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Matrix Decomposition of the r-Whitney Numbers

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Abstract. In this study, we gave a decomposition method used to calculate r-Whitney numbers of the first and second kind in an explicit and non-recursive mode. Then, we expressed the generalized factorial matrices in terms of the r-Whitney transform matrix which may be regarded as a generalization in the form of the Vandermonde matrices. Consequently, we presented some properties of the generalized factorial matrices, particularly the triangular matrix factors of their inverse matrices.

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Key Words and Phrases: r-Whitney numbers, matrix decomposition, generalized factorial matrices, Whitney transform matrix, Vandermonde matrices

1. Introduction

The study of Whitney numbers has long been a cornerstone in combinatorial mathematics, tracing its origins to Hassler Whitney's work on geometric lattices. In 1973, Dowling [1] introduced Dowling lattices $Q_n(G)$, indexed by a positive integer n and a finite group G of order $m > 1$, and characterized their structure using the Möbius function of finite partially ordered sets. The Whitney numbers of the first kind $w_m(n, k)$ emerge as coefficients in the characteristic polynomial of $Q_n(G)$, while the second kind $W_m(n, k)$ enumerate elements of corank k in the lattice. Dowling established foundational properties including orthogonality relations and recursive formulas.

Benoumhani [2] later expanded this framework by deriving generating functions, explicit formulas, and recurrence relations for Whitney numbers of Dowling lattices. Broder's [3] introduction of the r-Stirling numbers in 1984 further generalized classical Stirling numbers by enforcing distinct cycle or subset conditions on the elements $\{1, 2, ..., r\}$. Precisely, the r-Stirling numbers of the first and second kind, respectively, defined in [3] as the number of permutations of the set $\{1, 2, \ldots, n\}$ having n cycles such that the numbers $1, 2, \ldots, r$ are in distinct cycles and the number of partitions of the set $\{1, 2, \ldots, n\}$ into m

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distinct subsets, such that the numbers $1, 2, \ldots, r$ are in distinct subsets. Many properties of r-Stirling numbers can be seen in [4, 5].

These developments paved the way for Mező [6] in the formulation of the r-Whitney numbers in 2010, which unify the Whitney and r-Stirling numbers into a broader combinatorial class. According to [6, 7], the nth power of $mx + r$ can be expressed in terms of the falling factorial as follows:

$$(mx + r)^n = \sum_{k=0}^{n} m^k W_{m,r}(n, k)(x)_k,$$

where the coefficients $W_{m,r}(n, k)$ are called the r-Whitney numbers of the second kind. The r-Whitney numbers of the first kind, on the other hand, are the coefficients of $(mx + r)^k$ in the reverse relation

$$m^n(x)_n = \sum_{k=0}^{n} w_{m,r}(n, k)(mx + r)^k.$$

We note that the r-Whitney numbers of both kinds may be reduced to the r-Stirling numbers of both kinds by setting $m = 1$. Also, the case $r = 1$ gives the Whitney numbers of Dowling lattices.

Recent studies have explored various algebraic and combinatorial properties of r-Whitney numbers, including their matrix representations and transformations. Cheon and Jung [7] provided combinatorial interpretations and identities for both kinds of r-Whitney numbers, while Jiaqiang Pan [8] proposed decomposition formulas for matrix transforms of generalized Stirling numbers, inspiring the current investigation.

To further enrich the theoretical landscape, researchers have examined related structures such as r-Dowling numbers and matrices containing r-Whitney and Lah numbers [9], asymptotic estimates [10], and associated polynomials [11]. These works underscore the growing interest in matrix-based approaches to combinatorial enumeration.

This paper contributes to the growing field of combinatorial enumeration by presenting a matrix decomposition method for computing r-Whitney numbers of both kinds in an explicit, non-recursive manner. Specifically, it aims to define the matrix transforms of the r-Whitney numbers of the first and second kind, derive decomposition formulas for these transforms, and investigate the properties of generalized factorial matrices through the lens of r-Whitney transforms. In addition, the study explores convolution-type identities to obtain further matrix decompositions.

These results not only generalize existing combinatorial frameworks but also offer new computational tools for analyzing related number sequences and transforms.

2. Preliminaries

The r-Whitney numbers of the first and second kind, denoted by $w_{m,r}(n, k)$ and $W_{m,r}(n, k)$, respectively, are defined by means of the horizontal generating functions given as follows:

$$m^n(x)_n = \sum_{k=0}^{n} w_{m,r}(n,k)(mx+r)^k, \tag{1}$$

$$(mx+r)^n = \sum_{k=0}^{n} m^k W_{m,r}(n,k)(x)_k. \tag{2}$$

Note that, from equation (2), the coefficients $W_{m,r}(n,k)$ give back the Stirling, r-Stirling and Whitney numbers of the second kind, respectively. That is,

$$\begin{aligned} W_{1,0}(n,k) &= S(n,k), \\ W_{1,r}(n,k) &= \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r, \\ W_{m,0}(n,k) &= W_m(n,k). \end{aligned}$$

Similarly, by using equation (1), we obtain

$$\begin{aligned} w_{1,0}(n,k) &= s(n,k), \\ w_{1,r}(n,k) &= \begin{bmatrix} n+r \\ k+r \end{bmatrix}_r, \\ w_{m,0}(n,k) &= w_m(n,k). \end{aligned}$$

In 2012, G.S. Cheon and J.H. Jung [7] gave the combinatorial interpretation of the r-Whitney numbers of the first and second kind, respectively, as the sum of all possible products of $n-k$ different factors taken from the set $\{r, r+m, \ldots, r+(n-1)m\}$, and the sum of all possible products of $n-k$ factors, not necessarily distinct, taken from the set $\{r, r+m, \ldots, r+km\}$.

Several properties of these numbers were already established and can be read in references [6, 7, 12, 13].

