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Dirac Operators Their Eigenfunctions and Heat Kernels for Two Geometries

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Abstract. The heat kernel of the Dirac operator is obtained with respect to a particular geometry. This is done by calculating eigenfunctions of the Dirac operator on both spheres and hyperbolic spaces of arbitrary dimensions. They are computed by separating variables in geodesic polar coordinates. These eigenfunctions are used to derive the heat kernel of the Dirac operator on these spaces. The kernels can be studied as cross sections of homogeneous vector bundles.

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1. Introduction

The Dirac operator is of great importance across the area of mathematical physics [1], [2], [3]. The operator and its eigenfunctions, for example, have applications to the study of de Sitter and anti-de Sitter spacetimes [4], [5]. The sphere S^4 and the hyperbolic space H^4 are Euclidean sections of these space-times. The spaces S^3 and H^3 appear as the spatial sections of cosmological models as well. The intention is to examine the Dirac operator and its eigenfunctions on the N-dimensional sphere, S^N , and real hyperbolic spaces, H^N, respectively. These are dual to each other as symmetric spaces and are maximally symmetric [6], [7].

Dirac operators are first-order elliptic operators on Riemannian spin manifolds. They are fundamental and they contain crucial topological and geometrical information[8]. They can be used in geomtries such as Riemannian geometry, in index theory, noncommutative geometries and others that will not arise here [9]. Compared to Laplacian methods, they provide high sensitivity to surface features and curvature. The square of the Dirac operator is an operator of Laplace type and the connection ∇ as well also yields the Laplacian $\nabla^*\nabla$ on the manifold.

A spin structure on an oriented n-dimensional Riemannian manifold M^n is a principal bundle over M^n with structure group $\mathrm{Spin}(M^n)$, referred to as the bundle of spinor forms

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[10]. It has a two-to-one bundle homomorphism of Spin(M^n) onto $SO(M^n)$ associated to it which preserves fibers and is group equivariant with respect to the standard two-to-one covering homomorphism. There is a unique fundamental spinor representation of Spin(n) in dimension $2^{(n-1)/2}$ for n-odd and $\tau_+ \oplus \tau_-$, where $\tau_\pm$ are the two fundamental spinor representations of dimension $2^{n/2-1}$ each, for n even. A Dirac spinor field on M^n is defined as a cross section of the vector bundle E^τ over M^n with typical fiber $\mathbb{C}^{2\lfloor n/2 \rfloor}$ and structure group Spin(n) associated with Spin(M^n) by way of represention τ, spinor means Dirac spinor. Among the spaces mentioned here, S^N and H^N, only S^1 admits more than one spin structure. There exist two inequivalent spin structures on S^1, and the nontrivial one can be chosen [11], [12].

To carry out calculations, a spinor field and its covariant derivative have to be expressed with respect to a local moving frame $\{e_b\}_1^n$ on M^n in components [6−10]. A spin connection is induced by a metric connection ω on $Spin(M^n)$ and a covariant derivative for the cross-sections. Let ψ be a spinor, then the components of the spinor $\nabla_{e_a} \psi$ in a closed spinor frame $\{\theta_A\}_{A=1,\cdots,2^{\lfloor n/2 \rfloor}}$ are a set of linearly independent local cross sections of E^τ,

$$(\nabla_{e_b} \psi)^A = e_b(\psi^A) - \frac{1}{2}\omega_{abc}(\Sigma^{bc})^A_B \, \psi^B \tag{1.1}$$

The ω_{abc} in (1.1) are the connection coefficients in the frame specified by e_b,

$$\nabla_{e_a} e_b = \omega_{abc} \, e_c. \tag{1.2}$$

The Σ^{bc} are the generators of the representation τ of $Spin(n)$ and are expressed as follows

$$\Sigma^{bc} = \frac{1}{4}[\Gamma^b, \Gamma^c], \tag{1.3}$$

where $\Gamma^a = (\Gamma^a)^A_B$, $a = 1, \ldots, n$; $A, B = 1, \ldots, 2^{\lfloor n/2 \rfloor}$ are called Dirac matrices which obey

$$\Gamma^a \Gamma^b + \Gamma^b \Gamma^a = 2\,\delta^{ab}\,\mathbf{1}, \qquad a, b = 1, \ldots, n. \tag{1.4}$$

where $\mathbf{1}$ is the unit matrix. The Dirac operator is defined to be

$$\not\nabla \psi = \Gamma^a \, \nabla_{e_a} \psi. \tag{1.5}$$

2. Operators on the N-Sphere

In polar coordinates, the N-sphere S^N has the metric

$$ds_N^2 = d\theta^2 + f^2(\theta)\, ds_{N-1}^2 = d\theta^2 + f^2(\theta) \sum_{i,j=1}^{N-1} \bar{g}_{ij} \, d\omega^i \otimes d\omega^j. \tag{2.1}$$

In (2.1), θ is the geodesic distance from the origin, $f(\theta) = \sin\theta$ and $\{\omega^i\}$ coordinates on S^{N-1} with metric tensor $\bar{g}_{ij}(\omega) = \langle \partial/\partial\omega^i, \partial/\partial\omega^j \rangle$. Let $\{\bar{e}_j\}$ be a set of vector fields on S^{N-1} with anholonomy and Levi-Civita connection coefficients given as

$$[\bar{e}_i, \bar{e}_j] = \sum_{k=1}^{N-1} \bar{C}_{ijk} \, \bar{e}_k, \qquad \bar{\omega}_{ijk} = \langle \nabla_{\bar{e}_i} \bar{e}_j, \bar{e}_k \rangle - \frac{1}{2}(\bar{C}_{ijk} - \bar{C}_{ikj} - \bar{C}_{jki}). \tag{2.2}$$

It is best to work with geodesic polar coordinate vector fields $\{e_a\}_{a=1}^N$ on S^N defined by

$$e_j = \frac{\bar{e}_j}{f(\theta)}, \qquad i = 1, \dots, N-1, \qquad e_N = \frac{\partial}{\partial \theta} = \partial_\theta. \tag{2.3}$$

The only nonvanishing components of the Levi-Civita connection ω_{abc} in the $\{e_a\}$ frame are

$$\omega_{ijk} = \frac{1}{f}\,\bar{\omega}_{ijk}, \qquad \omega_{iNk} = -\omega_{ikN} = \frac{f'}{f}\,\delta_{ik}, \quad i, j, k = 1, \dots, N-1. \tag{2.4}$$

In (2.4), f' denotes the derivative of f. The spin connection on S^N induced uniquely by the Levi-Civita connection ω. Let us solve for the eigenfuctions of $\nabla\!\!\!/$ on S^N by separating variables in geodesic polar coordinates.

