})$ is not a linear space, since it does not possess an additive inverse and therefore subtraction is not well defined (see [40], [34], [13], [39], [41], [42]).

Definition 2. *[43] **Interval-valued function :** Let $K_C(\mathbb{R})$ be the set of all nonempty compact interval of the real line $\mathbb{R}$. If $\mathrm{A} = [\mathrm{a}^-, \mathrm{a}^+], \mathrm{B} = [\mathrm{b}^-, \mathrm{b}^+] \in K_C(\mathbb{R})$ then the usual interval operation, i.e, Minkowski addition and scalar multiplication, are defined by*

$$\mathrm{A} + \mathrm{B} = [\mathrm{a}^-, \mathrm{a}^+] + [\mathrm{b}^-, \mathrm{b}^+] = [\mathrm{a}^- + b^-, a^+ + b^+], \tag{5}$$

and

$$\lambda \mathrm{A} = \lambda[\mathrm{a}^-, \mathrm{a}^+] = \begin{cases} \lambda \mathrm{a}^-, \lambda \mathrm{a}^+] & \text{if } \lambda > 0, \\ \{0\} & \text{if } \lambda = 0, \\ \lambda \mathrm{a}^+, \lambda \mathrm{a}^-] & \text{if } \lambda < 0, \end{cases} \tag{6}$$

accordingly. If $\lambda = -1$, scalar multiplication gives the opposite result

$$-\mathrm{A} = (-1)[\mathrm{a}^-, \mathrm{a}^+] = [-\mathrm{a}^+, -\mathrm{a}^-]. \tag{7}$$

Generally speaking, $\mathrm{A} + (-\mathrm{A}) \neq \{0\}$; meaning that, unless $\mathrm{A} = \{\mathrm{a}\}$ is a singleton, $-\mathrm{A}$ is not equal to inverse of A with regard to Minkowski's addition. $\mathrm{A} - \mathrm{B} = \mathrm{A} + (-1)\mathrm{B} = [\mathrm{a}^- - \mathrm{b}^+, \mathrm{a}^+ - \mathrm{b}^-]$ is the Minkowski's difference. As a first consequence of this fact, simplification is generally invalid, even if it is true that $(\mathrm{A} + \mathrm{C} = \mathrm{B} + \mathrm{C}) \Longleftrightarrow \mathrm{A} = \mathrm{B}$, [33] i.e.

$$(\mathrm{A} + \mathrm{B}) - \mathrm{B} \neq \mathrm{A}.$$

Definition 3. *([26], [33])**Hukuhara-difference:** Hukuhara introduced the concept of the Hukuhara-difference (H-difference)*

$$\mathrm{A} \ominus \mathrm{B} = \mathrm{C} \Longleftrightarrow \mathrm{A} = \mathrm{B} + \mathrm{C}, \tag{8}$$

and an important property of $\ominus$ is that $\mathrm{A} \ominus \mathrm{A} = \{0\}, \forall \mathrm{A} \in K_C(\mathbb{R})$ and

$$(\mathrm{A} + \mathrm{B}) \ominus \mathrm{B} = \mathrm{A}, \forall \mathrm{A}, \mathrm{B} \in K_C(\mathbb{R}). \tag{9}$$

H-difference is unique, but a necessary condition for $\mathrm{A} \ominus \mathrm{B}$ to exists is that A contains a translate $\{\mathrm{c}\} + \mathrm{B}$ of B.

Definition 4. *([23], [30], [43]) **Generalized Hukuhara-difference(gH-difference):** The generalized Hukuhara-difference of two intervals $\mathrm{A} = [\mathrm{a}^-, \mathrm{a}^+]$, $\mathrm{B} = [\mathrm{b}^-, \mathrm{b}^+] \in K_C(\mathbb{R})$ is defined as follows:*

$$\mathrm{A} \ominus_{gH} \mathrm{B} = [min\{\mathrm{a}^- - \mathrm{b}^-, \mathrm{a}^+ - \mathrm{b}^+\}, max\{\mathrm{a}^- - \mathrm{b}^-, \mathrm{a}^+ - \mathrm{b}^+\}]. \tag{10}$$

If the width of A is written as $w(\mathrm{A}) = \mathrm{a}^+ - \mathrm{a}^-$, then

$$\mathrm{A} \ominus_{gH} \mathrm{B} = \mathrm{C} \Leftrightarrow \begin{cases} (a)\mathrm{A} = \mathrm{B} + \mathrm{C}, \text{ if } w(\mathrm{A}) \geq w(\mathrm{B}), \\ \\ (b)\mathrm{B} = \mathrm{A} + (-1)\mathrm{C}, \text{ if } w(\mathrm{A}) \leq w(\mathrm{B}). \end{cases} \tag{11}$$

Also $\mathrm{A} \ominus_{gH} \mathrm{B} \neq \mathrm{A} + (-1)\mathrm{B}$. Remember that *(i)* $w(-A) = w(A)$,

(ii) $w(A+B) = w(A) + w(B)$,

(iii) $w(A \ominus_{gH} B) = |w(A) - w(B)|$,

for any $\mathrm{A}, \mathrm{B} \in K_{\mathrm{C}}(\mathbb{R})$. *For additional attributes pertaining to the operation on* $K_{\mathrm{C}}(\mathbb{R})$*, see [36].*

To verify the Definition 2.4 we will now solve the examples.

Example 1. *(For* $w(A) \geq w(B)$*).*
Let

$$A = [1,4], \qquad B = [2,3].$$

Check the widths

$$w(A) = 4-1 = 3, \; w(B) = 3-2 = 1,$$

so

$$w(A) \geq w(B).$$

Compute the gH-difference using equation (10):

$$A \ominus_{gH} B = \big[\min\{1-2,\, 4-3\},\, \max\{1-2,\, 4-3\}\big] = [\min\{-1,1\}, \max\{-1,1\}] = [-1,1].$$

Take $C = [-1,1]$.
Compute equation (11) (a): $A = B + C$.

$$B + C = [2+(-1),\, 3+1] = [1,4] = A$$

Check that $A + (-1)B$ *is not equal to* $A \ominus_{gH} B$*:*

$$-B = [-3,-2], \qquad A + (-B) = [1-3, 4-2] = [-2,2] \neq [-1,1].$$

Check the width identities:

(i) $w(-A) = w(A) = 3$ *(since* $-A = [-4,-1]$ *has width* 3*).*

(ii) $w(A+B) = w([1+2, 4+3] = [3,7]) = 4 = w(A) + w(B) = 3+1$.

(iii) $w(A \ominus_{gH} B) = w([-1,1]) = 2 = |3-1|$.

Example 2. *(For $w(A) \leq w(B)$).*
Let

$$A = [1,2], \qquad B = [0,4].$$

Compute the widths of the intervals $w(A) = 1,\ w(B) = 4$,
so

$$w(A) \leq w(B).$$

Compute the gH−difference using equation (10):

$$A \ominus_{gH} B = \left[\min\{1-0,\ 2-4\},\ \max\{1-0,\ 2-4\}\right] = [\min\{1,-2\},\max\{1,-2\}] = [-2,1].$$

Take $C = [-2,1]$. Compute the equation (11) (b): $B = A + (-1)C$.

$$-C = [-1,2], \qquad A + (-C) = [1+(-1), 2+2] = [0,4] = B$$

The widths satisfy $w(-A) = w(A) = 1$, $w(A+B) = 1+4 = 5$, and $w(A \ominus_{gH} B) = w([-2,1]) = 3 = |1-4|$, confirming the three width identities.

These two examples illustrate how to compute $A \ominus_{gH} B$ using equation (10). They also show that when $w(A) \geq w(B)$, one can express $A = B + C$, whereas when $w(A) \leq w(B)$, the relation $B = A + (-1)C$ holds, as stated in equation (11). Moreover, the examples confirm that $A \ominus_{gH} B$ is generally not equal to $A + (-1)B$, and they verify that the three width identities are satisfied numerically.

Definition 5. *([44], [23],[43])**Hausdorff-Pompeiu Metric:** The Hausdorff-Pompeiu metric $\mathscr{H}$ in $K_{\mathrm{C}}(\mathbb{R})$ is defined as follows:*

$$\mathscr{H}(\mathrm{A},\mathrm{B}) = \max\{\left|\mathrm{a}^- - \mathrm{b}^-\right|, \left|\mathrm{a}^+ - \mathrm{b}^+\right|\}, \tag{12}$$

where $\mathrm{A} = [\mathrm{a}^-, \mathrm{a}^+]$ and $B = [\mathrm{b}^-, \mathrm{b}^+]$. The Hausdorff-Pompeiu measure has the following well-known characteristics:

$$\begin{cases} (1).\ \mathscr{H}(\mathrm{A},\mathrm{B}) = 0 \Longleftrightarrow \mathrm{A} = \mathrm{B}, \\ (2).\ \mathscr{H}(\lambda\mathrm{A},\lambda\mathrm{B}) = |\lambda|\,\mathscr{H}(\mathrm{A},\mathrm{B}) \forall \lambda \in \mathbb{R}, \\ (3).\ \mathscr{H}(\mathrm{A}+\mathrm{C},\mathrm{B}+\mathrm{C}) = \mathscr{H}(\mathrm{A},\mathrm{B}), \\ (4).\ \mathscr{H}(\mathrm{A}+\mathrm{B},\mathrm{C}+\mathrm{D}) \leq \mathscr{H}(\mathrm{A},\mathrm{C}) + \mathscr{H}(\mathrm{B},\mathrm{D}), \\ (5).\ \mathscr{H}(\mathrm{A},\mathrm{B}) = \mathscr{H}(\mathrm{A} \ominus_{gH} \mathrm{B}, 0), \\ (6).\ \mathscr{H}(\mathrm{A} \ominus_{gH} \mathrm{B}, \mathrm{A} \ominus_{gH} \mathrm{C}) = \mathscr{H}(\mathrm{B} \ominus_{gH} \mathrm{A}, \mathrm{C} \ominus_{gH} \mathrm{A}) = \mathscr{H}(\mathrm{B},\mathrm{C}). \end{cases} \tag{13}$$

Also note that $(K_{\mathrm{C}}(\mathbb{R}), \mathscr{H})$ is complete, separable and locally compact metric space ([23],[43]).

Another concept is of the set of all continuous interval-valued functions denoted by $\mathrm{C}([\mathrm{a},\mathrm{b}], K_{\mathrm{C}}(\mathbb{R}))$ is a quasi-linear space, see([40], [13], [42]).
On the quasi-linear space $(K_C(\mathbb{R}), +, ., \mathscr{H})$, the well-known quasi-norm is defined as

$$||\mathrm{F}||_\infty = \sup_{t\in[a,b]} \mathscr{H}(\mathrm{F(t)}, 0), \tag{14}$$

where $\mathrm{F} : [\mathrm{a},\mathrm{b}] \to K_{\mathrm{C}}(\mathbb{R})$ and $\mathrm{F(t)}$ is an interval-valued function.

Definition 6. *[34] **(Hukuhara-derivative):** Consider the interval-valued function* $\mathrm{F}:[\mathrm{a},\mathrm{b}]\to K_{\mathrm{C}}$, *state that at* $\mathrm{t}_0\in[\mathrm{a},\mathrm{b}]$, F *is differentiable, if an element* $\mathrm{F}'(\mathrm{t}_0)\in K_{\mathrm{C}}$ *exists such that the limits*

$$\lim_{\mathrm{h}\longrightarrow 0^+}\frac{\mathrm{F}(\mathrm{t}_0+\mathrm{h})\ominus_{\mathrm{H}}(\mathrm{F}(\mathrm{t}_0)}{\mathrm{h}}, \tag{15}$$

and

$$\lim_{\mathrm{h}\longrightarrow 0^+}\frac{(\mathrm{F}(\mathrm{t}_0)\ominus_{\mathrm{H}}\mathrm{F}(\mathrm{t}_0-\mathrm{h})}{\mathrm{h}}, \tag{16}$$

exist and are equals to $\mathrm{F}'(\mathrm{t}_0)$.

Definition 7. *[35]**Generalized Hukuhara Derivative:** Let* $\mathrm{F}:[\mathrm{a},\mathrm{b}]\to K_{\mathrm{C}}$ *be an interval-valued function, the gH-**derivative** of* F *at* $\mathrm{t}_0\in[\mathrm{a},\mathrm{b}]$ *is defined as:*

$$\mathrm{F}'(\mathrm{t}_0)=\lim_{\mathrm{h}\longrightarrow 0}\frac{\mathrm{F}(\mathrm{t}_0+\mathrm{h})\ominus_g\mathrm{F}(\mathrm{t}_0)}{\mathrm{h}}. \tag{17}$$

If $\mathrm{F}'(\mathrm{t}_0)\in K_{\mathrm{C}}$ *satisfying (17) exists, we say that* F *is generalized Hukuhara differentiable at* $\mathrm{t}_0\in[\mathrm{a},\mathrm{b}]$.

Definition 8. *[35] Let* $\mathrm{F}:[\mathrm{a},\mathrm{b}]\to K_{\mathrm{C}}$ *be an interval-valued function for which* $\mathrm{F}(\mathrm{t})=[\mathrm{f}^-(\mathrm{t}),\mathrm{f}^+(\mathrm{t})]$:

(**I**:) F is gH-differentiable at $\mathrm{t}_0\in[\mathrm{a},\mathrm{b}]$ in the first form if f^- and f^+ are differentiable at t_0 and

$$\mathrm{F}^{'}(\mathrm{t}_0)=[\mathrm{f}^{'-}(\mathrm{t}_0),\mathrm{f}^{'+}(\mathrm{t}_0)]. \tag{18}$$

(**II**:) F is gH-differentiable at $\mathrm{t}_0\in[\mathrm{a},\mathrm{b}]$ in the second form if $\mathrm{f}^-$and f^+ are differentiable at t_0 and

$$\mathrm{F}^{'}(\mathrm{t}_0)=[\mathrm{f}^{'+}(\mathrm{t}_0),\mathrm{f}^{'-}(\mathrm{t}_0)]. \tag{19}$$

Definition 9. *([35],[23],[43]) Let* $\mathrm{F}:[\mathrm{a},\mathrm{b}]\to K_{\mathrm{C}}$ *be an interval-valued function.* F *is known as w -increasing or w-decreasing on* $[a,b]$ *if the real function* $\mathrm{t}\longrightarrow w_{\mathrm{F}}(\mathrm{t}):=w(\mathrm{F}(\mathrm{t}))$ *is increasing or decreasing on* $[\mathrm{a},\mathrm{b}]$. *If* F *is w-increasing or w-decreasing on* $[a,b]$, *then we say that* F *is w-monotone on* $[a,b]$.