2.1. Table values for $w_{m,r}(n,k)$

Now, for the following table of values, we make use of the triangular recurrence relation of $w_{m,r}(n,k)$ given by

$$w_{m,r}(n,k) = w_{m,r}(n-1,k-1) + (m - nm - r)w_{m,r}(n-1,k). \tag{3}$$

Let $w_{0^+,r}(n,k)$ denote the limit of $w_{m,r}(n,k)$ (for $n,k = 0,1,2,3,\ldots$) as m approaches 0 from the right, that is,

$$\begin{aligned} w_{0^+,r}(n,k) &= \lim_{m\to 0^+} w_{m,r}(n,k) \\ &= \lim_{m\to 0^+} w_{m,r}(n-1,k-1) + (m - nm - r)\; w_{m,r}(n-1,k) \end{aligned}$$

$$= w_{0^+,r}(n-1,k-1) - r\ w_{0^+,r}(n-1,k).$$

For small integer values of n and k (specifically, $n,k = 0,1,2,3,4,5$), the corresponding values of $w_{m,r}(n,k)$ for chosen constants m and r are presented in the subsequent tables.

$n \backslash k$	0	1	2	3	4	5
0	1					
1	-2	1				
2	4	-4	1			
3	-8	12	-6	1		
4	16	-32	24	-8	1	
5	-32	80	-80	40	-10	1

Table 1. Table of values for $w_{0^+,2}(n,k)$

$n \backslash k$	0	1	2	3	4	5
0	1					
1	0	1				
2	0	-1	1			
3	0	2	-3	1		
4	0	-6	11	-6	1	
5	0	24	-50	35	-10	1

Table 2. Table of values for $w_{1,0}(n,k) = s(n,k)$

$n \backslash k$	0	1	2	3	4	5
0	1					
1	-1	1				
2	2	-3	1			
3	-6	11	-6	1		
4	24	-50	35	-10	1	
5	-120	274	-225	85	-15	1

Table 3. Table of values for $w_{1,1}(n,k)$

$n \backslash k$	0	1	2	3	4	5
0	1					
1	−2	1				
2	6	−5	1			
3	−24	26	−9	1		
4	120	−154	71	−14	1	
5	−720	1044	−580	155	−20	1

Table 4. Table of values for $w_{1,2}(n,k)$

$n \backslash k$	0	1	2	3	4	5
0	1					
1	−1	1				
2	3	−4	1			
3	−15	23	−9	1		
4	105	−176	86	−16	1	
5	−945	1689	−950	230	−25	1

Table 5. Table of values for $w_{2,1}(n,k)$

$n \backslash k$	0	1	2	3	4	5
0	1					
1	−1	1				
2	4	−5	1			
3	−28	39	−12	1		
4	280	−418	159	−22	1	
5	−3640	5714	−2485	445	−35	1

Table 6. Table of values for $w_{3,1}(n,k)$

$n \backslash k$	0	1	2	3	4	5
0	1					
1	−1	1				
2	5	−6	1			
3	−45	59	−15	1		
4	585	−812	254	−28	1	
5	−9945	14389	−5130	730	−45	1

Table 7. Table of values for $w_{4,1}(n,k)$

2.2. Table values for $W_{m,r}(n,k)$

Now, for the following table of values, we make use of the triangular recurrence relation of $W_{m,r}(n,k)$ given by

$$W_{m,r}(n,k) = W_{m,r}(n-1,k-1) + (km+r)W_{m,r}(n-1,k). \tag{4}$$

Let $W_{0^+,r}(n,k)$ denote the limit of $W_{m,r}(n,k)$ (for $n,k = 0,1,2,3,\ldots$) as m approaches 0 from the right, that is,

$$\begin{aligned} W_{0^+,r}(n,k) &= \lim_{m\to 0^+} W_{m,r}(n,k) \\ &= \lim_{m\to 0^+} W_{m,r}(n-1,k-1) + (km+r)\ W_{m,r}(n-1,k) \\ &= W_{0^+,r}(n-1,k-1) + r\ W_{0^+,r}(n-1,k). \end{aligned}$$

In particular, for $n,k = 0,1,2,3,4,5$ and for some m and r, the corresponding values of $W_{m,r}(n,k)$ are shown in the following tables:

n \ k	0	1	2	3	4	5
0	1					
1	2	1				
2	4	4	1			
3	8	12	6	1		
4	16	32	24	8	1	
5	32	80	80	40	10	1

Table 8. Table of values for $W_{0^+,2}(n,k)$

n \ k	0	1	2	3	4	5
0	1					
1	0	1				
2	0	1	1			
3	0	1	3	1		
4	0	1	7	6	1	
5	0	1	15	25	10	1

Table 9. Table of values for $W_{1,0}(n,k)$

$n \backslash k$	0	1	2	3	4	5
0	1					
1	2	1				
2	4	5	1			
3	8	19	9	1		
4	16	65	55	14	1	
5	32	211	285	125	20	1

Table 10. Table of values for $W_{1,2}(n, k)$

$n \backslash k$	0	1	2	3	4	5
0	1					
1	0	1				
2	0	2	1			
3	0	4	6	1		
4	0	8	28	12	1	
5	0	16	120	100	20	1

Table 11. Table of values for $W_{2,0}(n, k)$

$n \backslash k$	0	1	2	3	4	5
0	1					
1	0	1				
2	0	3	1			
3	0	9	9	1		
4	0	27	63	18	1	
5	0	18	405	225	30	1

Table 12. Table of values for $W_{3,0}(n, k)$

$n \backslash k$	0	1	2	3	4	5
0	1					
1	0	1				
2	0	4	1			
3	0	16	12	1		
4	0	64	112	24	1	
5	0	256	960	400	40	1

Table 13. Table of values for $W_{4,0}(n, k)$

$n \backslash k$	0	1	2	3	4	5
0	1					
1	1	1				
2	1	6	1			
3	1	31	15	1		
4	1	156	166	28	1	
5	1	781	1650	530	45	1

Table 14. Table of values for $W_{4,1}(n,k)$

Remark 1. Let $A = [w_{m,r}(n,k)]_{n,k\geq 0}$ and $B = [W_{m,r}(n,k)]_{n,k\geq 0}$. Then the product AB is

$$AB = [c_{ij}]_{i,j\geq 0},$$

where

$$c_{ij} = \sum_{k=0}^{n} w_{m,r}(i,k)W_{m,r}(k,j). \tag{5}$$

By orthogonality relation,

$$c_{ij} = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}.$$

Hence,

$$AB = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

This implies that A and B are inverses.

3. Main Results

In this section, we introduce the r-Whitney matrix of the first and second kind. Further, we give the decomposition formula for these matrices.