1. Suppose N is *even*. The Γ matrices are selected such that

$$\Gamma^N = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}. \qquad \Gamma^i = \begin{pmatrix} 0 & i\bar{\Gamma}^j \\ -i\bar{\Gamma}^j & 0 \end{pmatrix}, \quad j = 1, \dots, N-1, \tag{2.5}$$

where $\{\tilde{\Gamma}^j, \tilde{\Gamma}^k\} = 2\,\delta^{jk}\mathbf{1}$. Using (2.4) and (1.5), the Dirac operator can be expressed in the moving frame

$$\begin{aligned} \nabla\!\!\!/\,\psi &= \Gamma^N \left(e_N \psi^A - \frac{1}{2}\omega_{Nst}(\Sigma^{st})^A_B\,\psi^B\right) + \Gamma^b\left(e_b\psi^A - \frac{1}{2}\omega_{bst}(\Sigma^{st})^A_B\,\psi^B\right) \\ &= \Gamma^N\Big(\frac{\partial}{\partial\theta}\psi^A - \frac{f'}{2f}\delta_{st}(\Sigma^{st})^A_B\psi^B\Big) + \begin{pmatrix} 0 & i\bar{\Gamma}^j \\ -i\bar{\Gamma}^j & 0 \end{pmatrix}\Big(\frac{1}{f}\bar{e}_j\,\psi^A - \frac{1}{f}\bar{\omega}_{jst}(\Sigma^{st})^A_B\,\psi^B\Big). \\ &= (\partial_\theta + \frac{1}{2}\,(N-1)\frac{f'}{f})\,\Gamma^N\psi + \frac{1}{f}\begin{pmatrix} 0 & i\bar{\nabla}_b \\ -i\Gamma^b\bar{\nabla}_b & 0 \end{pmatrix}\psi \\ &= (\partial_\theta + \frac{1}{2}(N-1)\frac{f'}{f})\Gamma^N\,\psi + \frac{1}{f}\begin{pmatrix} 0 & i\,\bar{\nabla}\!\!\!/ \\ -i\,\bar{\nabla}\!\!\!/ & 0 \end{pmatrix}\psi, \end{aligned} \tag{2.6}$$

where $\bar{\nabla}\!\!\!/ = \bar{\Gamma}^j\,\bar{\nabla}_{e_j}$ is the Dirac operator on S^{N-1}.

Suppose that the eigenvalue equation has been solved on S^{N-1} so

$$\bar{\nabla}\!\!\!/\,\chi^{(\pm)}_{lm} = \pm i(l+\rho)\,\chi^{(\pm)}_{lm}. \tag{2.7}$$

The index $l = 0, 1, \dots,$ indicates the eigenvalues of the Dirac operator on the $(N-1)$-sphere, and $\rho = (N-1)/2$, while the index m runs from 1 to the degeneracy d_i. Define the spinor

$$\psi = \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix}. \tag{2.8}$$

Since the dimension of χ is the same as the dimension of $\phi_\pm$, that is, $2^{\lfloor N/2 \rfloor - 1}$, one can separate variables in the following way

$$^{(1)}\phi_{+nlm}(\theta,\Omega) = \phi_{nl}(\theta)\,\chi^{(-)}_{lm}(\Omega), \qquad ^{(2)}\phi_{+nlm}(\theta,\Omega) = \psi_{nl}(\theta)\chi^{(+)}_{lm}(\Omega), \tag{2.9}$$

and similarly for the lower spinors ϕ_-. Here $\Omega \in S^{N-1}$, and $n = 0, 1, \dots$ labels the eigenvalues $-\lambda_{n,N}^2$ of $\not{\nabla}^2$ on S^N and $n \geq D$. Subjecting ψ in (2.8) to the operator $\not{\nabla}$, we obtain

$$\not{\nabla} \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} = (\frac{\partial}{\partial \theta} + \rho \cot \theta) \, \Gamma^N \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} + \frac{i}{\sin \theta} \begin{pmatrix} \bar{\nabla} \phi_- \\ -\bar{\nabla} \phi_+ \end{pmatrix} = (\frac{\partial}{\partial \theta} + \rho \cot \theta) \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} + \frac{l+\rho}{\sin \theta} \begin{pmatrix} \phi_- \\ \phi_+ \end{pmatrix}.$$

and

$$\not{\nabla}^2 \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} = (\frac{\partial}{\partial \theta} + \rho \cot \theta)^2 \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} + \frac{i}{\sin \theta} \begin{pmatrix} 0 & \bar{\not{\nabla}} \\ -\bar{\not{\nabla}} & 0 \end{pmatrix} (\frac{\partial}{\partial \theta} + \rho \cot \theta) \begin{pmatrix} \phi_= \\ \phi_+ \end{pmatrix}$$
$$+ \frac{l+\rho}{\sin \theta} (\frac{\partial}{\partial \theta} + \rho \cot \theta) \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} + i \frac{l+\rho}{\sin^2 \theta} \begin{pmatrix} 0 & \tilde{\not{\nabla}} \\ -\tilde{\not{\nabla}} & 0 \end{pmatrix} \begin{pmatrix} \phi_- \\ \phi_+ \end{pmatrix} \tag{2.10}$$

Since

$$\begin{pmatrix} 0 & -i\bar{\not{\nabla}} \\ -i\bar{\not{\nabla}} & 0 \end{pmatrix} \begin{pmatrix} \phi_- \\ \phi_+ \end{pmatrix} = -(l+\rho) \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix},$$

the second derivative becomes

$$(\frac{\partial}{\partial \theta} + \rho \cot \theta)^2 \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} - \frac{l+\rho}{\sin \theta} (\frac{\partial}{\partial \theta} - \rho \cot \theta) \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} + \frac{l+\rho}{\sin \theta} (\frac{\partial}{\partial \theta} + \rho \cot \theta)) \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} - \frac{(l+\rho)^2}{\sin^2 \theta} \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix}$$
$$= [(\frac{\partial}{\partial \theta} + \rho \cot \theta)^2 + \rho(l+\rho) \frac{\cos \theta}{\sin^2 \theta} - \frac{(l+\rho)^2}{\sin^2 \theta}] \psi. \tag{2.11}$$

The equation $\not{\nabla}^2 \psi = -\lambda_{n,N}^2 \psi$ gives the following equation for the scalar functions ϕ_{nl}

$$[(\frac{\partial}{\partial \theta} + \rho \cot \theta)^2 + \rho(l+\rho) \frac{\cos \theta}{\sin^2 \theta} - \frac{(l+\rho)^2}{\sin^2 \theta}] \phi_{nl} = -\lambda_{n,N} \phi_{n,l}. \tag{2.12}$$

The unique regular solution to this equation is given in terms of Jacobi polynomials $P_n^{(a,b)}(x)$ up to a normalization constant,

$$\phi_{nl}(\theta) = (\cos \frac{\theta}{2})^{l+1} (\sin \frac{\theta}{2})^l \, P_{n-l}^{(N/2+l-1, N/2+l)}(\cos \theta), \tag{2.13}$$

with $n - l \geq 0$. This condition is needed for the regularity of the eigenfunctions with the eigenvalues

$$\lambda_{n,N}^2 = (n + \frac{N}{2})^2. \tag{2.14}$$

By proceeding similarly with the ψ_{nl} we find

$$\psi_{nl}(\theta) = (\cos \frac{\theta}{2})^l (\sin \frac{\theta}{2})^{n+1} \, P_{n-l}^{(N/2+1, N/2+l-1)}(\cos \theta) = (-1)^{n-l} \, \phi_{nl}(\pi - \theta). \tag{2.15}$$