In this paper we will denote the notations for w-increasing : $w\uparrow$, w-decreasing : $w\downarrow$, increasing function : $(\uparrow)$, decreasing function : $(\downarrow)$, w-monotone : $mon(w)$.

Proposition 1. *([23],[43]) Let* $\mathrm{F}:[\mathrm{a},\mathrm{b}]\to K_{\mathrm{C}}$ *be an interval-valued function.*

- *If* F *is gH-differentiable in the first form then* $w(\mathrm{F})$ *is* $\uparrow$.
- *If* F *is gH-differentiable in the second form then* $w(\mathrm{F})$ *is* $\downarrow$.

Proposition 2. *([23],[36]) Let* $\widehat{\mathrm{F}},\widehat{\mathrm{G}}:[\mathrm{a},\mathrm{b}]\to K_{\mathrm{C}}$ *be* $mon(w)$ *and gH-differentiable on* $[\mathrm{a},\mathrm{b}]$. *The characteristics listed below are true:*

- $\widehat{\mathrm{F}}+\widehat{\mathrm{C}}$, $\widehat{\mathrm{F}}\ominus_{gH}\widehat{\mathrm{C}}$, *and* $\lambda\widehat{\mathrm{F}}$ *being an interval-valued functions, are gH-differentiable on* $[\mathrm{a},\mathrm{b}]$ *for every* $\widehat{\mathrm{C}}\in K_{\mathrm{C}}$ *and* $\forall\ \lambda\in R$. *So* $(\widehat{\mathrm{F}}+\widehat{\mathrm{C}})'=\widehat{\mathrm{F}}'$ *and* $(\lambda\widehat{\mathrm{F}})'=\lambda\widehat{\mathrm{F}}'$, *accordingly.*

- *If* $\widehat{\mathrm{F}}$ *and* $\widehat{\mathrm{G}}$ *are both mon*(w)*, implying they are either* $w\uparrow$ *or* $w\downarrow$*,* $(\widehat{\mathrm{F}}+\widehat{\mathrm{C}})'=\widehat{\mathrm{F}}'+\widehat{\mathrm{G}}'$ *and* $(\widehat{\mathrm{F}}\ominus_{gH}\widehat{\mathrm{G}})'=\widehat{\mathrm{F}}'\ominus_{gH}\widehat{\mathrm{G}}'$.
- *If* $\widehat{\mathrm{F}}$ *and* $\widehat{\mathrm{G}}$ *are mon*(w) *(with one being* $w\uparrow$ *and the other* $w\downarrow$*), then* $(\widehat{\mathrm{F}}+\widehat{\mathrm{G}})'=\widehat{\mathrm{F}}'\ominus_{gH}(-\widehat{\mathrm{G}}')$ *and* $(\widehat{\mathrm{F}}\ominus_{gH}\widehat{\mathrm{G}})'=\widehat{\mathrm{F}}'+(-\widehat{\mathrm{G}}')$.

Definition 10. *[23] Let* $\exists$ *a sequence* $\mathrm{F_n}:[\mathrm{a,b}]\to K_{\mathrm{C}}$ *of step interval-valued functions such that*

$$\lim_{\mathrm{n}\to\infty}\mathscr{H}(\mathrm{F_n(t)},\mathrm{f(t)})=0\forall \mathrm{t}\in[\mathrm{a,b}], \tag{20}$$

then an interval-valued function $\mathrm{F}:[\mathrm{a,b}]\to K_{\mathrm{C}}$ *is refer to be measurable.*
Provided that $\lim_{\mathrm{n}\to\infty}\mathscr{H}(\mathrm{F_n(t)},\mathrm{f(t)})=0 \iff \lim_{\mathrm{n}\to\infty}|\mathrm{f}_{\mathrm{n}}^{\pm}(\mathrm{t})-\mathrm{f}^{\pm}(\mathrm{t})|=0$*, then it is evident that an interval-valued function* $\mathrm{F}:[\mathrm{a,b}]\to K_{\mathrm{C}}$ *is measurable* $\iff$ f^- *and* f^+ *are measurable. Additionally, it is simple to demonstrate that* $\mathrm{F}:[\mathrm{a,b}]\to K_{\mathrm{C}}$ *is integrable on* $[\mathrm{a,b}]$ $\iff$ F *is measurable and the real function* $\mathrm{t}\mapsto\|\mathrm{F(t)}\|$ *is Lebesgue integrable on* $[\mathrm{a,b}]$ *(see [45],[46]).*

Lemma 1. *[47] Let* $\mathrm{f,g}:[\mathrm{a,b}]\to K_{\mathrm{C}}$ *be measurable and integrability bounded interval-valued functions. Then for any* $[\mathrm{c,d}]\subset[\mathrm{a,b}]$,

$$\begin{cases}(i)\ \int_{\mathrm{c}}^{\mathrm{d}}(\alpha\mathrm{f(t)}+\beta\mathrm{g(t)})\mathrm{dt}=\alpha\int_{\mathrm{c}}^{\mathrm{d}}\mathrm{f(t)dt}+\beta\int_{\mathrm{c}}^{\mathrm{d}}\mathrm{g(t)dt},\ \forall\alpha,\beta\in R,\\ (ii)\ \int_{\mathrm{c}}^{\mathrm{d}}\mathrm{f(t)dt}=\int_{\mathrm{c}}^{\tau}\mathrm{f(t)dt}+\int_{\tau}^{\mathrm{d}}\mathrm{f(t)dt},\ \forall\tau\in[\mathrm{c,d}],\\ (iii)\mathscr{H}(\int_{\mathrm{c}}^{\mathrm{d}}\mathrm{f(t)dt},\int_{\mathrm{c}}^{\mathrm{d}}\mathrm{g(t)dt})\le\int_{\mathrm{c}}^{\mathrm{d}}\mathscr{H}(\mathrm{f(t)},\mathrm{g(t)})\mathrm{dt}.\end{cases} \tag{21}$$

Proposition 3. *[23] Assume that* $\mathrm{F(t)}=[\mathrm{f^-(t)},\mathrm{f^+(t)}],\mathrm{t}\in[\mathrm{a,b}]$. *Make* $\mathrm{F}:[\mathrm{a,b}]\to K_{\mathrm{C}}$. *At* $\mathrm{t}\in[\mathrm{a,b}]$, F *is gH-differentiable if the real-valued functions* f^- *and* f^+ *are differentiable at* $\mathrm{t}\in[\mathrm{a,b}]$ *and*

$$\mathrm{F}'(\mathrm{t})=\left[min\{\frac{\mathrm{d}}{\mathrm{dt}}\mathrm{f^-(t)},\frac{\mathrm{d}}{\mathrm{dt}}\mathrm{f^+(t)}\},max\{\frac{\mathrm{d}}{\mathrm{dt}}\mathrm{f^-(t)},\frac{\mathrm{d}}{\mathrm{dt}}\mathrm{f^+(t)}\}\right].$$

Conversely, the differentiability of f^- *and* f^+ *is not implied by the gH-differentiability.*

Proposition 4. *[23] Assume that* $\mathrm{F}:[\mathrm{a,b}]\to K_{\mathrm{C}}$ *for which* $\mathrm{F(t)}=[\mathrm{f^-(t)},\mathrm{f^+(t)}],\mathrm{t}\in[\mathrm{a,b}]$. *For any* $\mathrm{t}\in[\mathrm{a,b}]$, $\mathrm{f'^-(t)}$ *and* $\mathrm{f'^+(t)}$ *exist if* F *is mon*(w) *and gH -differentiable on* $[\mathrm{a,b}]$. *Additionally, we have that:*

$$\begin{aligned} & l\mathrm{F}'(\mathrm{t})=[\mathrm{f'^-(t)},\mathrm{f'^+(t)}]\ \forall\mathrm{t}\in[\mathrm{a,b}],\ \textit{if } \mathrm{F} \textit{ is } w\uparrow,\\ & \mathrm{F}'(\mathrm{t})=[\mathrm{f'^+(t)},\mathrm{f'^-(t)}]\ \forall\mathrm{t}\in[\mathrm{a,b}],\ \textit{if } \mathrm{F} \textit{ is } w\downarrow. \end{aligned} \tag{22}$$

The Fundamental theorem of calculus for interval-valued function: [35] Differentiation and integration are the two main operations of calculus, and their relationship is specified by the fundamental theorem of calculus in classical (real) analysis. The first portion of the theorem, which is commonly referred to as the first fundamental theorem of calculus, demonstrates that differentiation can reverse an indefinite integration. The second portion, sometimes known as the second fundamental theorem of calculus, enables one to use any one of a function's infinitely many anti-derivatives to calculate the function's definite integral. An interval-valued version of these theorems was provided by Chalco Cano et al. [35].

Interval-valued function notions in fractional-order calculus: Now we will discuss some definitions and properties of fractional calculus for Interval-valued functions. Let $\mathrm{AC}([\mathrm{a},\mathrm{b}],K_{\mathrm{C}})$ represents the set of interval-valued functions from $[\mathrm{a},\mathrm{b}]$ into K_{C} that are absolutely continuous.

Proposition 5. *[23]* $\mathrm{F}:[\mathrm{a},\mathrm{b}]\to K_{\mathrm{C}}$ *is an interval valued function, will only be considered absolutely continuous* $\iff$ $\mathrm{f}^-(\mathrm{t})$ *and* $\mathrm{f}^+(\mathrm{t})$ *are both absolutely continuous.*

Definition 11. *[23]* ***Interval-valued Riemann-Liouville fractional-order integral:*** *Assume that* $\mathrm{F}\in \mathrm{L}^1([\mathrm{a},\mathrm{b}],K)$ *and* $\alpha>0$. *Given that the real-valued function* $\mathrm{t}\mapsto(\mathrm{x}-\mathrm{t})^{\alpha-1}$ *is Lebesgue integrable on* $[\mathrm{a},\mathrm{x}]$ *for* $\mathrm{t}<\mathrm{x}$, *the real-valued functions* $\mathrm{t}\mapsto(\mathrm{x}-\mathrm{t})^{\alpha-1}\mathrm{f}^{\pm}(\mathrm{t})$ *are likewise Lebesgue integrable on* $[0,\mathrm{x}]$ *every* $\mathrm{x}\in[\mathrm{a},\mathrm{b}]$. *The interval-valued function* $\mathrm{t}\mapsto(\mathrm{x}-\mathrm{t})^{\alpha-1}\mathrm{F}(\mathrm{t})$ *is therefore Lebesgue integrable on* $[0,\mathrm{x}]$ *for any* $\mathrm{x}\in[\mathrm{a},\mathrm{b}]$. *Thus,*

$$\mathfrak{I}^{\alpha}_{\mathrm{a}^+}\mathrm{F}(\mathrm{x}):=\frac{1}{\Gamma(\alpha)}\int_{\mathrm{a}}^{\mathrm{x}}(\mathrm{x}-\mathrm{t})^{\alpha-1}\mathrm{F}(\mathrm{t})d\mathrm{t}, \tag{23}$$

exists $\forall x\in[\mathrm{a},\mathrm{b}]$, *"*$\mathfrak{I}^{\alpha}_{\mathrm{a}^+}\mathrm{F}$*" is then called as the interval-valued Riemann-Liouville fractional integral of order* $\alpha>0$. *If* $\mathrm{F}=[\mathrm{f}^-,\mathrm{f}^+]\in L^1([\mathrm{a},\mathrm{b}],K_{\mathrm{C}})$ *and* $\alpha>0$, *then it is evident that*

$$\mathfrak{I}^{\alpha}_{\mathrm{a}^+}\mathrm{F}(\mathrm{x})=[\mathrm{I}^{\alpha}_{\mathrm{a}^+}\mathrm{f}^-(\mathrm{x}),\mathrm{I}^{\alpha}_{\mathrm{a}^+}\mathrm{f}^+(\mathrm{x}),\forall\mathrm{x}\in[\mathrm{a},\mathrm{b}]. \tag{24}$$

Definition 12. *[23]****Interval-valued Riemann-Liouville fractional-order derivatives:*** *Given an interval-valued function* $\mathrm{F}=[\mathrm{f}^-,\mathrm{f}^+]\in L^1([\mathrm{a},\mathrm{b}],K_{\mathrm{C}})$ *and* $\alpha\in(0,1]$, *we obtain the interval-valued function* $\mathscr{F}_{1-\alpha}:[\mathrm{a},\mathrm{b}]\longrightarrow K_{\mathrm{C}}$ *by*

$$\mathscr{F}_{1-\alpha}(\mathrm{x})=\mathfrak{I}^{1-\alpha}_{\mathrm{a}^+}\mathrm{F}(\mathrm{x})=\frac{1}{\Gamma(1-\alpha)}\int_{\mathrm{a}}^{\mathrm{x}}(\mathrm{x}-\mathrm{t})^{-\alpha}\mathrm{F}(\mathrm{t})d\mathrm{t},\forall\mathrm{x}\in[\mathrm{a},\mathrm{b}]. \tag{25}$$

If the gH-derivative $\mathscr{F}'_{1-\alpha}(\mathrm{x})$ exists $\forall\mathrm{x}\in[\mathrm{a},\mathrm{b}]$, then $\mathscr{F}'_{1-\alpha}(\mathrm{x})$ is said to be the interval-valued Riemann-Liouville fractional derivative (or Riemann-Liouville gH-fractional derivative) of order $\alpha\in(0,1]$. The Riemann-Liouville gH-fractional derivative of F will be denoted by $\mathscr{D}^{\alpha}_{\mathrm{a}^+}\mathrm{F}$. Therefore

$$\mathscr{D}^{\alpha}_{\mathrm{a}^+}\mathrm{F}(\mathrm{x}):=(\mathfrak{I}^{1-\alpha}_{\mathrm{a}^+}\mathrm{F})'(\mathrm{x})\ \forall\mathrm{x}\in[\mathrm{a},\mathrm{b}]. \tag{26}$$