3.1. r-Whitney Matrix of the First Kind

From the horizontal generating function of $w_{m,r}(n,k)$, we define its matrix form as follows:

Definition 1. The r-Whitney matrix (transform) of the first kind, denoted by $\mathcal{A}_{m,r}$, is defined to be an infinite-dimensional lower triangular matrix, the (n,k)th entries of which are $w_{m,r}(n,k)$ $(n,k=0,1,2,\ldots)$ such that

$$\mathcal{V}_m(mx)=\mathcal{A}_{m,r}\mathcal{V}_0(mx+r), \tag{6}$$

where

$$\mathcal{V}_\alpha(x)=(1,x,(x|\alpha)_2,(x|\alpha)_3,\cdots,(x|\alpha)_n,\cdots)^T.$$

Remark 2. The matrix equation in (6) can be written explicitly as

$$\begin{pmatrix} 1 \\ mx \\ (mx|m)_2 \\ (mx|m)_3 \\ \vdots \end{pmatrix}=\begin{pmatrix} w_{m,r}(0,0) & 0 & 0 & 0 & \cdots \\ w_{m,r}(1,0) & w_{m,r}(1,1) & 0 & 0 & \cdots \\ w_{m,r}(2,0) & w_{m,r}(2,1) & w_{m,r}(2,2) & 0 & \cdots \\ w_{m,r}(3,0) & w_{m,r}(3,1) & w_{m,r}(3,2) & w_{m,r}(3,3) & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} 1 \\ mx+r \\ (mx+r|0)_2 \\ (mx+r|0)_3 \\ \vdots \end{pmatrix}$$

or

$$\begin{pmatrix} 1 \\ mx \\ m^2(x)_2 \\ m^3(x)_3 \\ \vdots \end{pmatrix}=\begin{pmatrix} w_{m,r}(0,0) & 0 & 0 & 0 & \cdots \\ w_{m,r}(1,0) & w_{m,r}(1,1) & 0 & 0 & \cdots \\ w_{m,r}(2,0) & w_{m,r}(2,1) & w_{m,r}(2,2) & 0 & \cdots \\ w_{m,r}(3,0) & w_{m,r}(3,1) & w_{m,r}(3,2) & w_{m,r}(3,3) & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} 1 \\ mx+r \\ (mx+r)^2 \\ (mx+r)^3 \\ \vdots \end{pmatrix}.$$

Note that for any value of x, the vector $\mathcal{V}_m(x)$ and $\mathcal{V}_0(mx+r)$ are nonzero since they both have 1 as their first entry.

Now, let us consider the matrices $\mathcal{A}_{0^+,0},\mathcal{A}_{m,0}$ and $\mathcal{A}_{0^+,r}$.

Remark 3. When $m\to 0^+$ and $r=0$, we have

$$\mathcal{V}_0(mx)=\mathcal{A}_{0^+,0}\mathcal{V}_0(mx).$$

This implies that

$$\mathcal{A}_{0^+,0}=\mathcal{E},$$

where $\mathcal{E}$ is an infinite-dimensional identity matrix.

When $r=0$, from the generating function of $w_{m,r}(n,k)$, we have

$$m^n(x)_n=\sum_{k=0}^{n}w_{m,0}(n,k)(mx)^k$$

or

$$m^n(x)_n=\sum_{k=0}^{n}w_{m,0}(n,k)m^kx^k. \tag{7}$$

Setting $m = 1$ gives us

$$(x)_n = \sum_{k=0}^{n} w_{1,0}(n,k)x^k.$$

Note that this equation gives us the horizontal generating function of the Stirling numbers $s(n,k)$ of the first kind.

Now, multiplying both sides by m^n, we have

$$m^n(x)_n = \sum_{k=0}^{n} m^n w_{1,0}(n,k)x^k.$$

Thus, from (7),

$$\sum_{k=0}^{n} w_{m,0}(n,k)m^k x^k = \sum_{k=0}^{n} m^n w_{1,0}(n,k)x^k.$$

This implies

$$w_{m,0}(n,k)m^k = m^n w_{1,0}(n,k).$$

It follows that

$$w_{m,0}(n,k) = m^{n-k} w_{1,0}(n,k);$$

that is,

$$w_{m,0}(n,k) = m^{n-k} s(n,k).$$

Hence, the (n,k)th entry of matrix $\mathcal{A}_{m,0}$ is the Stirling numbers $s(n,k)$ of the first kind multiplied by m^{n-k}; that is

$$\mathcal{A}_{m,0} = m^{n-k}\mathcal{S}^1_{n,k},$$

where $\mathcal{S}^1_{n,k}$ is the matrix transform of the Stirling numbers of the first kind; i.e. the entries of $\mathcal{S}^1_{n,k}$ are the numbers $s(n,k)$.

When $m \to 0^+$ and $r \neq 0$, the generating function of $w_{m,r}(n,k)$ gives us

$$0 = \sum_{k=0}^{n} w_{0+,r}(n,k)r^k.$$

Using the binomial theorem,

$$0 = (-r+r)^n = \sum_{k=0}^{n} \binom{n}{k}(-r)^{n-k}r^k.$$

Hence, we have

$$\sum_{k=0}^{n} \binom{n}{k}(-r)^{n-k}r^k = \sum_{k=0}^{n} w_{0^+,r}(n,k)r^k.$$

This implies that

$$w_{0+,r}(n,k) = \binom{n}{k}(-r)^{n-k}.$$

Hence, the (n,k)th entry of matrix $\mathcal{A}_{0+,r}$ is the binomial coefficient $\binom{n}{k}$ multiplied by $(-r)^{n-k}$; that is,

$$w_{0^+,r}(n,k) = (-r)^{n-k}\binom{n}{k} \ (n,k = 0,1,2,\ldots).$$

3.2. Decomposition Formula for $\mathcal{A}_{m,r}$

The following theorem gives a general decomposition formula of the r-Whitney matrix of the first kind, $\mathcal{A}_{m,r}$.