Define the spinors,

$$\psi_{\pm nlm}(\theta, \Omega) = \frac{c_N(nl)}{\sqrt{2}} \begin{pmatrix} \phi_{nl}(\theta) \chi_{lm}^{(-)}(\Omega) \\ \pm i \psi_{nl}(\theta) \chi_{lm}^{(-)}(\Omega) \end{pmatrix}, \quad \psi_{\pm nlm}^{(\pm)}(\theta, \Omega) = \frac{c_N(nl)}{\sqrt{2}} \begin{pmatrix} i \psi_{nl}(\theta) \chi_{lm}^{(+)}(\Omega) \\ \pm \phi_{nl}(\theta) \chi_{lm}^{(+)}(\Omega) \end{pmatrix} \tag{2.16}$$

Then it is found that these spinors satisfy a first order Dirac equation

$$\not{\nabla}\, \psi^{(s)}_{\pm nlm} = \pm i(n + \frac{N}{2})\, \psi^{(s)}_{\pm nlm}, \quad s = \pm. \tag{2.17}$$

It is required that these functions satisfy a normalization condition,

$$\langle \psi^{(s)}_{+nlm}, \psi^{(s'}_{+n'l'm'} \rangle = \int_{S^N} d\Omega_N\, \psi^{(s)}_{nlm}(\theta,\Omega)^{\dagger}\, \psi^{(s')}_{n'l'm'}(\theta,\Omega) = \delta_{nn'}\delta_{ll'}\delta_{mm'}\delta_{ss'}. \tag{2.18}$$

There is an analogous relation for $\psi^{(s)}_{+} \to \psi^{(s)}_{-}$. Note that $\langle \psi^{(s)}_{nlm}, \psi^{(s')}_{-nlm} \rangle = 0$ for any choice of indices. Suppose that the spinors $\chi^{(\pm)}_{lm}(\Omega)$ are normalized by

$$\int_{S^{N-1}} d\Omega_{N-1}\, \chi^{(s)}_{lm}(\Omega)^{\dagger} \chi^{(s')}_{l'm'}(\Omega) = \delta_{ll'}\delta_{mm'}\delta_{ss'}.$$

The normalization factor required to normalize this can be calculated,

$$|c_N(nl)|^{-2} = \frac{2^{N-2}|\Gamma(\frac{N}{2}+n)|}{(n-l)!(N+n+l-1)!}. \tag{2.19}$$

The degenseracy for the eigenvalues $+i(n+N/2)$ or $-i(n+N/2)$ is given by letting $n = n'$, $m = m'$, $s = s'$ in (2.18) and summing over l, m, s

$$D_N(n) = \int_{S^N} d\Omega_N \sum_{lms} \psi^{(s)}_{+nlm}(\theta,\Omega)^{\dagger}\, \psi^{(s)}_{nlm}(\theta,\Omega).$$

The sum over l, m, s inside this integral is constant over S^N, so that the sum may be calculated for $\theta \to 0$ where only the $l = 0$ term survives. The volume of S^N is given by

$$\Omega_N = \frac{2\pi^{(N+1)/2}}{\Gamma(\frac{1}{2}(N+1))}.$$

The degeneracy d_0 of $+i\rho$ or $-i\rho$ on S^{N-1} coincides with the dimension of $2^{(N/2)-1}$ of the fundamental spinor representations $\tau_{\pm}$ of Spin(N). Thus $m = 1, 2, \dots, 2^{(N/2)-1}$ for $l = 0$ in $D_N(n)$ and there follows the identity

$$\Omega_{N-1} \sum_m \chi^{(s)}_{0m}(\Omega)^{\dagger} \chi^{(s)}_{0m}(\Omega) = 2^{(N/2)-1}. \tag{2.20}$$

From this equation as well as $D_N(n)$ and Ω_N, we get

$$D_N(\Omega) = \frac{2^{N/2}(N+n-1)!}{n!(N-1)!}. \tag{2.21}$$

2. Suppose N *odd* $N \geq 3$. In this case, a Dirac spinor on S^N is irreducible under $Spin(N)$ and the dimension of the Γ-matrices is $2^{(N-1)/2}$ which is the same as S^{N-1}. Thus for $a = 1, \dots, N-1$, use the same Γ^a as before and for $a = N$

$$\Gamma^N = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} = (-1)^{(N-1)/2}\, \Gamma^1\Gamma^2 \cdots \Gamma^{N-1}. \tag{2.22}$$

The Dirac operator in geodesic polar coordinate frame (2.7) has the form

$$\nabla\!\!\!/ \psi = (\partial_\theta + \rho \cot\theta)\,\Gamma^N \psi + \frac{1}{\sin\theta}\,\bar{\nabla\!\!\!/}\,\psi. \tag{2.23}$$

Now suppose functions $\chi_{lm}^{(-)}$ satisfy

$$\bar{\nabla\!\!\!/}\,\chi_{lm}^{(-)} = -i(l+\rho)\,\chi_{lm}^{(-)}, \qquad l = 0,1,2,\ldots. \tag{2.24}$$

Then $\chi_{lm}^{(+)} = \Gamma^{(N)}\,\chi_{lm}^{(-)}$ is the eigenfunction of operator $\bar{\nabla\!\!\!/}$ with eigenvalue $+i(l+\rho)$. Define $\hat{\chi}_{lm}^{(\pm)}$ by

$$\hat{\chi}_{lm}^{(-)} = \frac{1}{\sqrt{2}}(\mathbf{1} + i\Gamma^N)\chi_{lm}^{(-)}, \quad \hat{\chi}_{lm}^{(+)} = \Gamma^N \hat{\chi}_{lm}^{(-)}. \tag{2.25}$$

Normalized eigenfunctions of the first-order Dirac operator are determined to be

$$\psi_{\pm nlm}(\theta,\Omega) = \frac{1}{\sqrt{2}} c_N(nl)[\phi_{nl}(\theta)\hat{\chi}_{lm}^{(-)}(\Omega) \pm i\psi_{nl}(\theta)\hat{\chi}_{lm}^{(+)}(\Omega)], \tag{2.26}$$

where,

$$\nabla\!\!\!/\,\psi_{\pm nlm}(\theta,\Omega) = \pm i(n + \frac{N}{2})\,\psi_{\pm nlm}(\theta,\Omega). \tag{2.27}$$

The calculation of dimensionality goes like the case N even, and is

$$D_N(n) = 2^{(N-1)/2}\,\frac{(N+n-1)!}{n!(N-1)!}. \tag{2.28}$$

It has been seen that the Dirac equation admits separation of variables in these coordinates. In fact, an induction procedure can be established by which the spinor modes on S^N can be recursively calculated beginning from spinor modes on S^2. For $N=2$ and with $ds_{N=2}^2 = d\theta^2 + \sin^2\theta\, d\phi^2$, it is $\tilde{\nabla\!\!\!/} = \partial/\partial\phi$. It can be asked which eigenvalues of this operator are permitted. Spinors on S^2 should transform as a double-valued spin-1/2 representation of $SO(2)$ allowed under the rotation of the two-frame. Now the loop defined as $0 \le \phi \le 2\pi$ with θ constant in the bundle of frames over S^2 is homotopic to the 2π rotation at a point for the system frame. The spinor field may change sign when it goes around this loop, so we have $\partial/\partial\phi = \pm i/2, \pm 3i/2, \cdots$. This means using a nontrivial spin structure on S^1.