Theorem 1. *[23] Let* $\mathrm{F}=[\mathrm{f}^-,\mathrm{f}^+]\in\mathrm{AC}([\mathrm{a},\mathrm{b}],K_{\mathrm{C}})$, *then*
(a). $\mathscr{F}_{1-\alpha}\in\mathrm{AC}([\mathrm{a},\mathrm{b}],K_{\mathrm{C}})$ *and*

$$\mathscr{D}^{\alpha}_{\mathrm{a}^+}\mathrm{F}(\mathrm{x})=[min\{D^{\alpha}_{\mathrm{a}^+}\mathrm{f}^-(\mathrm{x}),\mathrm{D}^{\alpha}_{\mathrm{a}^+}\mathrm{f}^+(\mathrm{x})\},max\{\mathrm{D}^{\alpha}_{\mathrm{a}^+}\mathrm{f}^-(\mathrm{x}),\mathrm{D}^{\alpha}_{\mathrm{a}^+}\mathrm{f}^+(\mathrm{x})\}], \tag{27}$$

$\forall\mathrm{x}\in[\mathrm{a},\mathrm{b}]$ *and specifically, when* $\alpha=1$, *then*

$$\mathscr{D}^{\alpha}_{\mathrm{a}^+}\mathrm{F}(\mathrm{x})=\mathrm{F}'(\mathrm{x})\ \forall\mathrm{x}\in[\mathrm{a},\mathrm{b}].$$

(b). If either F *is* $w\uparrow$ *on* $[\mathrm{a},\mathrm{b}]$ *or* F *is* $w\downarrow$ *and* $\mathscr{F}_{1-\alpha}$ *is* $w\uparrow$ *on* $[\mathrm{a},\mathrm{b}]$*, then*

$$\mathscr{D}^{\alpha}_{\mathrm{a}^+}\mathrm{F}(\mathrm{x}) = [\mathrm{D}^{\alpha}_{\mathrm{a}^+}\mathrm{f}^-(\mathrm{x}), \mathrm{D}^{\alpha}_{\mathrm{a}^+}\mathrm{f}^+(\mathrm{x})],\ \forall \mathrm{x} \in [\mathrm{a},\mathrm{b}]. \tag{28}$$

(c). If $\mathscr{F}_{1-\alpha}$ *is* $w\downarrow$ *on* $[\mathrm{a},\mathrm{b}]$*, then*

$$\mathscr{D}^{\alpha}_{\mathrm{a}^+}\mathrm{F}(\mathrm{x}) = [\mathrm{D}^{\alpha}_{\mathrm{a}^+}\mathrm{f}^+(\mathrm{x}), \mathrm{D}^{\alpha}_{\mathrm{a}^+}\mathrm{f}^-(\mathrm{x})],\ \forall \mathrm{x} \in [\mathrm{a},\mathrm{b}]. \tag{29}$$

Theorem 2. *([23],[43]) Let* $\mathrm{F},\mathrm{G} \in \mathrm{AC}([\mathrm{a},\mathrm{b}],K_{\mathrm{C}})$*, be mon*$(w)$ *on* $[\mathrm{a},\mathrm{b}]$ *for* $\alpha \in (0,1]$*. The following are attributes:*
(a). If $\mathscr{F}_{1-\alpha}$ *and* $\mathscr{G}_{1-\alpha}$ *are both mon*(w) *on* $[\mathrm{a},\mathrm{b}]$*, then*

$$(\mathscr{D}^{\alpha}_{\mathrm{a}^+}(\mathrm{F} \ominus_{gH} \mathrm{G}))(\mathrm{x}) = (\mathscr{D}^{\alpha}_{\mathrm{a}^+}\mathrm{F})(\mathrm{x}) \ominus_{gH} (\mathscr{D}^{\alpha}_{\mathrm{a}^+}\mathrm{G})(\mathrm{x})\ \forall \mathrm{x} \in [\mathrm{a},\mathrm{b}]. \tag{30}$$

(b). If $\mathscr{F}_{1-\alpha}$ *and* $\mathscr{G}_{1-\alpha}$ *are alternatively mon*(w) *on* $[\mathrm{a},\mathrm{b}]$*, then*

$$(\mathscr{D}^{\alpha}_{\mathrm{a}^+}(\mathrm{F} \ominus_{gH} \mathrm{G}))(\mathrm{x}) = (\mathscr{D}^{\alpha}_{\mathrm{a}^+}\mathrm{F})(x)) + (-(\mathscr{D}^{\alpha}_{\mathrm{a}^+}\mathrm{G})(\mathrm{x}))\ \forall \mathrm{x} \in [\mathrm{a},\mathrm{b}]. \tag{31}$$

Theorem 3. *([23],[43]) Let* $\mathrm{F} \in L([\mathrm{a},\mathrm{b}],K_{\mathrm{C}})$*, then*

$$(\mathscr{D}^{\alpha}_{\mathrm{a}^+}\mathfrak{I}^{\alpha}_{\mathrm{a}^+}\mathrm{F})(\mathrm{x}) = \mathrm{F}(\mathrm{x}),\ \forall \mathrm{x} \in [\mathrm{a},\mathrm{b}].$$

Proposition 6. *([23],[43]) Let* $\mathrm{F} \in L^1([\mathrm{a},\mathrm{b}],K_{\mathrm{C}})$ *for which* $\mathscr{F}_{1-\alpha} \in \mathrm{AC}([\mathrm{a},\mathrm{b}],K_{\mathrm{C}})$*. If either* $\frac{\mathrm{d}}{\mathrm{dx}}w(\mathscr{F}_{1-\alpha}(\mathrm{x})) \geq 0\ \ \forall \mathrm{x} \in [\mathrm{a},\mathrm{b}]$ *or* $\frac{\mathrm{d}}{\mathrm{dx}}w(\mathscr{F}_{1-\alpha}(\mathrm{x})) \leq 0\ \ \forall \mathrm{x} \in [\mathrm{a},\mathrm{b}]$*, then gH-difference* $\mathrm{F}(\mathrm{x}) \ominus_{gH} \frac{(\mathrm{x}-\mathrm{a})^{\alpha-1}}{\Gamma(\alpha}) \mathscr{F}_{1-\alpha}(\mathrm{a})$ *accour* $\forall \mathrm{x} \in [\mathrm{a},\mathrm{b}]$ *and*

$$(\mathfrak{I}^{\alpha}_{\mathrm{a}^+}\mathscr{D}^{\alpha}_{\mathrm{a}^+})(\mathrm{x}) = \mathrm{F}(\mathrm{x}) \ominus_{gH} \frac{(\mathrm{x}-\mathrm{a})^{\alpha-1}}{\Gamma(\alpha)}\mathscr{F}_{1-\alpha}(\mathrm{a})\ \forall \mathrm{x} \in [\mathrm{a},\mathrm{b}]. \tag{32}$$

Definition 13. *[23]* ***Interval-valued Caputo fractional-order derivative:*** *If* $\mathrm{F} \in \mathrm{AC}([\mathrm{a},\mathrm{b}],K_{\mathrm{C}})$ *and* $\alpha \in (0,1]$*,*

$$^{\mathrm{C}}\mathscr{D}^{\alpha}_{\mathrm{a}^+}\mathrm{F}(\mathrm{x}) = \frac{1}{\Gamma(1-\alpha)}\int_{\mathrm{a}}^{\mathrm{x}} (\mathrm{x}-\mathrm{t})^{-\alpha}\mathrm{F}'(\mathrm{t})\mathrm{dt}, \tag{33}$$

exists $\forall \mathrm{x} \in [\mathrm{a},\mathrm{b}]$*. Obviously* $^{\mathrm{C}}\mathscr{D}^{\alpha}_{\mathrm{a}^+}\mathrm{F}(\mathrm{x}) = \mathfrak{I}^{1-\alpha}_{\mathrm{a}^+}\mathrm{F}'(\mathrm{x})\ \ \forall \mathrm{x} \in [\mathrm{a},\mathrm{b}]$*.*
$^{\mathrm{C}}\mathscr{D}^{\alpha}_{\mathrm{a}^+}\mathrm{F}$ *is said to be the interval-valued Caputo fractional derivative (or Caputo gH-fractional derivative) of order* $\alpha \in (0,1]$*.*

Theorem 4. *[43] If* $\mathrm{F} \in \mathrm{AC}([\mathrm{a},\mathrm{b}],K_{\mathrm{C}}(\mathbb{R}))$ *is a mon*(w) *interval-valued function and* $\alpha \in (0,1]$*, then*

$$(^{\mathrm{C}}\mathscr{D}^{\alpha}_{\mathrm{a}^+}\mathrm{F})(\mathrm{x}) = (\mathfrak{I}^{1-\alpha}_{\mathrm{a}^+}\mathrm{F}')(\mathrm{x}) = \frac{1}{\Gamma(1-\alpha)}\int_{\mathrm{a}}^{\mathrm{x}} (\mathrm{x}-\mathrm{t})^{-\alpha}\mathrm{F}'(\mathrm{t})\mathrm{dt}\ \forall \mathrm{x} \in [\mathrm{a},\mathrm{b}]. \tag{34}$$

Remark 1. *[23] If* $\mathrm{F} = [\mathrm{f}^-, \mathrm{f}^+] \in \mathrm{AC}([\mathrm{a},\mathrm{b}], K_\mathrm{C}(\mathbb{R}))$ *and* $\alpha \in (0,1]$, *then*

$$^{\mathrm{C}}\mathscr{D}^{\alpha}_{\mathrm{a}^+}\mathrm{F}(\mathrm{x}) = \left[\min\{{}^{\mathrm{C}}\mathrm{D}^{\alpha}_{\mathrm{a}^+}\mathrm{f}^-(\mathrm{x}), {}^{\mathrm{C}}\mathrm{D}^{\alpha}_{\mathrm{a}^+}\mathrm{f}^+(\mathrm{x})\}, \max\{{}^{\mathrm{C}}\mathrm{D}^{\alpha}_{\mathrm{a}^+}\mathrm{f}^-(\mathrm{x}), {}^{\mathrm{C}}\mathrm{D}^{\alpha}_{\mathrm{a}^+}\mathrm{f}^+(\mathrm{x})\}\right], \tag{35}$$

for $\mathrm{x} \in [\mathrm{a},\mathrm{b}]$. *If* F *is mon*$(w)$, *then*

$$\begin{aligned} &(i).\ {}^{\mathrm{C}}\mathscr{D}^{\alpha}_{\mathrm{a}^+}\mathrm{F}(\mathrm{x}) = \left[{}^{\mathrm{C}}\mathrm{D}^{\alpha}_{\mathrm{a}^+}\mathrm{f}^-(\mathrm{x}), {}^{\mathrm{C}}\mathrm{D}^{\alpha}_{\mathrm{a}^+}\mathrm{f}^+(\mathrm{x})\right] \forall \mathrm{x} \in [\mathrm{a},\mathrm{b}] \ \textit{if} \ \mathrm{F} \textit{ is } w\uparrow, \\ &(ii).\ {}^{\mathrm{C}}\mathscr{D}^{\alpha}_{\mathrm{a}^+}\mathrm{F}(\mathrm{x}) = \left[{}^{\mathrm{C}}\mathrm{D}^{\alpha}_{\mathrm{a}^+}\mathrm{f}^+(\mathrm{x}), {}^{\mathrm{C}}\mathrm{D}^{\alpha}_{\mathrm{a}^+}\mathrm{f}^-(\mathrm{x})\right] \forall \mathrm{x} \in [\mathrm{a},\mathrm{b}] \ \textit{if} \ \mathrm{F} \textit{ is } w\downarrow. \end{aligned} \tag{36}$$

Theorem 5. *[23] If* $\mathrm{F} \in \mathrm{AC}([\mathrm{a},\mathrm{b}], K_\mathrm{C}(\mathbb{R}))$ *is a mon*(w) *interval-function and* $\alpha \in (0,1]$, *then*

$$(\mathfrak{I}^{\alpha}_{\mathrm{a}^+}\,{}^{\mathrm{C}}\mathscr{D}^{\alpha}_{\mathrm{a}^+})\mathrm{F}(\mathrm{x}) = \mathrm{F}(\mathrm{x}) \ominus_{gH} \mathrm{F}(\mathrm{a}). \tag{37}$$

Also

$$\mathrm{F}(\mathrm{x}) = \mathrm{F}(\mathrm{a}) + \mathfrak{I}^{\alpha}_{\mathrm{a}^+}\,{}^{\mathrm{C}}\mathscr{D}^{\alpha}_{\mathrm{a}^+}\mathrm{F}(\mathrm{x}), \text{ if F is } w\uparrow \text{ on } [\mathrm{a},\mathrm{b}], \tag{38}$$

and

$$\mathrm{F}(\mathrm{x}) = \mathrm{F}(\mathrm{a}) \ominus (-1)\mathfrak{I}^{\alpha}_{\mathrm{a}^+}\,{}^{\mathrm{C}}\mathscr{D}^{\alpha}_{\mathrm{a}^+}\mathrm{F}(\mathrm{x}), \text{if F is } w\downarrow \text{ on } [\mathrm{a},\mathrm{b}]. \tag{39}$$

Theorem 6. *[23] Let* $\mathrm{F} \in L^\infty([\mathrm{a},\mathrm{b}], K_\mathrm{C})$ *be such that either* F *is* $w\uparrow$, *or* F *is* $w\downarrow$ *on* $[\mathrm{a},\mathrm{b}]$ *and* $\mathrm{x} \mapsto \mathrm{F}_\alpha(\mathrm{x}) := \mathfrak{I}^{\alpha}_{\mathrm{a}^+}\mathrm{F}(\mathrm{x})$, *also* $\mathrm{x} \mapsto \mathrm{F}_\alpha(\mathrm{x}) := \mathfrak{I}^{\alpha}_{\mathrm{b}^-}\mathrm{F}(\mathrm{x})$ *is* $w\uparrow$ *on* $[\mathrm{a},\mathrm{b}]$, *then*

$$^{\mathrm{C}}\mathscr{D}^{\alpha}_{\mathrm{a}^+}\mathfrak{I}^{\alpha}_{\mathrm{a}^+}\mathrm{F}(\mathrm{x}) = \mathrm{F}(\mathrm{x}) \forall \mathrm{x} \in [\mathrm{a},\mathrm{b}]. \tag{40}$$

Throughout this paper, we will use the definitions 11 and 13.