Theorem 1. *Let $\mathcal{A}_{m,r}$ denote the r-Whitney matrix of the first kind. Then*

$$\mathcal{A}_{m,r} = \mathcal{A}_{m,0}\mathcal{A}_{0+,r}.$$

Proof. By Definition 1,

$$\mathcal{V}_m(mx) = \mathcal{A}_{m,r}\mathcal{V}_0(mx+r).$$

Note that when the increment $m \to 0^+$ and $r \neq 0$, we have

$$\mathcal{V}_0(mx) = \mathcal{A}_{0+,r}\mathcal{V}_0(mx+r),$$

while when $m \geq 1$ and $r = 0$, we have

$$\mathcal{V}_m(mx) = \mathcal{A}_{m,0}\mathcal{V}_0(mx) = \mathcal{A}_{m,0}\mathcal{A}_{0^+,r}\mathcal{V}_0(mx+r).$$

Hence, with $\mathcal{V}_m(mx) = \mathcal{A}_{m,r}\mathcal{V}_0(mx+r)$ and $\mathcal{V}_m(mx) = \mathcal{A}_{m,0}\mathcal{A}_{0^+,r}\mathcal{V}_0(mx+r)$, by transitive property, we have the equality

$$\mathcal{A}_{m,r}\mathcal{V}_0(mx+r) = \mathcal{A}_{m,0}\mathcal{A}_{0^+,r}\mathcal{V}_0(mx+r).$$

This implies that

$$\mathcal{A}_{m,r}\mathcal{V}_0(mx+r) - \mathcal{A}_{m,0}\mathcal{A}_{0^+,r}\mathcal{V}_0(mx+r) = \mathcal{O},$$

where $\mathcal{O}$ denotes an infinite dimensional zero matrix. Consequently,

$$\left(\mathcal{A}_{m,r} - \mathcal{A}_{m,0}\mathcal{A}_{0^+,r}\right)\mathcal{V}_0(mx+r) = \mathcal{O}.$$

Since x is an arbitrary real or complex number and $\mathcal{V}_0(mx+r)$ is a nonzero vector, we have

$$\mathcal{A}_{m,r} - \mathcal{A}_{m,0}\mathcal{A}_{0^+,r} = \mathcal{O}.$$

Thus,

$$\mathcal{A}_{m,r} = \mathcal{A}_{m,0}\mathcal{A}_{0^+,r}.$$

Remark 4. Since for $i = 1, 2, \ldots$, the ith row of $\mathcal{A}_{m,0}$ contains a finite number of nonzero entries arranged as $w_{m,0}(i, 0), w_{m,0}(i, 1), \ldots, w_{m,0}(i, i)$ and that the rest of the entries $w_{m,0}(i, i+1), w_{m,0}(i, i+2), \ldots$ are all zero, the entries of the product $\mathcal{A}_{m,0}\mathcal{A}_{0+,r}$ exist.

Example 1. The matrix $\mathcal{A}_{1,2}$ whose entries are $w_{1,2}(n, k)$ $(n, k = 0, 1, 2, 3, \ldots)$ may be factored as

$$\mathcal{A}_{1,2} = \mathcal{A}_{1,0}\mathcal{A}_{0+,2};$$

that is,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \ldots \\ -2 & 1 & 0 & 0 & \ldots \\ 6 & -5 & 1 & 0 & \ldots \\ -24 & 26 & -9 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & \ldots \\ 0 & -1 & 1 & 0 & \ldots \\ 0 & 2 & -3 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & \ldots \\ -2 & 1 & 0 & 0 & \ldots \\ 4 & -4 & 1 & 0 & \ldots \\ -8 & 12 & -6 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

3.3. r-Whitney Matrix of the Second Kind

From the horizontal generating function of $W_{m,r}(n, k)$, we define its matrix form as follows:

Definition 2. The r-Whitney matrix (transform) of the second kind, denoted by $\mathcal{B}_{m,r}$, is defined to be an infinite dimensional lower triangular matrix, the (n, k)th entries of which are $W_{m,r}(n, k)$ $(n, k = 0, 1, 2, \ldots)$ such that

$$\mathcal{V}_0(mx + r) = \mathcal{B}_{m,r}\mathcal{V}_m(mx). \tag{8}$$

Remark 5. The matrix equation (8) can be written explicitly as

$$\begin{pmatrix} 1 \\ mx + r \\ (mx + r|0)_2 \\ (mx + r|0)_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} W_{m,r}(0,0) & 0 & 0 & 0 & \cdots \\ W_{m,r}(1,0) & W_{m,r}(1,1) & 0 & 0 & \cdots \\ W_{m,r}(2,0) & W_{m,r}(2,1) & W_{m,r}(2,2) & 0 & \cdots \\ W_{m,r}(3,0) & W_{m,r}(3,1) & W_{m,r}(3,2) & W_{m,r}(3,3) & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ mx \\ (mx|m)_2 \\ (mx|m)_3 \\ \vdots \end{pmatrix}$$

or

$$\begin{pmatrix} 1 \\ mx + r \\ (mx + r)^2 \\ (mx + r)^3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ W_{m,r}(1,0) & 1 & 0 & 0 & \cdots \\ W_{m,r}(2,0) & W_{m,r}(2,1) & 1 & 0 & \ldots \\ W_{m,r}(3,0) & W_{m,r}(3,1) & W_{m,r}(3,2) & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ mx \\ m^2(x)_2 \\ m^3(x)_3 \\ \vdots \end{pmatrix}.$$

From the generating function $(mx + r)^n = \sum_{k=0}^{n} m^k W_{m,r}(n, k)(x)_k$, we have

$$(mx)^n = \sum_{k=0}^{n} m^k W_{m,0}(n, k)(x)_k,$$

when $r = 0$. Equivalently,

$$m^n x^n = \sum_{k=0}^{n} m^k W_{m,0}(n,k)(x)_k.$$

Now, when $m = 1$, we get the generating function of the Stirling numbers $S(n,k)$ of the second kind. That is,

$$x^n = \sum_{n=k}^{n} W_{1,0}(n,k)(x)_k.$$

Multiplying m^n to both sides gives us

$$m^n x^n = \sum_{n=k}^{n} m^n W_{1,0}(n,k)(x)_k.$$

Hence,

$$\sum_{k=0}^{n} m^k W_{m,0}(n,k)(x)_k = \sum_{n=k}^{n} m^n W_{1,0}(n,k)(x)_k.$$

This implies that

$$m^k W_{m,0}(n,k) = m^n W_{1,0}(n,k).$$

Consequently,

$$W_{m,0}(n,k) = m^{n-k} W_{1,0}(n,k)$$

or

$$W_{m,0}(n,k) = m^{n-k} S(n,k).$$

Thus, the (n,k)th entry of matrix $\mathcal{B}_{m,0}$ is the Stirling numbers $S(n,k)$ of the second kind multiplied by m^{n-k}. That is,

$$W_{m,0}(n,k) = m^{n-k} S_{n,k}.$$

3.4. Decomposition Formula for $\mathcal{B}_{m,r}$

Now, we give a general decomposition formula of the r-Whitney matrix of the second kind $\mathcal{B}_{m,r}$.