3. Hyperbolic Space

The real hyperbolic space H^N is the noncompact partner of the N-sphere S^N. The netric on H^N in geodesic polar coordinates takes the form (2.1) with $\theta \to y$ and $f(y) = \mathrm{sech}(y)$, where y is the geodesic distance from the origin. Repeating the same steps as with S^N, one obtains a hypergeometric equation for the spherical modes $\phi_{\lambda l}(y)$. The spectrum of $\nabla\!\!\!/^2$ is given by $-\lambda^2$, where now λ is a real continuous parameter. The final results on

H^N and S^N' are related by analytic continuation in the geodesic distance. More precisely, the Jacobi polynomials are expressed in terms of the hypergeometric functions

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}\, F(n+\alpha+\beta+1, -n, \alpha+1, \frac{1-x}{2}),$$

with respect to the substitutions

$$\theta \to i\, y, \qquad n \to -i\,\lambda - \frac{N}{2},$$

in the spinor modes found already on S^N. This results in the pair,

$$\phi_{\lambda l}(y) = (\cosh\frac{y}{2})^{l+1}\,(\sinh\frac{y}{2})^{l}\, F(\frac{N}{2}+l+i\lambda, \frac{N}{2}+l-i\lambda, \frac{N}{2}+l, -\sinh^2\frac{y}{2}), \tag{3.1}$$

$$\Psi_{\lambda l}(y) = \frac{2\lambda}{N+2l}(\cosh\frac{y}{2})^{l}\,(\sinh\frac{y}{2})^{l+1}\, F(\frac{N}{2}+l+i\lambda, \frac{N}{2}+l-i\lambda, \frac{N}{2}+l+1, -\sinh^2\frac{y}{2}),$$

where $\lambda \in \mathbb{R}^+$ and $l = 0, 1, \ldots$. For N even, one writes

$$\psi^{(-)}_{\pm\lambda lm}(y,\Omega) = \frac{1}{\sqrt{2}}\, c_N(\lambda l) \begin{pmatrix} \phi_{\lambda l}(y)\, \chi^{(-)}_{lm}(\Omega) \\ \pm i\psi_{\lambda l}(y)\chi^{(-)}_{lm}(\Omega) \end{pmatrix}, \tag{3.2}$$

and $\psi^{(-)}_{\pm\lambda m}$ satisy

$$\nabla\!\!\!/\, \psi^{(-)}_{\pm\lambda lm} = \pm i\lambda\, \psi^{(-)}_{\pm\lambda lm}.$$

The modes $\psi^{(+)}_{\pm nlm}$ are found by interchanging $\phi_{\lambda l}(y)$ and $i\psi_{\lambda l}(y)$, and letting $\chi^{(-)}_{lm} \to \chi^{(+)}_{lm}$ on the right-hand side. The continuous spectrum normalization constant $c_N(\lambda l)$ is determined by the condition

$$\int_0^\infty \int_{S^{N-1}} \psi^{(s)}_{\sigma\lambda lm}(y,\Omega)^\dagger \psi^{(s')}_{\sigma'\lambda' l' m'}(y,\Omega)\,(\sinh\, y)^{N-1}\, dy\, d\Omega_{N-1} = \delta(\lambda-\lambda')\delta_{ll'}\delta_{mm'}\delta_{ss'}\delta_{\sigma\sigma'}.$$

It can be found by using the method for scalar and tensor fields as

$$|c_N(\lambda l)|^2 = c_N |c_l(\lambda)|^{-2},$$

where $c_N = 2^{N-2}/\pi$ and

$$c_l(\lambda) = \frac{2^{N-2}}{\sqrt{\pi}}\, \frac{\Gamma(\frac{N}{2}+l)\Gamma(\frac{1}{2}+i\lambda)}{\Gamma(\frac{N}{2}+l+i\lambda)}. \tag{3.3}$$

Define the spectral function $\mu(\lambda)$, which is needed to find the heat kernel

$$\mu(\lambda) = \frac{\Omega_{N-1}}{c_N g(\frac{1}{2})} \sum_{nlm} \psi^{(s)}_{+\lambda lm}(0)^\dagger \psi^{(s)}_{+\lambda lm}(0), \tag{3.4}$$

where again $s = \pm$ and the spin factor $g(1/2) = 2^{N/2}$ is the dimension of the spinor. Then we have

$$\mu(\lambda) = |c_0(\lambda)|^{-2} = \frac{\pi}{2^{2N-4}}\, |\frac{\Gamma(N/2+i\lambda)}{\Gamma(N/2)\,\Gamma(1/2+i\lambda)}|^2. \tag{3.5}$$

For N odd the calculations go almost the same way. The modes are obtained from (2.26) using (3.1).

4. The spinor heat kernel

In terms of a local moving frame $\{e_a\}$ on S^N and a local spinor frame $\{\theta_A\}$, the spinor heat kernel $K(x,x',t)$ on S^N is a $2^{[N/2]} \times 2^{[N/2]}$ matrix and satisfies the heat equation for $\not\nabla^2$

$$(-\frac{\partial}{\partial t} + \not\nabla_x^2)\, K(x,x',t) = 0, \tag{4.1}$$

with the initial condition

$$\lim_{t \to 0} \int_{S^N} K(x,x',t)\, \psi(x')\, dx' = \psi(x). \tag{4.2}$$

Consider the case N *odd*. The heat kernel is given by the mode expansion

$$K(\theta',\Omega',\theta,\Omega,t) = \sum_{nlm} [\psi_{-nlm}(\theta',\Omega') \otimes \psi_{-nlm}(\theta,\Omega)^* + \psi_{nlm}(\theta',\Omega') \otimes \psi_{nlm}(\theta,\Omega)^*]\, e^{-t(n+N/2)^2}, \tag{4.3}$$

where $\psi_{nlm}(\theta,\Omega)$ is given in (2.26). In order to simplify the expression, we let $\Omega = \Omega'$; it is assumed that the two points $x = (t,\Omega)$ and $x' = (\theta',\Omega')$ lie on the same meridean, and then take the limit $\theta' \to 0$. Only the terms with $l = 0$ remain nonzero in this limit. Using (2.26) with $l = 0$, we find that

$$\lim_{\theta' \to 0} K(\theta',\Omega,\theta,\Omega,t) = f_N(\theta,t), \qquad f_N(\theta,t) = \frac{1}{\Omega_N} \sum_{n=0}^{\infty} d_n \phi_n(\theta)\, e^{-(n+N/2)^2 t}. \tag{4.4}$$

In (4.4), the functions $\phi_n(\theta)$ are the $\phi_{n0}(\theta)$ normalized by $\phi_n(0) = 1$,

$$\phi_n(\theta) = \frac{\phi_{n0}(\theta)}{\phi_{n0}(0)} = \frac{n!\Gamma(\frac{N}{2})}{\Gamma(n+\frac{N}{2})} \cos\frac{\theta}{2}\, P_n^{((N/2)-1,N/2)}(\cos\theta). \tag{4.5}$$

In (4.5), d_n accounts for the degeneracies of $\not\nabla^2$ on S^N without the spin factor $2^{[N/2]}$,

$$d_n = \frac{2(n+N-1)!}{n!(N-1)!}.$$

The spinor heat kernel $K(x',x,t)$ for the arbitrary pair of points (x',x) can be written using the invariance property as

$$K(x',x,t) = U(x',x)\, f_N(d(x',x),t). \tag{4.6}$$

and $d(x',x)$ represents the geodesic distance between x and x'. Also $U(x',x)$ is the spinor parallel propagator connecting x to x'. That $U(x',x) = \mathbf{1}$ for x, x' on the same meridian is used.