3. Novelty and Main Results of Research

This section introduces an Ostrowski-type inequality for interval-valued functions that are Caputo gH-differentiable, highlighting the novelty and importance of this approach. Ostrowski's inequality, which sets bounds on how much a function deviates from its mean value, is a key result in numerical analysis. By extending the classical Caputo derivative with gH-Caputo differentiability, this method accommodates functions with non-integer order dynamics, broadening the inequality's applicability to more complex systems. We begin by establishing the mean value theorem for Caputo gH-differentiable interval-valued functions, providing a basis for deriving the inequality.

4. Result formulation of Caputo gH-differentiable interval-valued functions and Application I

4.1. Mean value theorem for Caputo gH-differentiable interval-valued functions

In this section, we have proved the mean value theorem for Caputo gH-differentiability for interval-valued functions. The mean value theorem is essential in Ostrowski's type inequality as it relates a function's value to its derivative, enabling precise error bounds. This connection helps estimate a function's deviation from its mean value on an interval.

Theorem 7. *Let* $\mathrm{F}:[\mathrm{a},\mathrm{b}]\longrightarrow K_C$ *be a Caputo gH-differentiable interval-valued function on* $[\mathrm{a},\mathrm{b}]$ *and let* $[\mathrm{c},\mathrm{d}]\subseteq[\mathrm{a},\mathrm{b}]$, $\mathrm{x}\in[\mathrm{c},\mathrm{d}]$. *Assume that* F' *is continuous. Then*

$$\mathscr{H}\Big(\mathrm{F}(\mathrm{x}),\mathrm{F}(\mathrm{c})\Big)\leq\frac{1}{\Gamma(\alpha+1)}\left\|{}^{\mathrm{C}}\mathscr{D}^{\alpha}_{\mathrm{c}^+}\mathrm{F}(\mathrm{t})\right\|_{\infty}(\mathrm{x}-\mathrm{c})^{\alpha},\tag{41}$$

for $\alpha\in(0,1]$.

Proof. Consider $\mathscr{H}(\mathrm{F}(\mathrm{x}),\mathrm{F}(\mathrm{c}))$, from (13) we have

$$\mathscr{H}\Big(\mathrm{F}(\mathrm{x}),\mathrm{F}(\mathrm{c})\Big)=\mathscr{H}\Big(\mathrm{F}(\mathrm{x})\ominus_{gH}\mathrm{F}(\mathrm{c}),0\Big),\tag{42}$$

from equation (37) we have

$$\mathscr{H}\Big(\mathrm{F}(\mathrm{x}),\mathrm{F}(\mathrm{c})\Big)=\mathscr{H}\Big(\mathfrak{I}^{\alpha}_{\mathrm{c}^+}{}^{\mathrm{C}}\mathscr{D}^{\alpha}_{\mathrm{c}^+}\mathrm{F}(\mathrm{x}),0\Big),\tag{43}$$

from (23) and (13), we can write

$$\begin{aligned}\mathscr{H}\Big(\mathrm{F}(\mathrm{x}),\mathrm{F}(\mathrm{c})\Big)&=\mathscr{H}\left(\frac{1}{\Gamma(\alpha)}\int_{\mathrm{c}}^{\mathrm{x}}(\mathrm{x}-\mathrm{t})^{\alpha-1}\,{}^{\mathrm{C}}\mathscr{D}^{\alpha}_{\mathrm{c}^+}\mathrm{F}(\mathrm{t})\mathrm{dt},0\right),\\&\leq\frac{1}{\Gamma(\alpha)}\int_{\mathrm{c}}^{\mathrm{x}}(\mathrm{x}-\mathrm{t})^{\alpha-1}\mathscr{H}\Big({}^{\mathrm{C}}\mathscr{D}^{\alpha}_{\mathrm{c}^+}\mathrm{F}(\mathrm{t}),0\Big)\mathrm{dt},\\&=\frac{1}{\Gamma(\alpha)}\int_{\mathrm{c}}^{\mathrm{x}}(\mathrm{x}-\mathrm{t})^{\alpha-1}\sup_{\mathrm{t}\in[\mathrm{c},\mathrm{d}]}\mathscr{H}\Big({}^{\mathrm{C}}\mathscr{D}^{\alpha}_{\mathrm{c}^+}\mathrm{F}(\mathrm{t}),0\Big)\mathrm{dt},\end{aligned}\tag{44}$$

and from equation (14), we have

$$=\frac{1}{\Gamma(\alpha)}\left\|{}^{\mathrm{C}}\mathscr{D}^{\alpha}_{\mathrm{c}^+}\mathrm{F}(\mathrm{t})\right\|_{\infty}\int_{\mathrm{c}}^{\mathrm{x}}(\mathrm{x}-\mathrm{t})^{\alpha-1}\,\mathrm{dt},\tag{45}$$

and we obtain the form

$$\mathscr{H}(\mathrm{F}(\mathrm{x}),\mathrm{F}(\mathrm{c}))\leq\frac{1}{\Gamma(\alpha+1)}\left\|{}^{\mathrm{C}}\mathscr{D}^{\alpha}_{\mathrm{c}^+}\mathrm{F}(\mathrm{t})\right\|_{\infty}(\mathrm{x}-\mathrm{c})^{\alpha}.$$

Conclusively, we can write

$$\mathscr{H}(\mathrm{F}(\mathrm{x}),\mathrm{F}(\mathrm{c}))\leq\frac{1}{\Gamma(\alpha+1)}\left\|{}^{\mathrm{C}}\mathscr{D}^{\alpha}_{\mathrm{c}^+}\mathrm{F}(\mathrm{t})\right\|_{\infty}(\mathrm{x}-\mathrm{c})^{\alpha},\tag{46}$$

for $\alpha\in(0,1]$, which completes the proof.

Corollary 1. *If we take $\alpha = 1$ in (41), we obtain the following:*

$$\mathscr{H}(\mathrm{F}(\mathrm{x}),\mathrm{F}(\mathrm{c})) \leq \left\|\mathrm{F}'(\mathrm{t})\right\|_{\infty}(\mathrm{x}-\mathrm{c}), \tag{47}$$

which is the mean value theorem for gH-differentiable interval-valued function, as proved in [13].

We now solve an example to verify the (41).

Example 3. *Consider the interval-valued function* $\mathrm{F}(\mathrm{x}) = [-2\mathrm{x},\mathrm{x}]; \mathrm{F} : [0,1] \longrightarrow K_{\mathrm{C}}(\mathbb{R})$.
Now we consider the L.H.S of (41)

$$\mathscr{H}(\mathrm{F}(\mathrm{x}),\mathrm{F}(0)) = \mathscr{H}([-2\mathrm{x},\mathrm{x}],0) = 2\mathrm{x}.$$

Since $\mathrm{x} \in [0,1]$*, therefore maximum value is* $\mathrm{x} = 1$*, that is*

$$\mathscr{H}(\mathrm{F}(\mathrm{x}),\mathrm{F}(0)) = 2.000. \tag{48}$$

So, for our considerations in this example 3*, (41) becomes,*

$$2.000 \leq \frac{1}{\Gamma(\alpha+1)}\left\|{}^{C}D^{\alpha}_{c^+}[-2t,t]\right\|_{\infty}(1-0)^{\alpha}. \tag{49}$$

From (41), the gH−Caputo differentiability of $F(x)$ for $\alpha = 1/2$, gives $\left\|{}^{\mathrm{C}}\mathscr{D}^{(1/2)}_{0^+}\mathrm{F}(\mathrm{t})\right\|_{\infty} = 2.256.$
The R.H.S of (49) for $\alpha = 1/2$ becomes

$$\frac{1}{\Gamma(3/2)}\left\|{}^{\mathrm{C}}\mathscr{D}^{(1/2)}_{0^+}\mathrm{F}(\mathrm{t})\right\|_{\infty}(1-0)^{1/2} = 2.546, \tag{50}$$

thus from (48) and (50), we have

$$2.000 < 2.546$$

and the required inequality (41) is satisfied for $\alpha = 1/2$.
For other values of α, i.e, $\alpha = 1/3, 1/4, \ldots, 1/9$, the results of equation (49) are mentioned in Table (1) which clearly shows that (49) is satisfied.

4.2. Existence and Uniqueness

Consider

$$\begin{cases} {}^{\mathrm{C}}\mathscr{D}^{(1/2)}_{0^+}x(t) \in \mathrm{F}(\mathrm{x}(\mathrm{t})) = [-2\mathrm{x}(\mathrm{t}),\mathrm{x}(\mathrm{t})], \quad t \in [0,T], \\ x(0) = x_0. \end{cases} \tag{51}$$

Utilizing integral, we get

$$x(t) = x_0 + \frac{1}{\Gamma(\frac{1}{2})}\int_0^t \frac{1}{(t-\mu)^{\frac{1}{2}}} f(\mu)d\mu, \tag{52}$$

where $f(\mu) \in \mathrm{F}(\mathrm{x}(\mu)) = [-2\mathrm{x}(\mu), \mathrm{x}(\mu)]$. *Next, we define the operator*

$$\mathbf{A}x(t) = x_0 + \frac{1}{\Gamma(\frac{1}{2})}\int_0^t \frac{1}{(t-\mu)^{\frac{1}{2}}}\mathrm{F}(\mathrm{x}(\mu))d\mu. \tag{53}$$

Let $\mathscr{B} = \mathscr{C}([0,T],\mathbb{R})$ *with*

$$\sup_{t\in[0,T]} |x(t)| = \|x(t)\|_\infty. \tag{54}$$

We must demonstrate that the operator $\mathbf{A}$ *fulfils the following for any two functions* $x, y \in \mathscr{B}$.

$$\|\mathbf{A}x - \mathbf{A}y\|_\infty \le \mathrm{H}\|x-y\|_\infty, \quad \mathrm{H} < 1. \tag{55}$$

When comparing F(x) and F(y), the Hausdorff distance is:

$$d_H(F(x),F(y)) = [|(-2x-(-2y))|, |x-y|] = 2|x-y|. \tag{56}$$

Therefore, for any $f_x \in \mathrm{F}(\mathrm{x})$ *and* $f_y \in \mathrm{F}(\mathrm{y})$, *we find*

$$|f_x - f_y| \le 2|x-y|. \tag{57}$$

Now

$$|\mathbf{A}x(t) - \mathbf{A}y(t)| \le \frac{1}{\Gamma(\frac{1}{2})}\int_0^t \frac{1}{(t-\mu)^{\frac{1}{2}}}|f_x(\mu) - f_y(\mu)|d\mu. \tag{58}$$

Using F'*S Lipschitz property, we have*

$$|\mathbf{A}x(t) - \mathbf{A}y(t)| \le \frac{2}{\Gamma(\frac{1}{2})}\int_0^t \frac{1}{(t-\mu)^{\frac{1}{2}}}|x(\mu) - y(\mu)|d\mu. \tag{59}$$

Moreover, we have

$$\begin{aligned}|\mathbf{A}x(t) - \mathbf{A}y(t)|_\infty &\le \frac{2}{\Gamma(\frac{1}{2})}\|x(\mu)-y(\mu)\|_\infty \int_0^t \frac{1}{(t-\mu)^{\frac{1}{2}}}d\mu,\\ &\le \frac{4}{\Gamma(\frac{1}{2})}T^{\frac{1}{2}}\|x(\mu)-y(\mu)\|_\infty,\\ &\le \mathrm{H}\|x-y\|_\infty\end{aligned} \tag{60}$$

For sufficiently small T, $\mathbf{A}$ *is a contraction.* $\mathrm{H} < 1$ *if* $T \in \left(\frac{\Gamma(\frac{1}{2})}{4}\right)^{\frac{1}{2}}$.

4.2.1. Stability

The interval-valued function is $\mathrm{F}(\mathrm{x}) = [-2\mathrm{x}, \mathrm{x}]$ *and*

$${}^{\mathrm{C}}\mathscr{D}_{0^+}^{(1/2)}\mathrm{F}(\mathrm{t}) \in [-2\mathrm{x}(\mathrm{t}), \mathrm{x}(\mathrm{t})]. \tag{61}$$

Moreover, for $\mathrm{F}(\mathrm{x}) = [-2\mathrm{x}, \mathrm{x}]$*, the only equilibrium is* x^**. Here, we consider a Lyapunov function*

$$\mathrm{L} = \frac{1}{2}x^2. \tag{62}$$

Applying the Caputo derivative chain rule, we obtain

$${}^{\mathrm{C}}\mathscr{D}_{0^+}^{(1/2)}\mathrm{L}(x) = \frac{\partial \mathrm{L}}{\partial x}\, {}^{\mathrm{C}}\mathscr{D}_{0^+}^{(1/2)}x. \tag{63}$$

The differential inclusion ${}^{\mathrm{C}}\mathscr{D}_{0^+}^{(1/2)}x \in [2x, x]$ *provides us with*

$${}^{\mathrm{C}}\mathscr{D}_{0^+}^{(1/2)}\mathrm{L}(x) \in [2x^2, x^2]. \tag{64}$$

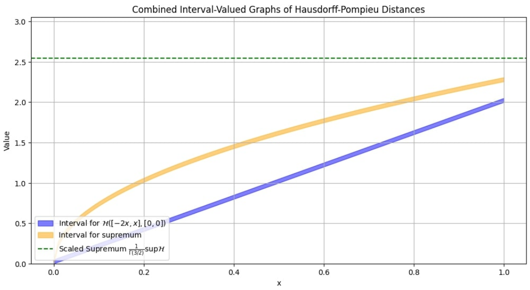
We have

$${}^{\mathrm{C}}\mathscr{D}_{0^+}^{(1/2)}\mathrm{L}(x) \le -2x^2 = -4\mathrm{L}(x). \tag{65}$$