Theorem 2. *Let $\mathcal{B}_{m,r}$ denote the r-Whitney matrix (transform) of the second kind. Then*

$$\mathcal{B}_{m,r} = \mathcal{B}_{0^+,r}\mathcal{B}_{m,0}.$$

Proof. When $m \geq 1$ and $r = 0$, equation (8) gives us

$$\mathcal{V}_0(mx) = \mathcal{B}_{m,0}\mathcal{V}_m(mx).$$

On the other hand, when the increment $m \to 0^+$ and $r \neq 0$, we have

$$\mathcal{V}_0(mx + r) = \mathcal{B}_{0^+,r}\mathcal{V}_0(mx) = \mathcal{B}_{0^+,r}\mathcal{B}_{m,0}\mathcal{V}_m(mx).$$

Hence, with $\mathcal{V}_0(mx+r) = \mathcal{B}_{m,r}\mathcal{V}_m(mx)$ and $\mathcal{V}_0(mx+r) = \mathcal{B}_{0^+,r}\mathcal{B}_{m,0}\mathcal{V}_m(mx)$, by transitivity, we have

$$\mathcal{B}_{0^+,r}\mathcal{B}_{m,0}\mathcal{V}_m(mx) = \mathcal{B}_{m,r}\mathcal{V}_m(mx).$$

This implies that

$$\mathcal{B}_{0^+,r}\mathcal{B}_{m,0}\mathcal{V}_m(mx) - \mathcal{B}_{m,r}\mathcal{V}_m(mx) = \mathcal{O}.$$

Hence,

$$\left(\mathcal{B}_{0^+,r}\mathcal{B}_{m,0} - \mathcal{B}_{m,r}\right)\mathcal{V}_m(mx) = \mathcal{O}.$$

Since x is an arbitrary real or complex number and $\mathcal{V}_m(mx)$ is a nonzero vector, we have

$$\mathcal{B}_{0^+,r}\mathcal{B}_{m,0} - \mathcal{B}_{m,r} = \mathcal{O}.$$

Thus,

$$\mathcal{B}_{0^+,r}\mathcal{B}_{m,0} = \mathcal{B}_{m,r}.$$

Example 2. The matrix $\mathcal{B}_{1,2}$ whose entries are $W_{1,2}(n,k)$ $(n,k = 0,1,2,3,\ldots)$ may be factored as

$$\mathcal{B}_{1,2} = \mathcal{B}_{0^+,2}\mathcal{B}_{1,0};$$

that is,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & \cdots \\ 4 & 5 & 1 & 0 & \cdots \\ 8 & 19 & 9 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & \cdots \\ 4 & 4 & 1 & 0 & \cdots \\ 8 & 12 & 6 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & \cdots \\ 0 & 1 & 3 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

3.5. α-Generalized Whitney Transform

In this section, we give another property related to the recurrent integer sequences, but before that, let us consider first the following definition:

Definition 3. Let α be a positive integer, and $a(n)$ $(n = 0,1,2,\ldots)$ be a linear homogeneous recurrent integer sequence of order q. The α-generalized Whitney transform of the first kind of $a(n)$ is defined by

$$b(n) = \sum_{k=0}^{n} w_{\alpha,0}(n,k)a(k).$$

Theorem 3. *The α-generalized Whitney transform of the first kind of a linear homogeneous recurrent integer sequence*

$$a(n) = \sum_{i=1}^{q} c_i\lambda_i^n,\ n = 0,1,2,\ldots,$$

is an integer sequence with the general term

$$b(n) = \sum_{i=1}^{q} c_i(\lambda_i|\alpha)_n, \ n = 0, 1, 2, \ldots,$$

where λ_i, $(i = 1, \ldots, q)$ *are* q *real or complex characteristic values of* $a(n)$.

Proof. Let $\mathfrak{a}$ and $\mathfrak{b}$ denote the vectors corresponding to $a(n)$ and $b(n)$; that is,

$$\mathfrak{a} = (a(0), a(1), \ldots, a(n), \ldots)^T \text{ and } \mathfrak{b} = (b(0), b(1), \ldots, b(n), \ldots)^T.$$

Then, we have

$$\mathfrak{a} = \sum_{i=1}^{q} c_i \mathcal{V}_0(\lambda_i).$$

By Definition 3, we get the equality

$$\mathfrak{b} = \mathcal{A}_{\alpha,0}\mathfrak{a}. \tag{9}$$

On the other hand, taking $\alpha = m, r = 0$ and $\lambda = mx$, Definition 3 gives us

$$\mathcal{V}_\alpha(\lambda) = \mathcal{A}_{\alpha,0}\mathcal{V}_0(\lambda).$$

Thus, we have

$$\begin{aligned}
\mathfrak{b} = \mathcal{A}_{\alpha,0}\mathfrak{a} = \mathcal{A}_{\alpha,0}\sum_{i=1}^{q} c_i \mathcal{V}_0(\lambda_i) \\
&= \sum_{i=1}^{q} c_i \mathcal{A}_{\alpha,0}\mathcal{V}_0(\lambda_i) \\
&= \sum_{i=1}^{q} c_i \mathcal{V}_\alpha(\lambda_i).
\end{aligned}$$

This implies that

$$b(n) = \sum_{i=1}^{q} c_i(\lambda_i|\alpha)_n, \ (n = 0, 1, 2, \ldots).$$

Since α is an integer and each entry of $\mathcal{A}_{\alpha,0}$ is also an integer, it implies that $b(n)$ is an integer sequence.