The case where N is even can be treated the same way. The result is that the heat kernel takes the same form (4.6).

For the non-compact case, by using the continuous mode expansion of the heat kernel and working through this procedure, as in the case S^N, it is found for arbitrary points $x, x' \in H^N$ with arbitrary N, and U the parallel propagator by going through the same procedure,

$$K(x', x, t) = U(x', x)\, h_N(d(x', x), t), \quad h_N(y, t) = \frac{2^{N-3}\Gamma(N/2)}{\pi^{(N/2)+1}} \int_0^\infty \phi_\lambda(y)\, e^{-t\lambda^2} \mu(\lambda)\, d\lambda. \tag{4.7}$$

In (4.7), $\phi_\lambda = \phi_{\lambda 0}$ and $\mu(\lambda)$ is the spinor spectral function, and U is often called the parallel propagator.

5. Derivation of the Heat Kernel Based on Group Theory

The spectrum, the spherical eigenfunctions and the heat kernel of the Dirac operator can be obtained by means of a group-theoretic derivation on S^N and H^N. Basic facts from harmonic analysis for homogeneous vector bundles over compact symmetric spaces U/K are used. The exact nature of the spaces U and K will be specified shortly. The cases S^N and H^N will each be considered.

Spinors on S^N.

The vector bundle on U/K associated with the principal bundle $U(U/K, K)$ by a representation τ of K coincides with the homogeneous vector bundle E^τ on U/K defined by τ. Therefore Dirac spinor fields on $S^N = \mathrm{Spin}(N+1)/\mathrm{Spin}(N)$ are considered as cross sections of the homogeneous vector bundle E^τ, where if N is odd, τ is the unique fundamental representation of $\mathrm{Spin}(N)$ of dimension $2^{(N-1)/2}$, and N even $\tau = \tau_+ \oplus \tau_-$. The $\tau_\pm$ are the two fundamental spinor representations of $Spin(N)$ of dimension $2^{(N/2)-1}$ each. For the analysis here, let $U = Spin(N+1)$ and $K = Spin(N)$, with $\mathcal{U}$ and $\mathcal{K}$ the corresponding Lie algebras. There is a direct sum decomposition $\mathcal{U} = \mathcal{K} \oplus \mathcal{K}^\perp$, where $\mathcal{K}^\perp$ is the orthogonal complement of $\mathcal{K}$ with respect to the Killing form. Consider the element $x_0 = e\,K \in U/K$ and in terms of it, define the local cross section of $U(U/K, K)$ as $\sigma : U/K \backslash$ (antipodal point of x_0) $\to U$ in the following way. Let Exp and exp denote the exponential mappings on U/K and U, respectively. Then as in any symmetric space $Exp\, X = \pi(\exp X) = \exp(X)\, x_0$, $X \in \mathcal{K}^\perp$, where the mapping $\pi : U \to U/K$ is the projection map. Hence let us take,

$$\sigma(Exp\, X) = \exp X, \qquad X \in \mathcal{P}_0, \tag{5.1}$$

$\mathcal{P} = \{X \in \mathcal{K}^\perp | \langle X, X \rangle < \pi\}$, where $\langle , \rangle = -c\,B$, with B the Killing form on $\mathcal{U}$ and $c > 0$ a normalization constant determined by requiring that the radius of S^N be one.

Case 1. N even. The sets $\hat{U}(\tau_\pm)$ of the equivalence classes of the irreducible representations or irreps of $\mathrm{Spin}(N+1)$ that contain $\tau_\pm$ are

$$\hat{U}(\tau_+) = \hat{U}(\tau_-) = \{\lambda_n = (n + \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}), n = 0, 1, \ldots\}. \tag{5.2}$$

The multiplicity of $\tau_\pm$ in $\lambda_n|_K$ is 1 for all n. The iterated Dirac operator and the spinor Laplacian $L = \sum_{a=1}^N \nabla^a \nabla_a$ on a manifold M are related in general by the formula of

Lichnerowicz

$$\nabla\!\!\!/^2 = L - \frac{1}{4} R, \tag{5.3}$$

where R is the scalar curvature of M. The eigenvalues $\lambda_{n,N}^2$ of $-\nabla\!\!\!/^2$ on S^N are given in terms of the eigenvalues of the operator $-L$ by the equation,

$$\lambda_{n,N}^2 = \omega_n + \frac{1}{4} N(N-1). \tag{5.4}$$

Thw general formula for the eigenvalues of $-L$ acting on the space $L^2(U/K, E^\tau)$ can be used, so for $E^{\tau+}$, where $C_2(\mu)$ denotes the Casimir number pertaining to irrep μ, we have

$$\omega_n = \omega_{\lambda_n} = C_2(\lambda_n) - C_2(\tau_+). \tag{5.5}$$

This arises from the statement $L_{U/K} = \Omega_D - \Omega_K$, where $\Omega_U = \sum_i T_i^2$ and $\Omega_K = \sum_i X_i^2$ are the Casimir elements of U and K, respectively. Thus $\{T_i\}$ and $\{X_i\}$ are orthonormal bases of $\mathcal{U}$ and $\mathcal{K}$. For $E^{\tau-}$, the same eigenvalues hold since $C_2(\tau_+) = C_2(\tau_-)$. The Casimir number of an irrep with highest weight λ is given by

$$C_2(\lambda) = (\lambda + \rho)^2 - \rho^2 = \lambda \cdot (\lambda + 2\rho), \tag{5.6}$$

where ρ is half the sum of the positive roots of the group. The dimensions are given by the Weyl result

$$d_\lambda = \prod_{\alpha > 0} \frac{\alpha \cdot (\lambda + \rho)}{\alpha \cdot \rho}.$$

The product is taken over the positive roots α of the group. From these one verifies the formulas for the eigenvalues and degeneracies of $\nabla\!\!\!/^2$ obtained already.