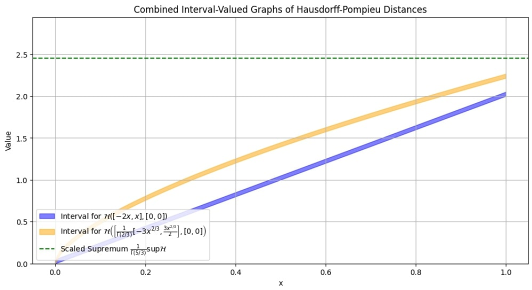
For the given interval-valued function, the equilibrium $x^* = 0$ *is hence asymptotically stable.*

Table 1: Analysis of (49) for Different Values of α

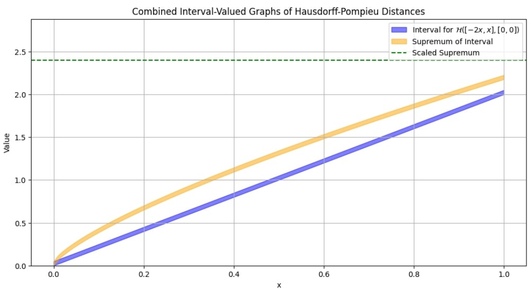
α	$\mathscr{H}(\mathrm{F}(\mathrm{x}),\mathrm{F}(\mathrm{c}))$	$\frac{1}{\Gamma(\alpha+1)}\left\|{}^{\mathrm{C}}\mathscr{D}^{\alpha}_{\mathrm{c}^+}\mathrm{F}(\mathrm{t})\right\|_{\infty}(\mathrm{x}-\mathrm{c})^{\alpha}$
1/2	2.000	2.546
1/3	2.000	2.480
1/4	2.000	2.400
1/5	2.000	2.338
1/6	2.000	2.291
1/7	2.000	2.255
1/8	2.000	2.227
1/9	2.000	2.204



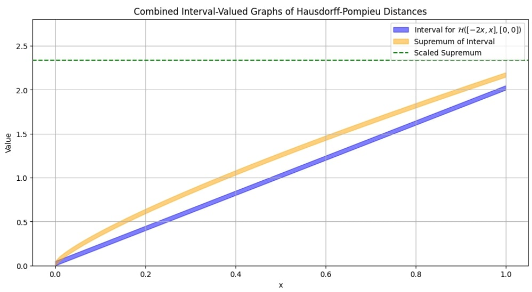
(a) $\alpha = 1/2$



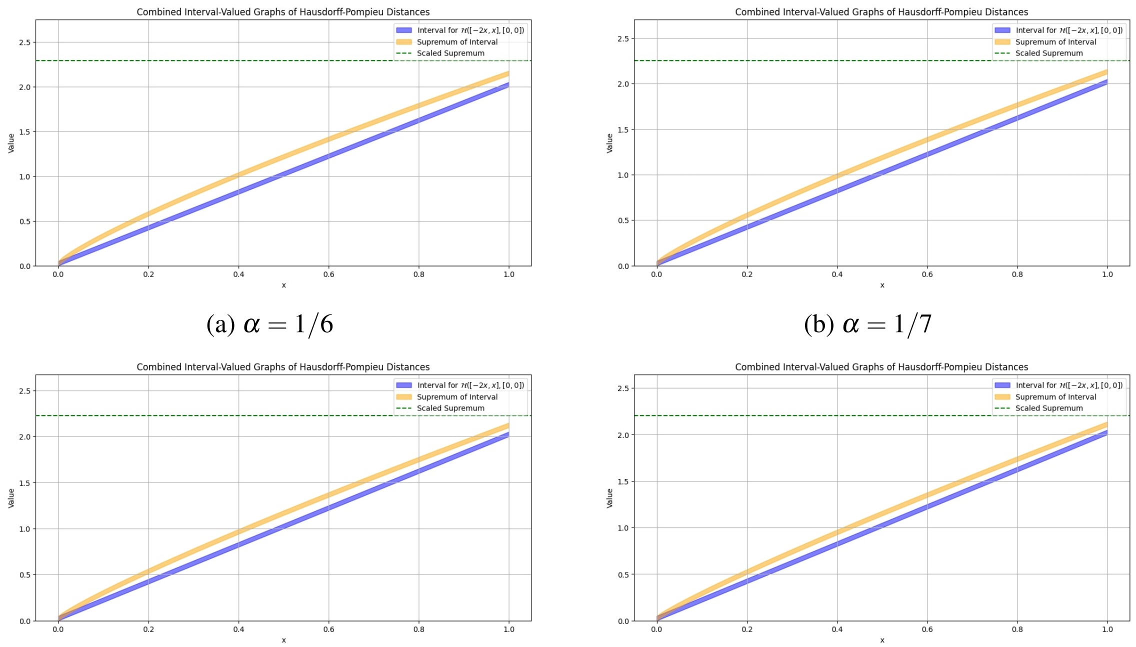
(b) $\alpha = 1/3$



(c) $\alpha = 1/4$



(d) $\alpha = 1/5$



(a) $\alpha = 1/6$ (b) $\alpha = 1/7$

(c) $\alpha = 1/8$ (d) $\alpha = 1/9$

Figure 2: Simulation of Model Inequality given in (41) at Different α Values

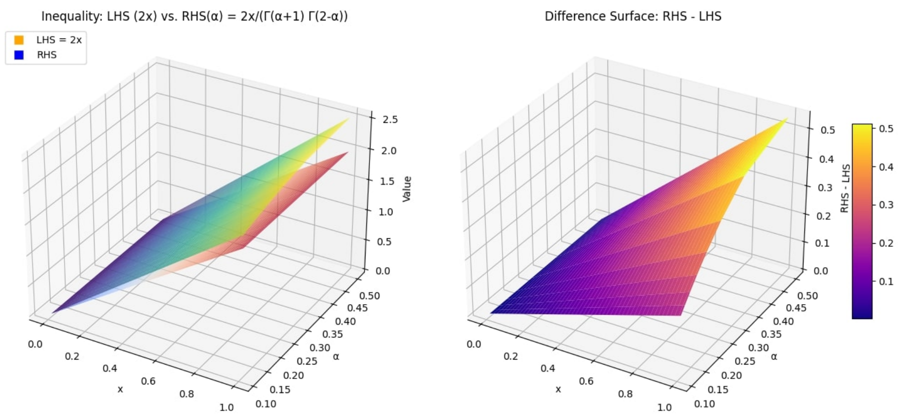


Figure 3: Graphical Representation of (41) for $\alpha = 1/2, 1/3, ..., 1/9$.

The plots show that when the fractional order α decreases from $\frac{1}{2}$ to $\frac{1}{9}$, the corresponding solution curves become progressively smoother with a slower rate of increase, demonstrating the influence of the fractional-order memory effect. The 3D visualization further highlights a uniform decay and convergence pattern of the model inequality (41) for all considered values of α.

For the solution of Example 3 *by classical derivative, we consider the interval-valued function*

$$F(x) = [-2x, x] \quad \textit{for } x \in [0, 1]$$

We apply the result of (Theorem 3.10, see [13]) under the assumption that F is $gH-$differentiable with no switching point in $[0,1]$. The theorem provides a bound,

$$\mathscr{H}(F(b),F(a)) \leq \|F'\|_\infty (b-a) \tag{66}$$

where F' denotes the $gH-$derivative, and H is the Hausdorff distance. We can check that $F'(x) = [-2,1]$ and $\|F'\|_\infty = 2$. The L.H.S of equation (66) becomes,

$$\mathscr{H}(F(1),F(0)) = \mathscr{H}\left(\int_0^1 F'(t)\,dt, \{0\}\right) = 2$$

Now R.H.S of equation (66) gives,

$$\|F'\|_\infty (b-a) = 2\cdot(1-0) = 2\cdot 1 = 2$$

So the inequality holds as equality,

$$\mathscr{H}(F(1),F(0)) = 2 = \|F'\|_\infty (b-a) = 2.$$

So the mean value theorem for Caputo $gH-$differentiable interval-valued function is satisfied, demonstrating the validity of the theorem in this context.

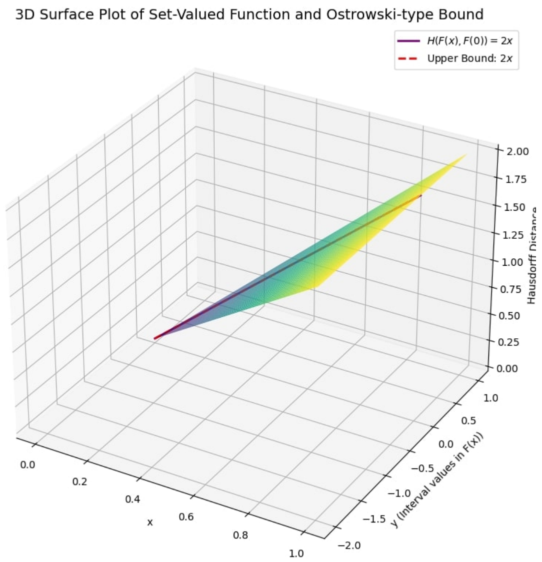


Figure 4: Graphical Representation of (41) by Classical Derivative

This example demonstrates that for $F(x) = [-2x, x]$, the mean value theorem for $gH-$ differentiable interval-valued function by classical derivative is sharp, while fractional bounds offer a smooth generalization dependent on the order of α

5. Result formulation of Caputo gH-differentiable interval-valued functions and Application II

5.1. Ostrowski's type inequality for Caputo gH-differentiable interval-valued functions

Now we prove Ostrowski's type inequality for Caputo gH-differentiable interval-valued function.

Theorem 8. *Let* $\mathrm{F} : [\mathrm{a},\mathrm{b}] \longrightarrow K_{\mathrm{C}}$ *be a Caputo gH-differentiability interval-valued function on* $[\mathrm{a},\mathrm{b}]$ *with* $\mathrm{x} \in [\mathrm{a},\mathrm{b}]$. *Assume that* F' *is continuous. Then*

$$\left\{ \mathscr{H}\left(\frac{1}{(\mathrm{b}-\mathrm{a})}\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{F}(\mathrm{x})\mathrm{dx}, \mathrm{F}(\mathrm{a})\right) \leq \frac{1}{\Gamma(\alpha+2)} \left\|{}^{\mathrm{C}}\mathscr{D}_{a^+}^{\alpha}\mathrm{F}(\mathrm{t})\right\|_{\infty} (\mathrm{b}-\mathrm{a})^{\alpha}, \right. \tag{67}$$

where $0 < \alpha \leq 1$.

Proof. From (Theorem 3.13 [13]), the L.H.S of (67) becomes

$$\mathscr{H}\left(\frac{1}{(\mathrm{b}-\mathrm{a})}\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{F}(\mathrm{x})\mathrm{dx}, \mathrm{F}(\mathrm{a})\right) = \mathscr{H}\left(\frac{1}{(\mathrm{b}-\mathrm{a})}\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{F}(\mathrm{x})\mathrm{dx}, \frac{\mathrm{F}(\mathrm{a})}{(\mathrm{b}-\mathrm{a})}\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{dx}\right)$$

$$= \mathscr{H}\left(\frac{1}{(\mathrm{b}-\mathrm{a})}\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{F}(\mathrm{x})\mathrm{dx}, \frac{1}{(\mathrm{b}-\mathrm{a})}\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{F}(\mathrm{a})\mathrm{dx}\right) = \frac{1}{(\mathrm{b}-\mathrm{a})}\mathscr{H}\left(\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{F}(\mathrm{x})\mathrm{dx}, \int_{\mathrm{a}}^{\mathrm{b}} \mathrm{F}(\mathrm{a})\mathrm{dx}\right),$$

from property (*iii*) in (21)

$$\mathscr{H}\left(\frac{1}{(\mathrm{b}-\mathrm{a})}\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{F}(\mathrm{x})\mathrm{dx}, \mathrm{F}(\mathrm{a})\right) \leq \frac{1}{(\mathrm{b}-\mathrm{a})}\int_{\mathrm{a}}^{\mathrm{b}} \mathscr{H}(\mathrm{F}(\mathrm{x}), \mathrm{F}(\mathrm{a}))\mathrm{dx}. \tag{68}$$

From Theorem 7, we have

$$\mathscr{H}(\mathrm{F}(\mathrm{x}), \mathrm{F}(\mathrm{c})) \leq \frac{1}{\Gamma(\alpha+1)} \left\|{}^{\mathrm{C}}\mathscr{D}_{\mathrm{c}^+}^{\alpha}\mathrm{F}(\mathrm{t})\right\|_{\infty} (\mathrm{x}-\mathrm{c})^{\alpha} \text{ for } \alpha \in (0,1],$$

therefore from (68), we have

$$\frac{1}{(\mathrm{b}-\mathrm{a})}\int_{\mathrm{a}}^{\mathrm{b}} \mathscr{H}(\mathrm{F}(\mathrm{x}), \mathrm{F}(\mathrm{a}))dx \leq \frac{1}{(\mathrm{b}-\mathrm{a})}\int_{\mathrm{a}}^{\mathrm{b}} \frac{1}{\Gamma(\alpha+1)} \left\|{}^{\mathrm{C}}\mathscr{D}_{a^+}^{\alpha}\mathrm{F}(\mathrm{t})\right\|_{\infty} (\mathrm{x}-\mathrm{a})^{\alpha}\mathrm{dx}$$

$$= \frac{1}{(\mathrm{b}-\mathrm{a})\Gamma(\alpha+1)} \left\|{}^{\mathrm{C}}\mathscr{D}_{\mathrm{a}^+}^{\alpha}\mathrm{F}(\mathrm{t})\right\|_{\infty} \int_{\mathrm{a}}^{\mathrm{b}} (\mathrm{x}-\mathrm{a})^{\alpha}\mathrm{dx},$$

$$\left\{ \begin{array}{l} \frac{1}{(\mathrm{b}-\mathrm{a})}\int_{\mathrm{a}}^{\mathrm{b}} \mathscr{H}(\mathrm{F}(\mathrm{x}), \mathrm{F}(\mathrm{a}))\mathrm{dx} \\ \leq \frac{1}{(\mathrm{b}-\mathrm{a})\Gamma(\alpha+1)} \left\|{}^{\mathrm{C}}\mathscr{D}_{\mathrm{a}^+}^{\alpha}\mathrm{F}(\mathrm{t})\right\|_{\infty} \int_{\mathrm{a}}^{\mathrm{b}} (\mathrm{x}-\mathrm{a})^{\alpha}\mathrm{dx}. \end{array} \right. \tag{69}$$

By further simplifications equation (69) becomes,

$$\frac{1}{(b-a)}\int_a^b \mathscr{H}\Big(F(x),F(a)\Big)dx \le \frac{1}{(b-a)\Gamma(\alpha+1)}\left\|{}^{C}\mathscr{D}^{\alpha}_{a^+}F(t)\right\|_\infty\left(\frac{(b-a)^{\alpha+1}}{(\alpha+1)}\right)$$

$$= \frac{1}{(b-a)(\alpha+1)\Gamma(\alpha+1)}\left\|{}^{C}\mathscr{D}^{\alpha}_{a^+}F(t)\right\|_\infty\Big((b-a)^{\alpha+1}\Big).$$

So,

$$\frac{1}{(b-a)}\int_a^b \mathscr{H}\Big(F(x),F(a)\Big)dx \le \frac{1}{(b-a)\Gamma(\alpha+2)}\left\|{}^{C}\mathscr{D}^{\alpha}_{a^+}F(t)\right\|_\infty\Big((b-a)^{\alpha+1}\Big).$$

$$\left\{\ \mathscr{H}\left(\frac{1}{(b-a)}\int_a^b F(x)dx,F(a)\right)\le \frac{1}{\Gamma(\alpha+2)}\left\|{}^{C}\mathscr{D}^{\alpha}_{a^+}F(t)\right\|_\infty\Big((b-a)^{\alpha}\Big) \right. \tag{70}$$

which is the required result.