Example 3. The general term of the recurrent integer sequence of order 2 of Lucas numbers $L(n) = 2, 1, 3, 4, 7, 11, 18, \ldots$ is

$$L(n) = c_1\lambda_1^n + c_2\lambda_2^n = \left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)^n + \left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)^n.$$

Hence, we find that the general term of its Whitney transform of the first kind is

$$L_\alpha(n) = \left(\frac{1}{2} + \frac{\sqrt{5}}{2}\middle|\alpha\right)_n + \left(\frac{1}{2} - \frac{\sqrt{5}}{2}\middle|\alpha\right)_n, \quad \text{for } n = 0, 1, 2, \ldots.$$

In particular, when $\alpha = 1$, we have

$$L_1(n) = \left(\frac{1}{2} + \frac{\sqrt{5}}{2}\middle|1\right)_n + \left(\frac{1}{2} - \frac{\sqrt{5}}{2}\middle|1\right)_n, \ n = 0, 1, 2, \ldots,$$

that is, $L_1(n) = 2, 1, 2, -3, 10, -45, 250, -1645, \ldots$.

3.6. Properties of the Generalized Factorial Matrices

The following theorem gives some basic properties of the generalized factorial matrices.

Theorem 4. *Let $y_1, y_2, \ldots, y_p$ be p distinct real or complex numbers. Then*

i. Two generalized factorial matrices can be connected with suitable r-Whitney matrix transforms.

$$\mathcal{U}_m(y_1, y_2, \ldots, y_p) = \mathcal{A}_{m,r}\ \mathcal{U}_0(y_1 + r, y_2 + r, \ldots, y_p + r) \tag{10}$$

and

$$\mathcal{U}_0(y_1 + r, y_2 + r, \ldots, y_p + r) = \mathcal{B}_{m,r}\mathcal{U}_m(y_1, y_2, \ldots, y_p); \tag{11}$$

ii. Determinant of $\mathcal{U}_m(y_1, y_2, \ldots, y_p)$ is

$$\det\ \mathcal{U}_m(y_1, y_2, \ldots, y_p) = \det\ \mathcal{U}_0(y_1, y_2, \ldots, y_p) = \prod_{1\le i<j\le p}(y_j - y_i); \ \textit{and} \tag{12}$$

iii. The inverse of $\mathcal{U}_m(y_1, y_2, \ldots, y_p)$ is

$$\mathcal{U}_m^{-1}(y_1, y_2, \ldots, y_p) = \mathcal{U}_0^{-1}(y_1, y_2, \ldots, y_p)\mathcal{B}_{m,0}, \tag{13}$$

namely, the inverse of the Vandermonde matrix, right multiplied by the r-Whitney matrix of the second kind.

Proof.

(i) From the right-hand side of equality (10), we have

$$\begin{aligned}
\mathcal{A}_{m,r}\mathcal{U}_0(y_1 + r, \ldots, y_p + r) &= \mathcal{A}_{m,r}\left[\mathcal{V}_0(y_1 + r), \ldots, \mathcal{V}_0(y_p + r)\right] \\
&= \left[\mathcal{A}_{m,r}\mathcal{V}_0(y_1 + r), \ldots, \mathcal{A}_{m,r}\mathcal{V}_0(y_p + r)\right] \\
&= \left[\mathcal{V}_m(y_1), \ldots, \mathcal{V}_m(y_p)\right] \\
&= \mathcal{U}_m(y_1, \ldots, y_p).
\end{aligned}$$

Thus, equality (10) holds.

Now, multiplying $\mathcal{B}_{m,r}$ to equation (10) yields

$$\mathcal{B}_{m,r}\mathcal{U}_m(y_1, y_2, \ldots, y_p) = \mathcal{B}_{m,r}\mathcal{A}_{m,r}\mathcal{U}_0(y_1 + ry_2 + r, \ldots, y_p + r).$$

Since the product of $\mathcal{B}_{m,r}\mathcal{A}_{m,r}$ is an infinite dimensional identity matrix $\mathcal{E}$, we have

$$\mathcal{B}_{m,r}\mathcal{U}_m(y_1, y_2, \ldots, y_p) = \mathcal{E}\ \mathcal{U}_0(y_1 + ry_2 + r, \ldots, y_p + r),$$

implying that

$$\mathcal{B}_{m,r}\mathcal{U}_m(y_1, y_2, \ldots, y_p) = \mathcal{U}_0(y_1 + ry_2 + r, \ldots, y_p + r).$$

Thus, equality (11) holds.

(ii) It is noted that the determinants of equation (10) is given by

$$\det\ \mathcal{U}_m(y_1, y_2, \ldots, y_p) = \det\ \mathcal{A}_{m,r} \det\ \mathcal{U}_0(y_1 + r, y_2 + r, \ldots, y_p + r). \tag{14}$$

Since $\mathcal{A}_{m,r}$ is a lower triangular matrix with 1's on the main diagonal and the matrix $\mathcal{U}_0(y_1, y_2, \ldots, y_p)$ a Vandermonde matrix, we have the following:

$$\det\ \mathcal{A}_{m,0} = 1,$$
$$\det\ \mathcal{U}_0(y_1, y_2, \ldots, y_p) = \prod_{1\le i<j\le p} (y_j - y_i).$$

This implies that equation (14) can further be equate to

$$\begin{aligned}\det\ \mathcal{U}_m(y_1, y_2, \ldots, y_p) &= \det\ \mathcal{A}_{m,r}\ \det\ \mathcal{U}_0(y_1 + r, y_2 + r, \ldots, y_p + r) \\ &= 1 \cdot \prod_{1\le i<j\le p} (y_j - y_i) \\ &= \prod_{1\le i<j\le p} (y_j - y_i).\end{aligned}$$

Hence, equality (12) holds.