If τ is any irrep of K, the heat kernel $K(x,y,t)$ of the Laplacian L_x on E^2 is in general an element of $\mathrm{Hom}(E_y, E_x) \equiv E_x \otimes E_y$ for each $t \in \mathbb{R}^+$ and $x, y \in U/K$. It satisfies

$$\Big(- \frac{\partial}{\partial t} + L_x \Big) K(x,y,t) = 0, \tag{5.7}$$

subject to the initial condition

$$\lim_{t \to 0^+} \int_{U/K} K(x,y,t)\, \psi(y)\, dy = \psi(x), \tag{5.8}$$

for any continuous section $\psi \in \Gamma(E^\tau)$. Let $U^\lambda(u)^i_j$ for $u \in U$, $i, j = 1, \dots, d_\lambda$, be the representation matrix for the irrep $\lambda \in \hat{U}(\tau)$ in a given orthonormal basis of V_λ. For simplicity, suppose that the representation τ appears only once in $\lambda|_K$, and choose the basis $\{e_j\}$ of V_λ such that the vectors $\{e_a\}_{a=1,\dots,d_r}$ span V_τ, the subspace of vectors of V_λ which transform under K according to τ. The matrix-valued function on U given by $f^a_{\lambda j} : u \to U^\lambda(u^{-1})^a_j$ satisfies $f^a_{\lambda j}(uk) = \tau(k^{-1})^a_b\, f^b_{\lambda j}(u)$ for $k \in K$. Thus for each $j = 1, \dots, d_\lambda$ $f^a_{\lambda j}$ defines a cross section of E^π. Let the eigenvalue of L_x acting on $f^a_{\lambda j}$ be $-\omega_\lambda$. Then the mode expansion of the matrix heat kernel in the local basis $\theta_a(x) = \sigma(x)\, e_a$

of $\Gamma(E^\tau)$ can be obtained in general by using the Frobenius reciprocity theorem in the form

$$K(x,y,t)^a_b = \frac{1}{d_\tau V_{U/K}} \sum_{\lambda \in \hat{U}(\tau)} d_\lambda U^\lambda(\sigma^{-1}(x)\sigma(y))^a_b e^{-i\omega_\lambda}, \quad a,b = 1,\dots,d_\tau. \tag{5.9}$$

In (5.9) $\sigma^{-1}(x) = (\sigma(x))^{-1}$ and $V_{U/K}$ is the volume of U/K. One can express this in a coordinate invariant way as well

$$K(x,y,t) = \frac{1}{d_\tau V_{U/K}} \sum_{\lambda \in \hat{U}(\tau)} d_\lambda U(x,x_0) P_\tau U^\lambda(\sigma^{-1}(x)\sigma(y)) P_\tau U(x_0,y) e^{-t\omega_\lambda}. \tag{5.10}$$

The projector of V_λ onto V_τ is P_τ and $U(x,y)$ is the parallel propagator from E_y, the fibre at y to E_x along the shortest geodesic between them. The U-invariance of the heat kernel permits setting $x = x_0$ without loss of generality. Letting $U = \mathrm{Spin}(N+1)$ and $K = \mathrm{Spin}(N)$ and using (5.3) it is found that the heat kernel of $\not\nabla^2$, with one point at the origin, is the direct sum $K = K^+ \oplus K^-$, where the heat kernels $K^\pm$ on E^{τ_+} are given by

$$K^\pm(x_0,x,t) = \frac{1}{\Omega_N d_{\tau_\pm}} \sum_{n=0}^\infty d_{\lambda_n} \Phi^n_\pm(\sigma(x)) U(x_0,x) e^{-t\lambda^2_{n,N}}. \tag{5.11}$$

The $\tau_+(\tau_-)$ spherical functions $\Phi^n_+(\Phi^n_-)$ which have been introduced are the linear operators in $V_{\tau_+}(V_{\tau_-})$ defined by

$$\Phi^n_+(u) = P_{\tau_+} U^{\lambda_n} P_{\tau_+}, \qquad u \in U.$$

There is a similar relation for Φ^n_-, and P_{τ_+}, P_{τ_-} are projectors of V_{λ_n} onto the subspace where $\lambda_n|_K$ is equivalent to τ_+ and τ_-, respectively. To determine Φ^n_+, note first that

$$\Phi^n_+(k_1\, u k_2) = \tau_+(k_1)\, \Phi^n_+(u)\, \tau_+(k_2), \quad u \in U, \quad k_1, k_2 \in K. \tag{5.12}$$

With respect to the polar decomposition of U, it is enough to calculate the restrictions $\Phi^n_+(a), a \in A = \exp \mathcal{A}$, where $\mathcal{A}$ is any maximal abelian subspace of $\mathcal{K}^\perp$. In this case, A is a one-dimensional group. and so $A \equiv S^1$. It is well known that M, the centralizer of A in K, may be identified with $\mathrm{Spin}(N-1)$. The relation $a\,m = m\,a$ where $a \in A, m \in M$ and (5.12), it is found that operators $\Phi^n_+(a)$ commute with all the operators of the representation, $\tau_+|_M = \sigma$, that is, the unique fundamental spinor representation of $\mathrm{Spin}(N-1)$. Since σ is irreducible, it follows by Schur's lemma that $\Phi^n_+(a)$ must be proportional to the identity operator in V_{τ_+} as

$$\Phi^n_+(a) = f_n(a)\mathbf{1},$$

where $a \in A$ and f_n is a scalar function on A.

Choose $H \in \mathcal{A}$ by requiring $\alpha(H) = 1$, where α is the unique positive restricted root of U relative to $\mathcal{A}$. The normalization of the scalar product $\langle , \rangle$ on $\mathcal{A}$, induced from that on $K^{\perp}$ is such that $\langle H, H \rangle = 1$. and corresponds to the choice $V_{U/K} = \Omega_N$. For $a \in A$ write $a = a_B \exp(\theta H)$, where $\theta \in \mathbb{R}$. If $d(x, x')$ denotes the geodesic distance between $x, x' \in S^N$, then

$$d(a_\theta x_0, x_0) = d(k a_\theta x_0, x_0) = \theta, \qquad k \in K,$$

where x_0 is the origin, the north pole here. It is to be shown that,

$$f_n(a_\theta) = \phi_n(\theta) \tag{5.13}$$

where $\phi_n(\theta)$ is given by (4.5).

To accomplish this, note the second order differential equation for $h_\lambda(\theta) = U^\lambda(a_\theta)$, $\lambda \in \hat{U}$ with an analytic continuation to the compact case with $t \to i\,\theta$. Let Ω_U, Ω_K and Ω_M be the quadratic Casimir operators of the algebras corresponding to the groups U, K and M. Then it is found that $h_\lambda(\theta)$ satisfies the following differential equation

$$\left(\frac{d^2}{d\theta^2} + m_\alpha \cot\theta \, \frac{d}{d\theta}\right) h_\lambda(\theta) + h_\lambda(\theta)\Omega_M - \frac{1}{\sin^2\theta}[(\Omega_M - \Omega_K)\, h_\lambda(\theta)(\Omega_M - \Omega_K) + 2\cos\theta \sum_{i=1}^{N-1} X_{\alpha_i} h_\lambda(\theta) X_{\alpha_i}]$$

$$= \Omega_U \, h_\lambda(\theta) = -C_2(\lambda)\, h_\lambda(\theta). \tag{5.14}$$

where $\{X_{\alpha i}\}$ is an orthonormal basis of

$$K_\alpha = \{X \in K : (ad\, H)^2\, X = -X\}.$$

and $X_{\alpha i}$ is an abbreviation of $dU^\lambda\,(X_{\alpha i})$ or $d\tau_+(X_{\alpha i})$. Now let $\lambda \in \hat{U}(\tau_+)$, that is $\lambda = \lambda_n$ and $\Phi_+^n(a_\theta) = P_{\tau_+} h_{\lambda_n}(\theta) P_{\tau_+}$. By acting with P_{τ_+} from the left and right in (5.14), an equation for $\Phi_+^n(a_\theta)$ is obtained. It is clear that Ω_K and Ω_M commute with P_{τ_+}. It is also the case that

$$\Omega_K \Phi_+^n(a_\theta) = \Phi_+^n(a_\theta)\Omega_K = -C_2(\tau_+)\Phi_+^n(a_\theta), \quad \Omega_M \Phi_+^n(a_\theta) = \Phi_+^n(a_\theta)\, \Omega_M = -C_2(\sigma)\Phi_+^n(a_\theta). \tag{5.15}$$