Following implications have been observed.
Case 1: For $\alpha = 1$, equation (67), yields,

$$\mathscr{H}\left(\frac{1}{(b-a)}\int_a^b F'(x)dx,F(a)\right)\le \frac{1}{2}\left\|F'(t)\right\|_\infty(b-a).$$

Case 2: In the limiting case $\alpha \to 0$, equation (67),

$$\mathscr{H}\left(\frac{1}{(b-a)}\int_a^b F(x)dx,F(a)\right)\le \left\|F(t)\right\|_\infty.$$

Next we primarily utilize the above analysis to provide straightforward examples for solving fractional Ostrowski's type inequality using gH-Caputo derivative.
In the following examples we are going to satisfy the inequality (67) for
$\alpha = 1/2, 1/3, 1/4, ..., 1/9$.

Example 4. *Consider the interval-valued function* $F(x) = [x^2, x^3]; F : [0,1] \longrightarrow K_C(\mathbb{R})$.
It is easy to check that $F'(x) = [2x, 3x^2]$. *We see that* $w(F'(x))$ *has a constant sign on interval* $[0,1]$, *so* $F(x)$ *is increasing in* $[0,1]$. *Further, for* $\alpha = 1/2$ *the L.H.S of the inequality (67) gets the form,*

$$\mathscr{H}\left(\frac{1}{(1-0)}\int_0^1 F(x)dx, F(0)\right) = \mathscr{H}\left(\int_0^1 [x^2,x^3],\ \mathbf{0}dx\right) = 0.33,$$

$$\mathscr{H}\left(\frac{1}{(1-0)}\int_0^1 F(x)dx, F(0)\right) = 0.33, \tag{71}$$

So, for our considerations in this example 4*, (67) becomes*

$$0.33 \le \frac{1}{\Gamma(\alpha+2)}\left\|{}^{C}\mathscr{D}^{\alpha}_{a^+}[t^2,t^3]\right\|_\infty(1-0)^{\alpha} \tag{72}$$

Also, for $\alpha = 1/2$ $\left\| {}^{C}\mathscr{D}_{0^+}^{1/2}\mathrm{F(t)} \right\|_\infty = 1.805$. *The R.H.S of inequality (72) for* $\alpha = 1/2$ *becomes,*

$$\frac{1}{\Gamma(3/2)} \left\| {}^{C}\mathscr{D}_{a^+}^{\alpha} F(t) \right\|_\infty (1-0)^\alpha = 1.35, \tag{73}$$

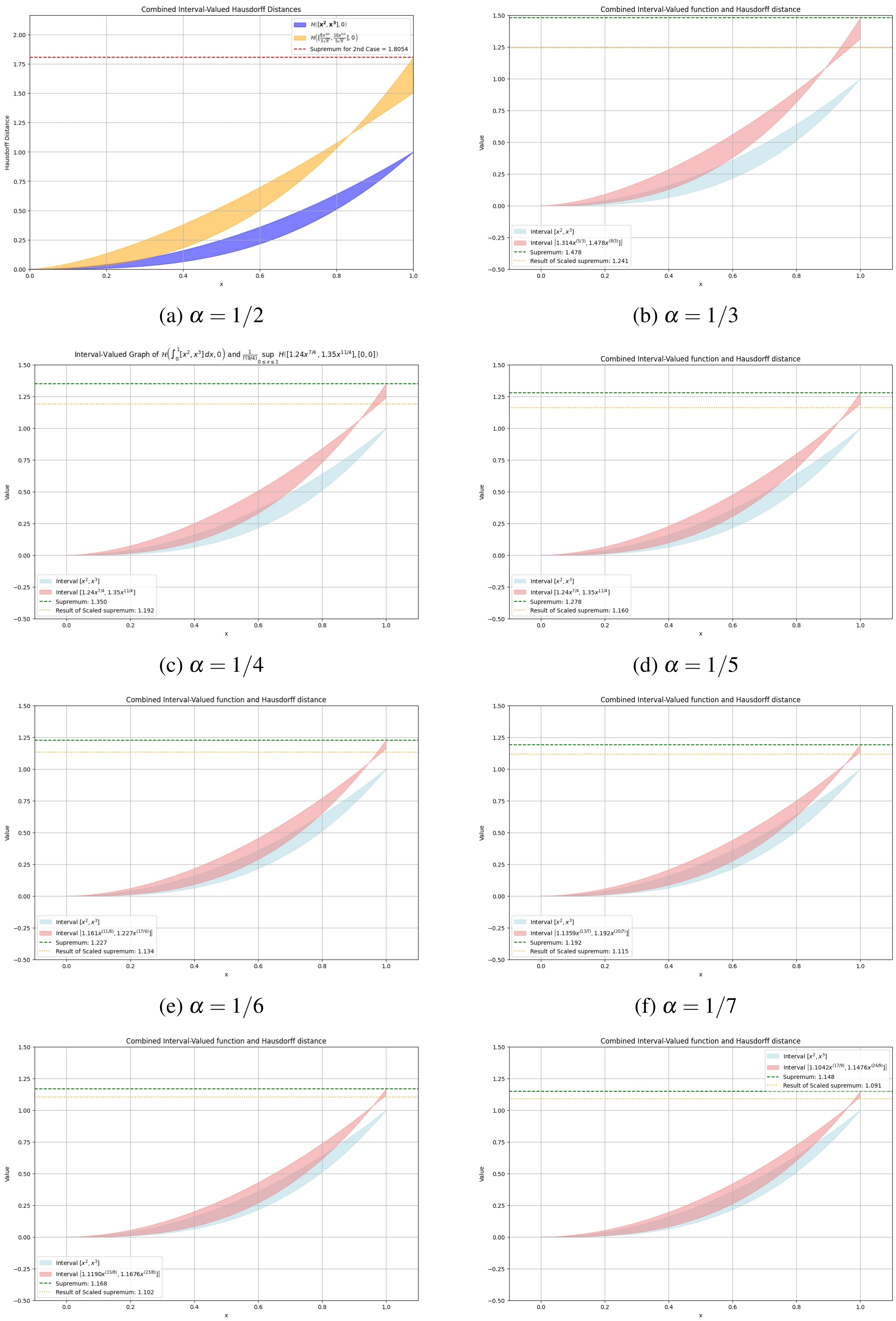
thus, from(71) and (79), we have,

$$0.33 < 1.35$$

Hence the required inequality (67) is satisfied for $\alpha = 1/2$.
For other values of α*, i.e,* $\alpha = 1/3, 1/4, ..., 1/9$*, the results of equation (72) are mentioned in Table (2) which clearly shows that (72) is satisfied.*

Table 2: Analysis of (72) for Different Values of α.

α	$\mathscr{H}\left(\frac{1}{(b-a)}\int_a^b F(x)dx, F(a)\right)$	$\frac{1}{\Gamma(\alpha+2)} \left\| {}^{C}\mathscr{D}_{a^+}^{\alpha} F(t) \right\|_\infty (b-a)^\alpha$
1/2	0.333	1.350
1/3	0.333	1.241
1/4	0.333	1.191
1/5	0.333	1.160
1/6	0.333	1.134
1/7	0.333	1.115
1/8	0.333	1.102
1/9	0.333	1.109



(a) $\alpha = 1/2$ (b) $\alpha = 1/3$

(c) $\alpha = 1/4$ (d) $\alpha = 1/5$

(e) $\alpha = 1/6$ (f) $\alpha = 1/7$

(g) $\alpha = 1/8$ (h) $\alpha = 1/9$

Figure 5: Simulation of Model Inequality given in (67) at Different α Values

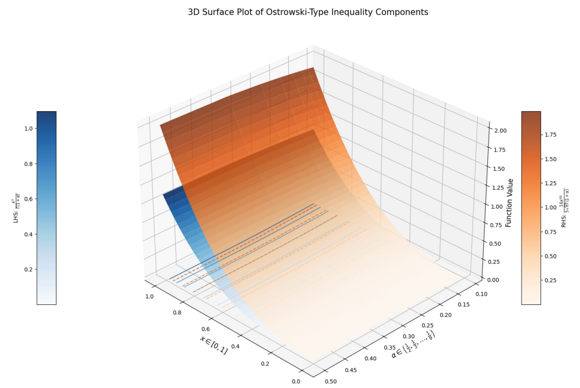


Figure 6: Graphical Representation of (67) for $\alpha = 1/2, 1/3, ..., 1/9$.

The plots demonstrate that as the fractional order α decreases from $\frac{1}{2}$ to $\frac{1}{9}$, the solutions display reduced amplitude and slower variation, emphasizing the damping influence of lower fractional orders. The 3D plot further validates this trend, showing smooth convergence of the model inequality (67) across all α values. For the solution of example 4 *by classical derivative, we analyze the inequality mentioned in (Theorem* 4.1*, see [13])*

$$\mathscr{H}\left(\frac{1}{b-a}\int_a^b F(y)dy,\ F(x)\right) \leq \|F'\|_\infty \cdot \frac{(x-a)^2+(b-x)^2}{2(b-a)} \tag{74}$$

For the function $F(x) = [x^2, x^3]$. It is easy to check that $F'(x) = [2x, 3x^2]$ and $\|F'\|_\infty = 3$.
The L.H.S of equation (74) gives,

$$\mathscr{H}\left(\frac{1}{b-a}\int_a^b F(y)dy,\ F(x)\right) = \mathscr{H}\left(\left[\frac{1}{3}, \frac{1}{4}\right], \{0\}\right) = \frac{1}{3} \tag{75}$$

While the R.H.S of (74) becomes,

$$\|F'\|_\infty \cdot \frac{(x-a)^2+(b-x)^2}{2(b-a)} = 3 \cdot \frac{x^2+(1-x)^2}{2} = 3 \cdot \frac{0+1}{2} = 1.5 \tag{76}$$

From equation(75) and equation(76) we get,

$$\frac{1}{3} \leq 1.5$$

This confirms the validity of inequality (74) for the interval-valued function $F(x) = [x^2, x^3]$ over $[0, 1]$.

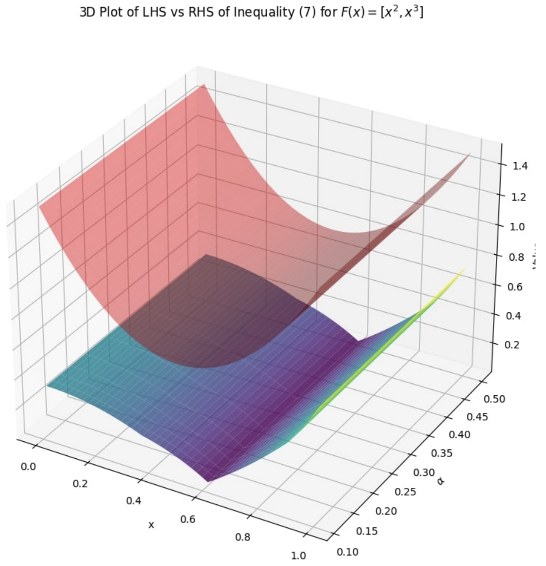


Figure 7: Graphical Representation of (74)(Classical Derivative)

Example 5. *Consider the interval-valued function* $\mathrm{F} : [1,2] \longrightarrow K_C(\mathbb{R})$ *defined by*

$$\mathrm{F(x)} = [-1,2](1-\mathrm{x}^2).$$

It is easy to check that $\mathrm{F}'(\mathrm{x}) = [-1,2](-2\mathrm{x})$.
The L.H.S of the inequality (67) gives,

$$\mathscr{H}\left(\frac{1}{(2-1)}\int_1^2 \mathrm{F(x)dx}, \mathrm{F}(1)\right) = 2.667. \tag{77}$$

So, for our considerations in this example 5, *(67) becomes*

$$2.667 \leq \frac{1}{\Gamma(\alpha+2)} \left\|{}^{\mathrm{C}}\mathscr{D}^{\alpha}_{\mathrm{a}^+}[-1,2](1-\mathrm{t}^2)\right\|_{\infty}(2-1)^{\alpha} \tag{78}$$

Also, for $\alpha = 1/2$ $\left\|{}^{\mathrm{C}}\mathscr{D}^{1/2}_{1^+}\mathrm{F(t)}\right\|_{\infty} = 6.387$.
The R.H.S of inequality (78) for $\alpha = 1/2$ *becomes,*

$$\frac{1}{\Gamma(3/2)} \left\|{}^{\mathrm{C}}\mathscr{D}^{\alpha}_{\mathrm{a}^+}F(t)\right\|_{\infty}(2-1)^{\alpha} = 8.489, \tag{79}$$

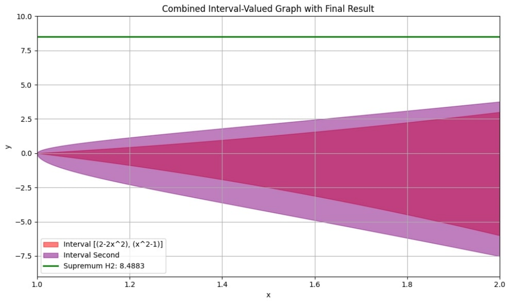
thus, from (77) and (79), we have,

$$2.667 < 8.489$$

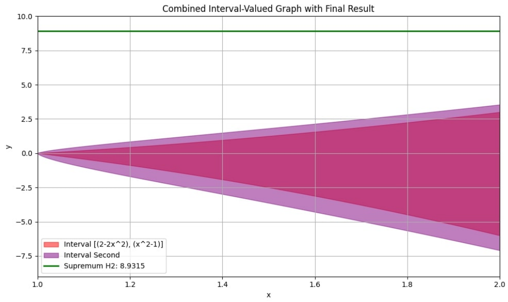
Hence the required inequality (67) is satisfied for $\alpha = 1/2$.
For other values of α, *i.e,* $\alpha = 1/3, 1/4, \ldots, 1/9$, *the results of equation (78) are mentioned in Table (3) which clearly shows that (78) is satisfied.*

Table 3: Analysis of (78) For Different Values of α.