(iii) From (ii), since $\det\ \mathcal{U}_m(y_1, y_2, \ldots, y_p) \neq 0$, matrix $\mathcal{U}_m(y_1, y_2, \ldots, y_p)$ is invertible. Hence, we obtain

$$\mathcal{U}_m^{-1}(y_1, y_2, \ldots, y_p) = \mathcal{U}_0^{-1}(y_1 + r, y_2 + r, \ldots, y_p + r)\ \mathcal{A}_{m,r}^{-1}. \tag{15}$$

We see from Remark 1 that $\mathcal{A}_{m,r}^{-1} = \mathcal{B}_{m,r}$. Hence,

$$\mathcal{U}_m^{-1}(y_1, y_2, \ldots, y_p) = \mathcal{U}_0^{-1}(y_1 + r, y_2 + r, \ldots, y_p + r)\ \mathcal{B}_{m,0}. \tag{16}$$

Thus, equality (13) holds.

Example 4. The 4×4 matrices $\mathcal{U}_m(0,1,2,3)$ $(m = 1,2,3,4)$ are, respectively,

$$\mathcal{U}_1(0,1,2,3) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 6 \end{bmatrix};$$

$$\mathcal{U}_2(0,1,2,3) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & 0 & 3 \\ 0 & 3 & 0 & -3 \end{bmatrix};$$

$$\mathcal{U}_3(0,1,2,3) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & -2 & -2 & 0 \\ 0 & 10 & 8 & 0 \end{bmatrix};\ \text{and}$$

$$\mathcal{U}_4(0,1,2,3) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & -3 & -4 & -3 \\ 0 & 21 & 24 & 15 \end{bmatrix}.$$

If $r = 1$, then we see that $\mathcal{U}_m(0,1,2,3)$ $(m = 1,2,3,4)$ may be factored as follows:

$$\mathcal{U}_1(0,1,2,3) = \mathcal{A}_{1,1}\,\mathcal{U}_0(1,2,3,4);\ \text{that is,}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ -6 & -11 & -6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{bmatrix},$$

$$\mathcal{U}_2(0,1,2,3) = \mathcal{A}_{2,1}\,\mathcal{U}_0(1,2,3,4);\ \text{that is,}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & 0 & 3 \\ 0 & 3 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & -4 & 1 & 0 \\ -15 & 23 & -9 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{bmatrix},$$

$$\mathcal{U}_3(0,1,2,3) = \mathcal{A}_{3,1}\,\mathcal{U}_0(1,2,3,4);\ \text{that is,}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & -2 & -2 & 0 \\ 0 & 10 & 8 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 4 & -5 & 1 & 0 \\ -28 & 39 & -12 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{bmatrix},$$

$$\mathcal{U}_4(0,1,2,3) = \mathcal{A}_{4,1}\,\mathcal{U}_0(1,2,3,4);\ \text{that is,}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & -3 & -4 & -3 \\ 0 & 21 & 24 & 15 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 5 & -6 & 1 & 0 \\ -45 & 59 & -15 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{bmatrix}.$$

Now, we have

$$\begin{aligned}\det\ \mathcal{U}_m(0,1,2,3) &= \prod_{1\le i<j\le 4}(x_j - x_i)\\ &= (1-0)(2-1)(2-0)(3-2)(3-1)(3-0)\\ &= 12,\ \text{for } m = 1,2,3,4.\end{aligned}$$

Next, we see that a Vandermonde matrix $\mathcal{U}_0(0,1,2,3)$ is

$$\mathcal{U}_0(0,1,2,3) = \begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 2 & 3\\ 0 & 1 & 4 & 9\\ 0 & 1 & 8 & 27\end{bmatrix},$$

and its inverse $\mathcal{U}_0^{-1}(0,1,2,3)$ is given by

$$\mathcal{U}_0^{-1}(0,1,2,3) = \begin{bmatrix}1 & -\frac{11}{6} & 1 & -\frac{1}{6}\\ 0 & 3 & -\frac{5}{2} & \frac{1}{2}\\ 0 & -\frac{3}{2} & 2 & -\frac{1}{2}\\ 0 & \frac{1}{3} & -\frac{1}{2} & \frac{1}{6}\end{bmatrix}.$$

Hence, by equation (13), we have the following decomposition formulas:

$$\begin{aligned}\mathcal{U}_1^{-1}(0,1,2,3) &= \mathcal{U}_0^{-1}(0,1,2,3)\mathcal{B}_{1,0};\\ \mathcal{U}_2^{-1}(0,1,2,3) &= \mathcal{U}_0^{-1}(0,1,2,3)\mathcal{B}_{2,0};\\ \mathcal{U}_3^{-1}(0,1,2,3) &= \mathcal{U}_0^{-1}(0,1,2,3)\mathcal{B}_{3,0};\\ \mathcal{U}_4^{-1}(0,1,2,3) &= \mathcal{U}_0^{-1}(0,1,2,3)\mathcal{B}_{4,0}.\end{aligned}$$

Corollary 1. *Let $\mathcal{U}_m(x_1, x_2, \dots, x_p)$ be a generalized factorial matrix of p distinct real or complex numbers $x_1, x_2, \dots, x_p$ and let $y_i = mx_i$ $(i = 1, \dots, p)$. Then we may express the inverse of matrix $\mathcal{U}_m(x_1, x_2, \dots, x_p)$ as*

$$\mathcal{U}_m^{-1}(x_1, x_2, \dots, x_p) = \mathcal{H}_p\mathcal{L}_p\mathcal{B}_{m,0},$$

where factor matrix $\mathcal{L}_p$ is a lower triangular matrix, the arbitrary entry $L_p(n,k)$ $(n,k = 0,1,2,\dots,p-1)$ of which is determined by the following relations

$$L_p(0,0) = 1 \text{ and } \prod_{i=1}^{n}(x - x_i) = \sum_{k=0}^{n} L_p(n,k)x^k, \tag{17}$$

and the matrix factor $\mathcal{H}_p$ is an upper triangular matrix, the arbitrary entry $h_p(n,k)$ $(n,k = 0,1,\dots,p-1)$ of which is given by

$$h_p(n,k) = \begin{cases}1, & \textit{if } n = k = 0\\ 0, & \textit{if } k < n\\ \left(\prod_{i=1, i\neq n+1}^{k+1}(x_{n+1} - x_i)\right)^{-1} & \textit{if } k \ge n,\ n,k \neq 0.\end{cases} \tag{18}$$

Proof. By using a triangular-matrix decomposition method given by Hou and Hou [14], we may obtain

$$\mathcal{U}_0^{-1}(x_1, x_2, \ldots, x_p) = \mathcal{H}_p\mathcal{L}_p,$$

where the lower triangular matrix $\mathcal{L}_p$ and the upper triangular matrix $\mathcal{H}_p$ are calculated by using equations (17) and (18). Hence, we see from equation (13) that

$$\mathcal{U}_m^{-1}(x_1, x_2, \ldots, x_p) = \mathcal{H}_p\mathcal{L}_p\mathcal{B}_{m,0}.$$

Thus, the corollary holds.