Furthermore, since $\Phi_+^n(a_\theta)$ is just a scalar operator, we have

$$\sum_{i=1}^{N-1} X_{\alpha i} \Phi_+^n(a_\theta) X_{\alpha i} = \sum_{i=1}^{N-1} X_{\alpha i} X_{\alpha i}\, \Phi_+^n(a_\theta) = (C_2(\sigma) - C_2(\tau_+))\Phi_+^n(a_\theta). \tag{5.16}$$

Using these relations and substituting the explicit values of $C_2(\lambda_0)$, $C_2(\tau_+)$ and $C_2(\sigma)$ in (5.14) it is found that $f_n(a_\theta)$ satisfies the same second-order differential equation as $\phi_n(\theta)$ given in (4.5). Clearly, $\Phi_+^n(e) = 1$, and f_n must satisfy $f_n(e) = 1$. Thus (5.13) is proved and $\Phi_+^n(a_\theta) = \phi_n(\theta)$. By proceeding in a similar way with τ_-, it is found that the same expression for $\Phi_-^n(a_\theta)$, results, namely $\Phi_-^n(a_\theta) \cdot \mathbf{1}$. Using these formulas in (5.11), the heat kernel $K(x, t)$ appears relevant for an arbitrary point x, which agrees with (4.6).

Case 2. N odd. Let τ be the fundamental spinor representation $\{1/2, 1/2, \ldots, 1/2\}$ of $K = \mathrm{Spin}(N)$. The branching rule for $\mathrm{Spin}(N+1) \supset \mathrm{Spin}(N)$ gives

$$\hat{U}(\tau) = \{\lambda_n^+ : n = 0, 1, \ldots\} \cup \{\lambda_n^- : n = 0, 1, \ldots\},$$

where λ_n^+ and λ_n^- are the spinor representations of Spin$(N+1)$, with finite weights,

$$\lambda_n^{\pm} = (n + \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \pm\frac{1}{2}\}. \tag{5.17}$$

One can reproduce the formulas that were obtained for the eigenvalues $\lambda_{n,N}^2$ of $-\not\nabla^2$ and the degeneracies using the Freudenthal and Weyl formulas.

The heat kernel of $\not\nabla^2$, with one point at the origin, takes the form,

$$K(x,t) = K(x_0, x, t) = \frac{1}{\Omega_N d_\tau} \sum_{n=0}^{\infty} d_{\lambda_n^+} \left(\Phi^{n+}(\sigma(x)) + \Phi^{n-}(\sigma(x))\right) U(x_0, x) e^{-t\lambda_{n,N}^2}, \tag{5.18}$$

where the τ-spherical functions $\Phi^{n\pm}$ are defined by,

$$\Phi^{n\pm}(u) = P_\tau \, U^{\lambda_n^{\pm}}(u) \, P_\tau, \qquad u \in U. \tag{5.19}$$

To calculate $\Phi^{n\pm}$, notice that the operators $\Phi^{n\pm}(a)$, $a \in \mathcal{A}$, commute with all $\tau(m)$, $m \in M \approx$ Spin$(N-1)$, which is the centralizer of A in K. The branching rule for Spin$(N) \supset$ Spin$(N-1)$ gives $\tau|_M = \sigma_+ \oplus \sigma_-$, where σ_+ and σ_- are the two fundamental spinor representations on Spin$(N-1)$. Suppose an orthonormal basis of V_τ is fixed and adapted to the direct sum decomposition $V_\tau = V_{\sigma_+} \oplus V_{\sigma_-}$. Applying Schur's lemma, we find

$$\Phi^{n\pm}(a) = f_{n^+}^{\pm}(a)\mathbf{1}_+ \oplus f_{n^-}^{\pm}(a)\mathbf{1}_-, \qquad a \in A, \tag{5.20}$$

where $f_{n^+}^{\pm}$ and $f_n^{\pm}$ are scalar functions on A and $\mathbf{1}_\pm$ is the identity operator in $V_{\sigma^\pm}$. Scalar functions can be determined now by using the radial part of the Casimir operator.

It suffices to define $h_\lambda(\theta) = U^\lambda(a_\theta)$ to obtain (5.14). Suppose $\lambda \in \hat{U}(\tau)$ so $\lambda = \lambda_n^+$ or λ_n^-. By acting with P_τ in (5.14), both from the left and from the right, we obtain an equation for $\Phi^{N+}(a_0)$ and $\Phi^{n-}(a_0)$. In this case, however, as $\Phi^{n\pm}(a_\theta)$ are not scalar operators, (5.16) is not valid. Let diag(p,q) denote the operator $p\,\mathbf{1}_+ \oplus \mathbf{1}_-$ in V_τ, where $p, q \in \mathbb{C}$. Then it is not difficult to show, using the explicit formula for $X_{\alpha i}$ in the *irrep* τ that

$$\sum_{i=1}^{N-1} X_{\alpha i} \, \mathrm{diag}(p,q) \, X_{\alpha i} = -\frac{1}{4}(N-1) \, \mathrm{diag}(q,p). \tag{5.21}$$

Using (5.21) in (5.14), the following set of coupled equations for the functions $f_{n+}^+(a_\theta)$ and $f_{n-}^+(a_\theta)$ come out,

$$[\partial_\theta^2 + (N-1) \cot\theta \, \partial_\theta - C_2(\sigma_+)] f_{n\pm}^+ - \frac{N-1}{2\sin^2\theta} f_{n\pm}^+ + (N-1) \frac{\cos\theta}{2\sin^2\theta} f_{n\pm}^+ = -C_2(\lambda_n^+) f_{n\pm}^+. \tag{5.22}$$

The same set of equations with $f_{n\pm}^+ \to f_{n\pm}^-$ is obtained for the irrep $\lambda = \lambda_n^-$.

By computing the sum and difference of the two equations in (5.20), it is then easy to see that the function $f_{n+}^+(a_\theta) + f_{n-}^+(a_\theta)$ satisfies the same differential equation (2.12) as $\phi_n(\theta)$ given by (4.5). The function $f^+_{n+}(a_\theta) - f^+_{n-}(a_\theta)$ satisifes the same equation that function $\psi_n(\theta)$ does and is given by

$$\psi_n(\theta) = \frac{\psi_{n0}(0)}{\phi_{n0}(0)} = \frac{n!\Gamma(N/2)}{\Gamma(n+N/2)} \sin\frac{\theta}{2} P_n^{(N/2,N/2-1)}(\cos\theta). \tag{5.23}$$

The equation for ψ_n is the same as (2.12) when $l = 0$ under the replacement $\theta \to \pi - \theta$.