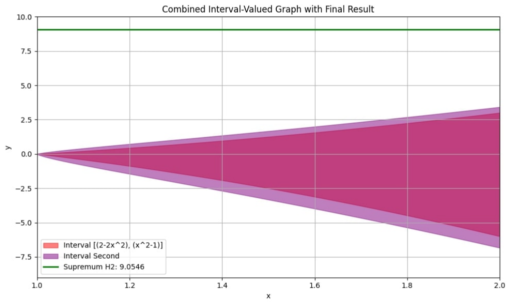
α	$\mathscr{H}\left(\frac{1}{(b-a)}\int_a^b F(x)dx, F(a)\right)$	$\frac{1}{\Gamma(\alpha+2)}\left\|{}^C\mathscr{D}_{a^+}^{\alpha}F(t)\right\|_\infty (b-a)^\alpha$
$1/2$	2.667	8.489
$1/3$	2.667	8.931
$1/4$	2.667	9.050
$1/5$	2.667	9.090
$1/6$	2.667	9.107
$1/7$	2.667	9.109
$1/8$	2.667	9.1077
$1/9$	2.667	9.1036



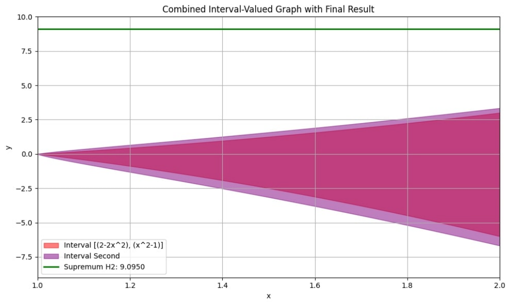
(a) $\alpha = 1/2$



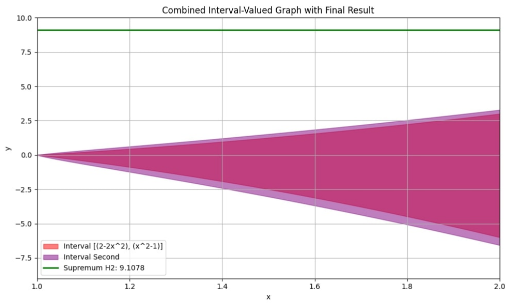
(b) $\alpha = 1/3$



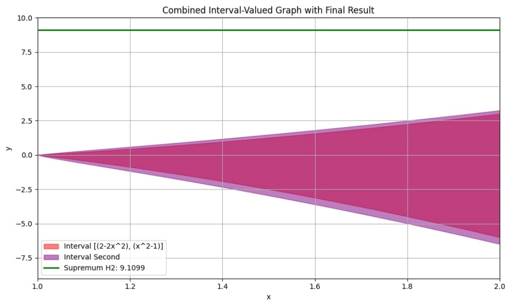
(c) $\alpha = 1/4$



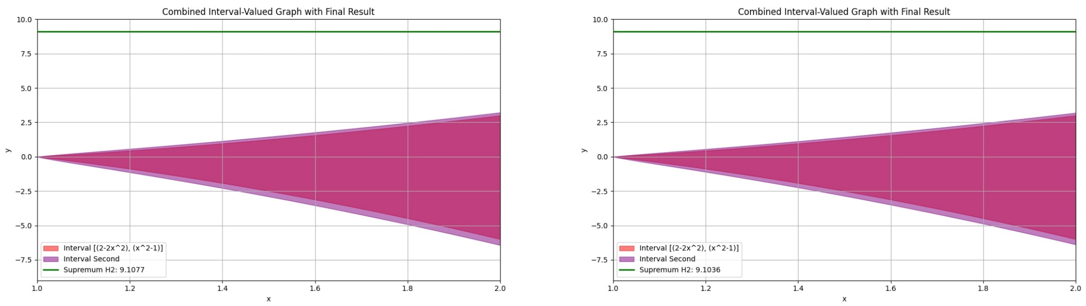
(d) $\alpha = 1/5$



(e) $\alpha = 1/6$



(f) $\alpha = 1/7$



(a) $\alpha = 1/8$ (b) $\alpha = 1/9$

Figure 9: Simulation of Model Inequality given in (67) at Different α Values

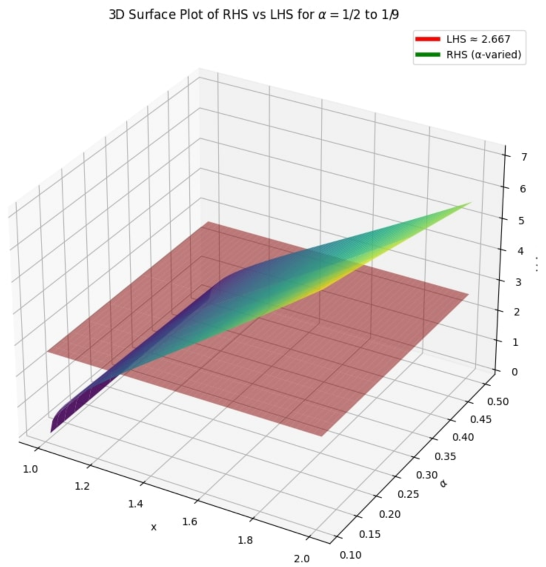


Figure 10: 3D Graphical Representation of (67) for $\alpha = 1/2, 1/3, ..., 1/9$.

The graphical results indicate that decreasing the fractional order α from $\frac{1}{2}$ to $\frac{1}{9}$ leads to smoother solution profiles with diminished oscillations, reflecting stronger memory and damping characteristics. The 3D visualization further confirms consistent stability and convergence of the model inequality (67) across the considered fractional orders.
For the solution of Example 5 *by classical derivative, we consider the interval-valued function,*

$$F(x) = [-1,2](1-x^2) = \left[-1(1-x^2),\ 2(1-x^2)\right] = [x^2-1,\ 2(1-x^2)]$$

for $x \in [1,2]$ as given in Example 5*. We apply the result of (Theorem 4.1, see [13]) under the assumption that F is $gH-$differentiable and* $\mathrm{F}'(\mathrm{x}) = [-1,2](-2\mathrm{x})$, *also* $\|F'\|_\infty = 8$.

The inequality is given by

$$\mathscr{H}\left(\frac{1}{b-a}\int_a^b F(t)dt, F(x)\right) \leq \|F'\|_\infty \cdot \frac{(x-a)^2+(b-x)^2}{2(b-a)} \tag{80}$$

The L.H.S of (80) gives,

$$\mathscr{H}\left(\frac{1}{b-a}\int_a^b F(t)dt, F(x)\right) = \frac{8}{3} \approx 2.667,$$

whereas the R.H.S of (80) becomes,

$$\|F'\|_\infty \cdot \frac{(x-a)^2+(b-x)^2}{2(b-a)} = 8 \cdot \frac{(1-1)^2+(2-1)^2}{2(2-1)} = 8 \cdot \frac{1}{2} = 4$$

Hence

$$\mathscr{H}\left(\frac{1}{b-a}\int_a^b F(y)\,dy,\ F(x)\right) = 2.667 \leq 4 = \|F_g'\|_\infty \cdot \frac{(x-a)^2+(b-x)^2}{2(b-a)}$$

The Ostrowski-type inequality is satisfied for $F(x) = [-1,2](1-x^2)$.

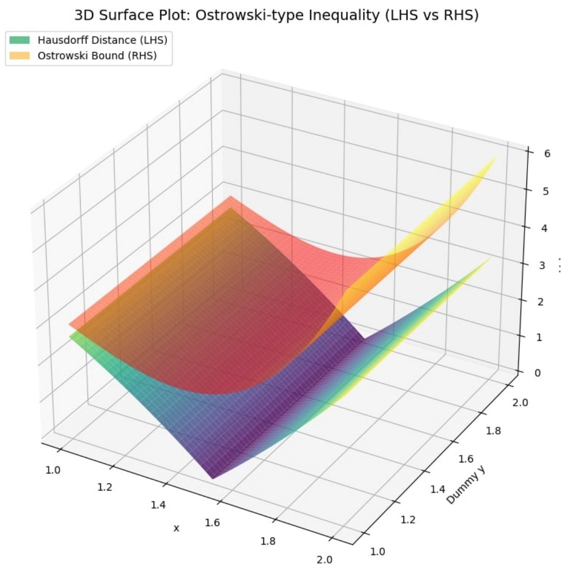


Figure 11: 3D Graphical Representation of (80)(Classical Derivative)

Both 3D plots demonstrate the sharpness and tightness of the inequalities, with minimal gaps between the bound and actual values, confirming their accuracy. The Caputo plot shows increasing tightness with higher α, while the Ostrowski plot highlights precise approximation of function variation across the domain.

6. Application III

Note that the inequality (67) is valid for any continuously Caputo gH-differentiable interval-valued function on $[\mathrm{a},\mathrm{b}]$. From the example we can see that F is Caputo gH-differentiable and (67) is valid, however the endpoint are not necessarily differentiable. For this special case, when endpoints are differentiable we have the following result.

Theorem 9. *Let* $\mathrm{F}:[\mathrm{a},\mathrm{b}]\longrightarrow K_{\mathrm{C}}$ *be a Caputo gH-differentiability interval-valued function such that the end points functions* f^- *and* f^+ *are continuously differentiable,*

$$\left\{\ \mathscr{H}\left(\frac{1}{(\mathrm{b}-\mathrm{a})}\int_{\mathrm{a}}^{\mathrm{b}}\mathrm{F}(\mathrm{x})\mathrm{dx},\mathrm{F}(\mathrm{a})\right)\leq\frac{1}{\Gamma(\alpha+2)}\left\|{}^{\mathrm{C}}\mathscr{D}_{\mathrm{a}^+}^{\alpha}\mathrm{F}(\mathrm{t})\right\|_{\infty}(\mathrm{b}-\mathrm{a})^{\alpha}. \right. \tag{81}$$

Proof. For $\mathrm{x}\in[\mathrm{a},\mathrm{b}]$, since f^- and f^+ are continuously differentiable, we have

$$\mathscr{H}\left(\frac{1}{(\mathrm{b}-\mathrm{a})}\int_{\mathrm{a}}^{\mathrm{b}}\mathrm{F}(\mathrm{x})\mathrm{dx},\mathrm{F}(\mathrm{a})\right)$$

$$=\mathscr{H}\left(\frac{1}{(\mathrm{b}-\mathrm{a})}\int_{\mathrm{a}}^{\mathrm{b}}[\mathrm{f}^-(\mathrm{x}),\mathrm{f}^+(\mathrm{x})]\mathrm{dx},[\mathrm{f}^-(\mathrm{a}),\mathrm{f}^+(\mathrm{a})]\right).$$

$$=\mathscr{H}\left(\left[\frac{1}{(\mathrm{b}-\mathrm{a})}\int_{\mathrm{a}}^{\mathrm{b}}\mathrm{f}^-(\mathrm{x})\mathrm{dx},\frac{1}{(\mathrm{b}-\mathrm{a})}\int_{\mathrm{a}}^{\mathrm{b}}\mathrm{f}^+(\mathrm{x})\right]\mathrm{dx},[\mathrm{f}^-(\mathrm{a}),\mathrm{f}^+(\mathrm{a})]\right)$$

from equation (67),

$$\leq\left[\frac{1}{\Gamma(\alpha+2)}\left\|{}^{\mathrm{C}}\mathscr{D}_{\mathrm{a}^+}^{\alpha}\mathrm{f}^-\right\|_{\infty}(\mathrm{b}-\mathrm{a})^{\alpha},\frac{1}{\Gamma(\alpha+2)}\left\|{}^{\mathrm{C}}\mathscr{D}_{\mathrm{a}^+}^{\alpha}\mathrm{f}^+\right\|_{\infty}(\mathrm{b}-\mathrm{a})^{\alpha}\right]$$

$$=\frac{1}{\Gamma(\alpha+2)}(\mathrm{b}-\mathrm{a})^{\alpha}\left[\left\|{}^{\mathrm{C}}\mathscr{D}_{\mathrm{a}^+}^{\alpha}\mathrm{f}^-\right\|_{\infty},\left\|{}^{\mathrm{C}}\mathscr{D}_{\mathrm{a}^+}^{\alpha}\mathrm{f}^+\right\|_{\infty}\right],$$

from Remark 35, with w-increasing case,

$$\mathscr{H}\left(\frac{1}{(\mathrm{b}-\mathrm{a})}\int_{\mathrm{a}}^{\mathrm{b}}\mathrm{F}(\mathrm{x})\mathrm{dx},\mathrm{F}(\mathrm{a})\right)=\frac{1}{\Gamma(\alpha+2)}(\mathrm{b}-\mathrm{a})^{\alpha}\left\|{}^{\mathrm{C}}\mathscr{D}_{\mathrm{a}^+}^{\alpha}\mathrm{F}(\mathrm{t})\right\|_{\infty}. \tag{82}$$

Thus the proof is completed. □

Next we have solved an example to justify our result.