Example 5. By Corollary 1, the inverse of $\mathcal{U}_m(0, 1, 2, 3)$ $(m = 1, 2, 3, 4)$ in Example 4 may be expressed as follows:

$$\begin{aligned}
\mathcal{U}_1^{-1}(0, 1, 2, 3) &= \mathcal{H}_4\mathcal{L}_4\mathcal{B}_{1,0};\\
\mathcal{U}_2^{-1}(0, 1, 2, 3) &= \mathcal{H}_4\mathcal{L}_4\mathcal{B}_{2,0};\\
\mathcal{U}_3^{-1}(0, 1, 2, 3) &= \mathcal{H}_4\mathcal{L}_4\mathcal{B}_{3,0};\\
\mathcal{U}_4^{-1}(0, 1, 2, 3) &= \mathcal{H}_4\mathcal{L}_4\mathcal{B}_{4,0}.
\end{aligned}$$

Using equations (17) and (18) for $\mathcal{U}_0(y_1, y_2, \ldots, y_p)$, we obtain

$$\mathcal{H}_4 = \begin{bmatrix} 1 & -1 & \frac{1}{2} & -\frac{1}{6} \\ 0 & 1 & -1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{6} \end{bmatrix} \quad \text{and} \quad \mathcal{L}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & -3 & 1 \end{bmatrix}.$$

Thus, we have

$$\begin{aligned}
\mathcal{U}_1^{-1}(0, 1, 2, 3) = \mathcal{H}_4\mathcal{L}_4\mathcal{B}_{1,0} &= \begin{bmatrix} 1 & -1 & \frac{1}{2} & -\frac{1}{6} \\ 0 & 1 & -1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{6} \end{bmatrix};\\
\mathcal{U}_2^{-1}(0, 1, 2, 3) = \mathcal{H}_4\mathcal{L}_4\mathcal{B}_{2,0} &= \begin{bmatrix} 1 & -\frac{1}{2} & 0 & -\frac{1}{6} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{6} \end{bmatrix};\\
\mathcal{U}_3^{-1}(0, 1, 2, 3) = \mathcal{H}_4\mathcal{L}_4\mathcal{B}_{3,0} &= \begin{bmatrix} 1 & -\frac{1}{3} & -\frac{1}{2} & -\frac{1}{6} \\ 0 & 0 & 2 & \frac{1}{2} \\ 0 & 0 & -\frac{5}{2} & -\frac{1}{2} \\ 0 & \frac{1}{3} & 1 & \frac{1}{6} \end{bmatrix}; \text{ and}\\
\mathcal{U}_4^{-1}(0, 1, 2, 3) = \mathcal{H}_4\mathcal{L}_4\mathcal{B}_{4,0} &= \begin{bmatrix} 1 & -\frac{1}{2} & -1 & -\frac{1}{6} \\ 0 & 1 & \frac{7}{2} & \frac{1}{2} \\ 0 & -\frac{3}{2} & -4 & -\frac{1}{2} \\ 0 & 1 & \frac{3}{2} & \frac{1}{6} \end{bmatrix}.
\end{aligned}$$

The next theorem gives another matrix decomposition formula using some convolution-type identity for the r-Whitney numbers of the second kind.

Theorem 5. *Let* $\mathcal{W} = [c_{pj}]$ *where* $c_{pj} = W_{m,r}(p + j, n)$*. Then*

$$\mathcal{W} = \mathbb{AB}, \tag{19}$$

where $\mathbb{A} = [a_{pk}]_{pk}$ *and* $\mathbb{B} = [b_{kj}]_{kj}$ *with*

$$a_{pk} = W_{m,r}(p, k) \quad \text{and} \quad b_{kj} = W_{m,km+r}(j, n - k).$$

Proof. Note that the pj-th entry in the product $\mathbb{AB}$ is the linear combination of the entries c_{ij} in the pth row of A and the entries in the jth column of B.

Thus,

$$c_{pj} = \sum_{k=p}^{n-j} W_{m,r}(m, k)W_{m,km+r}(j, n - k).$$

By the convolution-type identity of the r-Whitney numbers of the second kind, we have

$$c_{pj} = W_{m,r}(p + j, n).$$

Thus,

$$\mathbb{AB} = \mathcal{W}. \tag{20}$$

4. Conclusions

This study introduced a matrix decomposition method for computing r-Whitney numbers of the first and second kind in an explicit, non-recursive manner. By expressing generalized factorial matrices through the lens of r-Whitney transforms, we established connections to Vandermonde-type structures and derived key properties including determinant identities and inverse matrix factorizations. These results contribute to the algebraic and combinatorial understanding of Whitney-type transforms and offer computational tools for further exploration.

Given the growing interest in q-analogues of classical combinatorial numbers, a natural extension of this work is to investigate q-analogue versions of the r-Whitney numbers and their associated matrix transforms. Several studies have laid the groundwork for such generalizations, including q-analogues of Stirling numbers, Bell numbers, and Dowling numbers (see references [13, 15–17]). In particular, the Hankel transforms of (q, r)-Dowling and (q, r)-Whitney-Lah numbers (see references [18–20]), as well as q-noncentral Bell numbers [21], suggest rich algebraic structures that may parallel the decomposition techniques presented here.

Future research may focus on developing q-analogue matrix transforms for Whitney numbers and examining their decomposition properties, while also investigating orthogonality and convolution identities within the q-analogue framework. Another promising avenue is the extension of this framework to encompass broader combinatorial families, including q-Lah numbers and q-Fubini–Vogt numbers. In addition, these transforms could be applied to a range of problems in combinatorics, special functions, and symbolic computation, thereby deepening both theoretical understanding and practical applications in these areas.

Such directions promise to deepen the interplay between algebraic matrix theory and combinatorial enumeration, potentially revealing new identities and computational techniques across discrete mathematics.

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