Thus $f^+_{n+}(a_\theta) + f^+_{n-}(a_\theta)$ is proportional to $\phi_n(\theta)$ and since $\phi_n(0) = f^+_{n\pm}(e) = 1$,

$$f^+_{n+}(a_\theta) + f^+_{n-1}(a_\theta) = 2\phi_n(\theta). \tag{5.24}$$

Similarly, we find

$$f^+_{n+} - f^+_{n-}(a_\theta) = 2i\psi_n(\theta), \tag{5.25}$$

where the relation $f^+_{n+}(a_\pi) = -f^+_{n-}(a_\pi) = (-1)^n\, i$ has been used. The relation $\phi^{n\pm}(a_\pi) = \pm(-1)^n\,\Gamma^N$ can be established so that

$$f^+_{n\pm}(a_\theta) = \phi_n(\theta) \pm i\,\psi_n(\theta). \tag{5.26}$$

For the functions $f^-_{n\pm}$ we obtain

$$f^-_{n\pm}(a_\theta) = \phi_n(\theta) \mp i\,\psi_n(\theta). \tag{5.27}$$

Using (5.26)-(5.27) in (5.20), the heat kernel $K(x,x',t)$ of (4.6) is reproduced.

Spinors on H^N.

To conclude we briefly look at the case of H^N. Suppose G is a noncompact semisimple Lie group with finite center, K a maximal compact subgroup, and $G\backslash K$ the associated Riemannian symmetric space of noncompact type. For vector bundles L^2 harmonic analysis on $G\backslash K$ may be reduced to L^2-harmonic analysis on G by allowing $L^2(G/K, E^\tau)$ reside in $L^2(G)$ in a natural way. The heat kernel of the Laplacian acting on this space $L^2(G/K, E^\tau)$ may be expanded in terms of τ spherical functions similar to (5.11) replacing the sum by an integral

$$K(x,t) = K(x_0,x,t) = \frac{1}{d_\lambda} \int_{\hat{G}(\tau)} \Phi^\lambda_\tau(\sigma(x)) U(x_0,x)\, e^{-i\omega_\lambda} d\mu(\lambda), \tag{5.28}$$

where $d\mu(\lambda)$ is the Plancherel measure. The τ-spherical functions are the operator valued functions $g \to \Phi^\lambda_\tau(g)$ on G defined by

$$\Phi^\lambda_\tau(g) = P_\tau\, U^\lambda P_\tau, \qquad g \in G, \tag{5.29}$$

where $U^\lambda(g)$ denote the operators of the representation $\lambda \in \hat{G}(\tau)$ in a Hilbert space H_λ, and P_λ is the projector of H_λ onto V_τ, the subspace of vectors of H_λ which transform under K according to τ. As in the compact case, $\sigma(\mathrm{Exp}\, X) = \exp\, X$, such that $X \in \mathcal{K}^\perp$, and $U(x_0,x)$ denotes the vector bundle parallel transport operator from x to x_0 by way of the geodesic between them. The vector bundle Laplacian $L_{G/K}$ then has the form,

$$L_{G/K} = \Omega_G + \Omega_K, \tag{5.30}$$

where Ω_G and Ω_K are the quadratic Casimir operators of $\mathcal{G}$ and $\mathcal{K}$. The plus in front of Ω_K is due to the Killing form, and has opposite signature on $\mathcal{K}$ and $\mathcal{K}^{\perp}$. Denoting the Casimir value of λ by $-C_2(\lambda)$, the eigenvalues $-\omega_\lambda$ of $L_{G/K}$ arising from (5.30) are

$$-\omega_\lambda = -C_2(\lambda) - C_2(\tau). \tag{5.31}$$

There is thus a duality between the noncompact and the compact symmetric spaces. Without going into detail to finish, the following information is of interest. Consider the subspace $\mathcal{U} = \mathcal{K} \oplus i\,\mathcal{K}^{\perp}$ of $\mathcal{G}$. It is a compact Lie group as the Killing form is negative definite on $\mathcal{U}$. Suppose U is a simply conected compact Lie algebra $\mathcal{U}$. Then U/K is the compact symmetric space which is dual to G/K. The radial parts of the Casimir operators pertaining to G and U and acting on τ-spherical functions are related by analytic continuation $h \to i\,h$ for $h \in \mathcal{A}$.

6. Conclusions

Let us finally summarize and remark that these results have applications to many other areas involving elliptic operators. For example, this is relevant to the study of the Atiyah-Singer index theorem [13], [14]. This theorem states that for an elliptic operator on a compact manifold, the analytical index, related to the dimension of the solution space is equal to topological index. It includes many other theorems as subcases, and has applications to theoretical physics.

Spinor fields which satisfy a Dirac equation $\not\nabla\,\psi = i\lambda\,\psi$ have been investigated as well as the heat kernel for $\not\nabla^2$ on S^N and H^N. The gamma matrices are established needed to determine the eigenfunctions of the Dirac operator on S^N with N arbitrary using geodesic polar coordinates. The degeneracies of the Dirac operator can be found using these eigenfunctions. Eigenfunctions on H^N are obtained by analytically continuing those pertaining to S^N now known. They are used to produce the spinor spectral function on H^N. The results can be used to construct the heat kernel for the iterated Dirac operator on these spaces. Some group theoretic analysis of spinor fields on S^N and H^N is provided.

References

[1] T. Friedrich. *Dirac Operators in Riemannian Geometry.* AMS Graduate Studies in Mathematics, 25, Providence, RI, 2000.

[2] R. Camporesi and A. Higuchi. On the eigenfunctions of the dirac operator on spheres and real hyperbolic spaces. *Journal of Geometry and Physics*, 20:1–18, 1996.

[3] R. Camporesi. Harmonic analysis and propagators on homogeneous spaces. *Physics Reports*, 196:1–200, 1990.

[4] A. O. Barut and R. Raczka. *Theory of Group Representations.* World Scientific, Singapore, 1986.

[5] A. Trautman. *Spin structures on hypersurfaces and the spectrum of the Dirac operator on spheres, Spinors, Twistors, Clifford Algebras and Quantum Deformations.* Kluwer Academic Publishers, Dordrecht, 1991.

[6] R. Camporesi. The spinor heat kernel in maximally symmetric spaces. *Comm. Math. Phys.*, 148:283–308, 1992.

[7] A. Knapp. *Representation Theory of Semisimple Lie Groups: An Overview Based on Examples.* Princeton University Press, Princeton, NJ, 1986.

[8] Y. Choquet-Bruhat, C. DeWitt-Morette, and M. Dillard-Bleick. *Analysis, Manifolds and Physics.* North Holland, Amsterdam, 1982.

[9] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry, v. I and II.* Interscience, New York, 1969.

[10] N. Wallach. *Harmonic Analysis on Homogeneous Spaces.* Marcel Dekker, New York, 1973.

[11] S. Helgason. *Differential Geometry, Lie Groups and Symmetric Spaces.* Academic Press, New York, 1978.

[12] R. Godement. A theory of spherical functions. *Trans. Amer. Math. Soc.*, 73:496–556, 1952.

[13] M. F. Atiyah, V. K. Patodi, and I. M. Singer. Spectral asymmetry and riemannian geometry. I. *Math. Proc. Camb. Phil. Soc.*, 77:43–69, 1975.

[14] M. F. Atiyah and I. M. Singer. Dirac operators coupled to vector potentials. *Proc. Natl. Acad. Sci. USA*, 81:2597–2600, 1984.