Example 6. *Consider the interval-valued function* $\mathrm{F}:[1,2]\longrightarrow K_{\mathrm{C}}(\mathbb{R})$ *defined by*

$$\mathrm{F}(\mathrm{x})=[\mathrm{x}^2-1,2\mathrm{x}^2+1].$$

It is easy to check that $\mathrm{F}'(\mathrm{x}) = [2\mathrm{x}, 4\mathrm{x}]$ *and* $\left\|{}^{\mathrm{C}}\mathscr{D}_{1^+}^{1/2}\mathrm{F}(\mathrm{t})\right\|_\infty = 2.559$. *Further, for* $\alpha = 1/2$, *the L.H.S of the inequality (81) gets the form,*

$$\mathscr{H}\left(\frac{1}{(2-1)}\int_1^2 \mathrm{F}(\mathrm{x})\mathrm{dx}, \mathrm{F}(1)\right) = 2.667. \tag{83}$$

Also the right hand side of the inequality (81),

$$\frac{1}{\Gamma(\alpha+2)}\left\|{}^{\mathrm{C}}\mathscr{D}_{\mathrm{a}^+}^{\alpha}\mathrm{F}(\mathrm{t})\right\|_\infty (\mathrm{b}-\mathrm{a})^\alpha = 3.402. \tag{84}$$

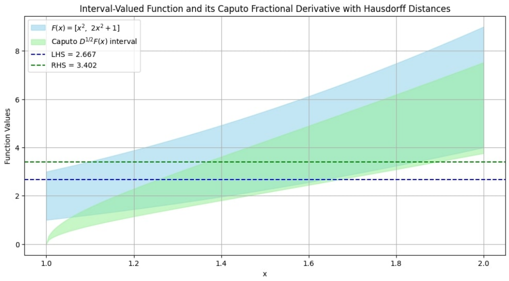
From (83) and (84), we have

$$2.667 \leq 3.402.$$

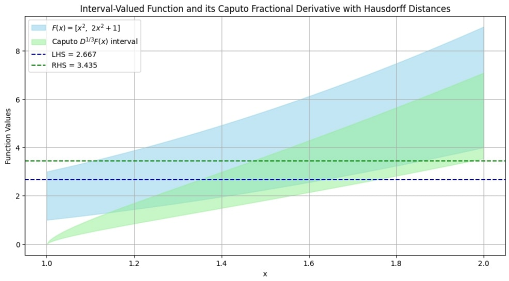
Hence inequality (81) satisfied.
Further we have solved example 6 *for various values of* $\alpha = 1/3, 1/4, ..., 1/9$ *and Table (4) is about the Ostrowski's type inequality for Caputo gH-differentiable interval-valued function defined in equation (81).*

Table 4: Analysis of (81) for Different α Values.

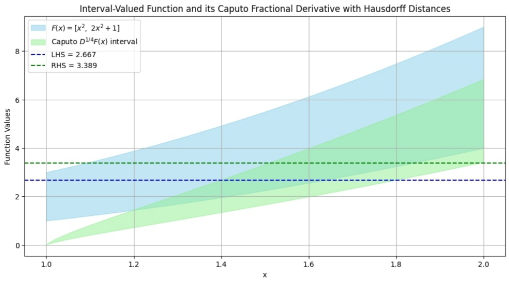
α	$\mathscr{H}\left(\frac{1}{(\mathrm{b}-\mathrm{a})}\int_\mathrm{a}^\mathrm{b} \mathrm{F}(\mathrm{x})\mathrm{dx}, \mathrm{F}(\mathrm{a})\right)$	$\frac{1}{\Gamma(\alpha+2)}\left\|{}^{\mathrm{C}}\mathscr{D}_{\mathrm{a}^+}^{\alpha}\mathrm{F}(\mathrm{t})\right\|_\infty (\mathrm{b}-\mathrm{a})^\alpha$
1/2	2.667	3.402
1/3	2.667	3.435
1/4	2.667	3.389
1/5	2.667	3.341
1/6	2.667	3.300
1/7	2.667	3.267
1/8	2.667	3.240
1/9	2.667	3.218



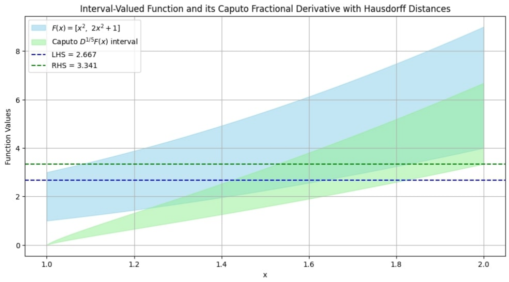
(a) $\alpha = 1/2$



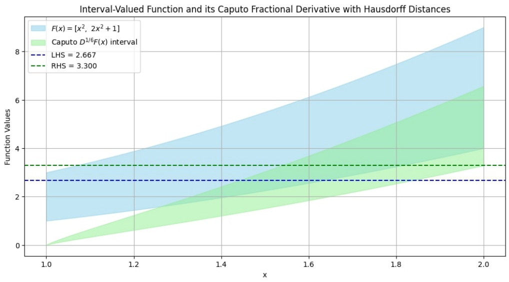
(b) $\alpha = 1/3$



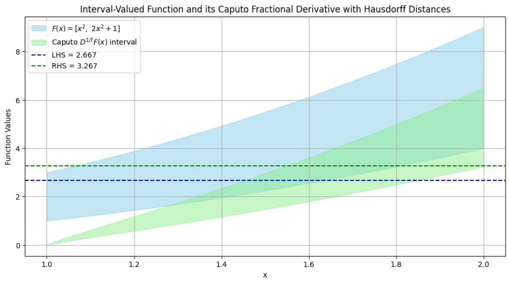
(c) $\alpha = 1/4$



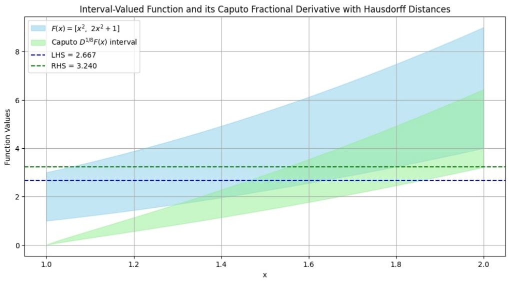
(d) $\alpha = 1/5$



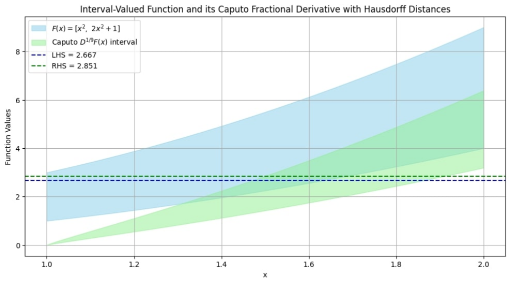
(e) $\alpha = 1/6$



(f) $\alpha = 1/7$



(g) $\alpha = 1/8$



(h) $\alpha = 1/9$

Figure 12: Simulation of Model Inequality given in (81) at Different α Values

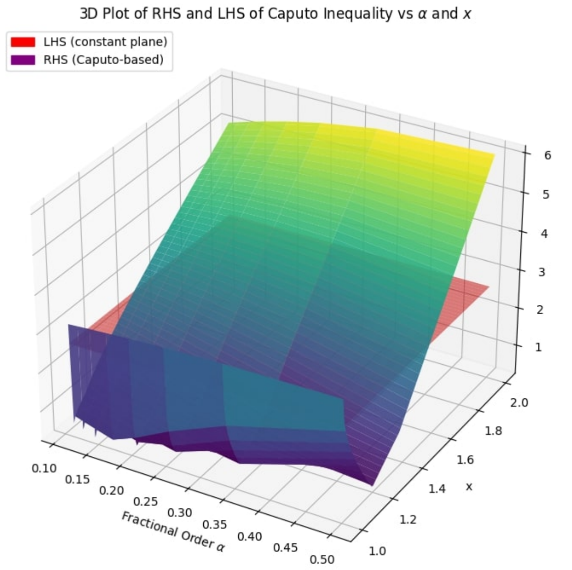


Figure 13: 3D Graphical Representation of (81) for $\alpha = 1/2, 1/3, ..., 1/9$

The simulation results reveal that as the fractional order α *decreases from* $\frac{1}{2}$ *to* $\frac{1}{9}$*, the solution curves exhibit slower growth and enhanced smoothness, indicating stronger fractional damping effects. The 3D plot further illustrates a uniform trend of convergence and stability for the model inequality (81) across all examined values of* α*.*
For the solution of Example 6 *by classical derivative, we verify that the function*

$$F(x) = [x^2,\ 2x^2+1], \quad x \in [1,2]$$

given in example 6 *also satisfy the inequality (Theorem* 4.4*, see [13])*

$$\mathscr{H}\left(\frac{1}{b-a}\int_a^b F(y)\,dy,\ F(x)\right) \le \|F'\|_\infty \cdot \frac{(x-a)^2+(b-x)^2}{2(b-a)}. \tag{85}$$

We have $\|F'\|_\infty = 8$*. The L.H.S of equation (85) gives,*

$$\mathscr{H}\left(\frac{1}{b-a}\int_a^b F(y)\,dy,\ F(x)\right) = \mathscr{H}\left(\left[\frac{7}{3},\frac{17}{3}\right],\ [1,3]\right) = \frac{8}{3} \approx 2.666,$$

and the R.H.S becomes,

$$\|F'\|_\infty \cdot \frac{(x-a)^2+(b-x)^2}{2(b-a)} = 8 \cdot \frac{(1-1)^2+(2-1)^2}{2(2-1)} = 8 \cdot 0.5 = 4 \tag{86}$$

Hence $2.666 \le 4$
Similarly at $x = 2$*, it can be calculated that*

$$\mathscr{H}\left(\frac{1}{b-a}\int_a^b F(y)\,dy,\ F(x)\right) = \frac{10}{3} \approx 3.333 \le \|F'\|_\infty \cdot \frac{(x-a)^2+(b-x)^2}{2(b-a)} = 4$$

Therefore, the inequality

$$\mathscr{H}\left(\frac{1}{b-a}\int_a^b F(y)\,dy,\ F(x)\right) \leq \|F'\|_\infty \cdot \frac{(x-a)^2+(b-x)^2}{2(b-a)}$$

is satisfied at both endpoints $x=1$ and $x=2$ for the interval-valued function

$$F(x) = [x^2,\ 2x^2+1], \quad x \in [1,2].$$

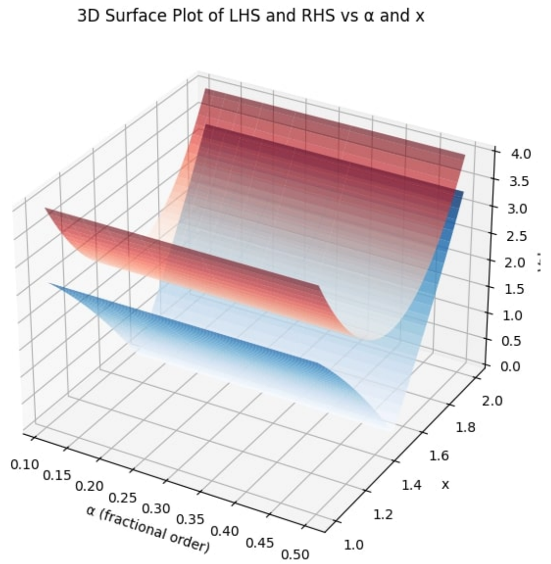


Figure 14: Graphical Representation of (81) for Classical Derivative

The classical derivative offers a simple, $\alpha-$independent bound, while the Caputo fractional approach, varying with both α and x, provides a more precise and rigorous tool for fractional interval analysis.

7. Conclusion

This article is about a novel study of a new concept of Ostrowski's type inequality for Caputo gH-differentiable interval-valued functions. We have discussed some of the interesting results along with examples. Reader can notice that L.H.S of (67) provides a classical approach to analyze interval-valued functions, emphasizing steady growth in endpoint gaps as x increases. Meanwhile, the R.H.S, involving Caputo gH-differentiable functions, exhibits more intricate behavior, where the variation in the bounds is quantified by the Hausdorff-Pompeiu distance.

Graphical comparisons show that classical systems follow predictable patterns, while fractional systems display more intricate and evolving structures with unpredictable graphs. This difference highlights the complexity of fractional systems add to the modeling and analysis. We also develop the comparison between fractional order Ostrowski's type inequality with the classical order Ostrowski's inequality through the examples and thier graphs. These plots underscore a fundamental distinction: while classical derivatives offer a fixed, linear estimate, fractional derivatives introduce an order-dependent flexibility that often yields sharper, more accurate bounds. Together, the graphs illustrate how fractional calculus extends the classical framework, offering nuanced control over inequality bounds in the analysis of gH-differentiable interval-valued functions. Overall, the 3D graphs illustrate not just the validity of the inequalities but their sharpness and tightness, meaning the bounds are closed to actual values and it reflects the true behaviour or variation of the function also it is practically useful, not just theoretically valid. These graphical insights reinforce the strength of the inequalities in practical applications involving uncertainty, fractional order dynamics, or interval analysis.

Classical systems are defined by their simplicity, predictable nature, making them ideal for modeling steady and uniform processes. In contrast, fractional systems capture intricate, memory-influenced, and non-linear dynamics, offering a more detailed representation of irregular or complex phenomena.

Our derived inequality generalizes the classical Ostrowski inequality by incorporating fractional derivatives, making it applicable to functions that may not be classically differentiable but have a well-defined fractional derivative. The inclusion of the fractional order α enhances flexibility, particularly for non-integer differentiability, while classical inequalities require at least one classical derivative. Compared to other fractional inequalities, the given inequality is more specific, the Caputo fractional derivatives allows a more nuanced and complex description of the function's behavior. It handles non-integer differentiability and offers refined estimates over simpler fractional inequalities like Ostrowski or Grüss.

This study will attract researchers utilize powerful tools to address the complexities associated with interval-valued functions. The usefulness of the suggested strategy is illustrated by solving a number of related instances, highlighting its practical relevance.

Declarations

Availability of data and materials: Not applicable

Competing interests: The authors declare that they have no competing interests.

Authors contributions: All the authors contributed equally and they read and approved the final manuscript for publication.

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