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Quantum Information Resources for Two Superconducting Artificial Atoms

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Abstract. This study investigates the emergent effects of coupling two superconducting charge qubits on thermal-equilibrium nonlocality, including Bell-inequality violation (BIV) and uncertainty-induced (UI) quantum nonlocality beyond logarithmic negativity entanglement (LN entanglement). Increasing the Josephson-qubit energy enhances the resistance to degradation at high temperatures. Large differences in Josephson-qubit energies increase the temperature at which nonlocality decreases. The coupling of the two qubits improves the robustness, preservation, and thermal interval of maximal and partial nonlocalities. The temperatures for the sudden death of the Bell-inequality violation and entanglement depend on the Josephson qubit energies, their differences, and mutual coupling. Finally, we explore the symmetrical dependence of the thermal two superconducting-qubit nonlocality on Josephson-qubit energies and mutual qubit coupling.

2020 Mathematics Subject Classifications: 81P45, 81V80

Key Words and Phrases: Superconducting charge qubits, uncertainty-induced quantum nonlocality, Bell nonlocality violation

1. Introduction

Exploring the generation behavior of nonlocal quantum in various proposed qubit systems has become increasingly important in several quantum information technologies [1–3], such as quantum computation [4] and quantum estimation [5].

Two-qubit nonlocality is an important resource for all key quantum information tasks, including quantum computation [6–8], quantum cryptography [9, 10], teleportation [11], and quantum memory [12].

The first type of nonlocality is entanglement. The Neumann entropy, negativity, entanglement of formation, and concurrence are useful quantifiers for evaluating entanglement

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[13–17] in different quantum systems, such as two driven quantum dots inside a coherent micro cavity [18], and neutrino oscillation systems [19].

Using concurrence, trade-off relations between coherence and entanglement have been demonstrated in arbitrary two- and three-qubit states [20, 21].

For arbitrary systems of two bits, an effective strategy has been proposed to recover maximal concurrence through operators [22].

By introducing another type of two-qubit nonlocality beyond entanglement depending on the implementation of quantum discord [23–25], several nonlocality quantifiers have been proposed to measure quantum coherence beyond entanglement based on quantum Fisher information [26, 27], Wigner–Yanase skew information [28], skew information quantifiers include local quantum uncertainty [29] and uncertainty-induced quantum nonlocality (UI) [30]. The two-qubit nonlocality beyond entanglement was estimated using the UI-quantum nonlocality [31–33].

Nonlocality is a concept that arises from violations. Particular inequalities, such as the Bell inequalities and the Clauser-Horne-Shimony-Holt (CHSH) inequalities, have been studied to characterize Bell nonlocality as a function of varying polarizer angles [34]. By having prior knowledge of a specific generated state, researchers discovered that ideal polarizer angles can be selected to maximize the violation of the Bell-CHSH inequality and determine the Bell nonlocality measure [35]. This study investigated the relationship between Bell's nonlocality, entanglement concurrence, and negativity. Various two-qubit states have been examined, revealing differing levels of entanglement while maintaining the same level of nonlocality [36].

Meanwhile, another study examined a setup that included magnetic field pulses, thermal effects, gravity, and nonlocal correlations [37]. It analyzes gravitational cat states under various conditions, including thermal environments, classical stochastic fields, general decay, and power-law-noise fields [38]. In addition, the thermal entanglement between two superconducting qubits was studied across various interaction strengths and ground state frequencies [39]. The evolution of entanglement, coherence, and mixedness was mapped in a four-qubit GHZ-class maximally entangled state affected by fractional Gaussian noise [40]. The model features two coupled superconducting qubits interacting during unitary charging in thermal equilibrium, demonstrating quantum advantages and discussing superconducting qubit batteries [41]. The collisions between the condensate and the thermal cloud influence regimes (Josephson plasma, phase-slip dissipation, and quantum self-trapping) in ultracold atomic Josephson junctions at subcritical temperatures [42]. The impact of the axion field on Josephson junctions was analyzed by examining its effect on the potential barrier that prevents switching from the superconducting state to the finite-voltage state [43]. Superconducting tunnel-junction thermoelectricity, often assumed in equilibrium and linear regimes, was modeled using a nonlinear, out-of-equilibrium numerical approach that reduces it to a linear model at low power [44]. Superconducting quantum circuits comprise Josephson junctions and behave like artificial atoms, exhibiting quantum mechanical behavior. This unique characteristic enables researchers to investigate quantum entanglement and coherence on the macroscopic scale, as demonstrated in several studies [45–49]. These synthetic atoms are promising for implementing quantum information technology and testing quantum-mechanical laws in larger systems [50]. Theoretical and experimental studies have explored the entanglement among different types of Josephson-junction qubits, including charge [51–53], flux [54–56], and phase qubits [57, 58].

Motivated by the above-mentioned issues, we investigated the behavior of nonlocality in a two-qubit superconducting charge (SCC) system at thermal equilibrium. To do this, we use uncertainty-induced (UI) quantum nonlocality, Bell-inequality-violation (BIV) nonlocality, and Log-negativity (LN) entanglement in our theoretical analysis. We examined the temperature dependence of BIV nonlocality, UI-quantum nonlocality, and LN entanglement at different Josephson and coupling energy values. In addition, the effects of the Hamiltonian parameters on the thermal BIV nonlocality, UI-quantum nonlocality, and LN entanglement are discussed in detail. Hence, the current model, which comprises two distinct Cooper interactions between two charge qubits and is formulated within the thermal equilibrium framework, is a significant factor in assessing entanglement in a two-qubit system.

The rest of the paper is organized as follows: In Section (2), the physical model of interactions between two charge qubits is addressed within the thermal equilibrium framework. In Section(3), the thermal charge-qubit nonlocality (BIV nonlocality, UI-quantum nonlocality, and entanglement) quantifiers are presented. The results of the numerical simulations of thermal charge-qubit nonlocality are presented and analyzed in Section (4). Finally, we conclude the paper in Section (5).

2. Thermal superconducting-qubits model

Here, we use the thermal equilibrium model to investigate the nonlocality behavior of two superconducting-charge qubits (two-SCC-qubits). The thermal equilibrium model describes two superconducting-charge-qubits (A and B) with Pauli matrices $\hat{\sigma}_i^k$ ($i = x, y, z$ and $k = A, B$) and is connected to the thermal reservoir in equilibrium with the bath temperature T. The associated thermal density matrix of the two-SCC-qubits $\hat{\rho}(T)$ is expressed as follows

$$\hat{\rho}(T) = \frac{1}{Z} \exp\Big[- \frac{\hat{H}}{K_B T}\Big], \tag{1}$$

where $Z = Tr[\exp(-\hat{H}/K_B T)]$ is a partition function with K_B is the Boltzmann constant (Set as unity). The Hamiltonian $\hat{H}$ describes two Cooper pair box-charge qubits coupled with a fixed capacitance C_m. When the operating point of the two superconducting-charge-qubits is fixed at the degeneracy point $ng_1 = ng_2 = \frac{1}{2}$ (which is the condition of insensitivity to noise [59], the Hamiltonian $\hat{H}$ is given by

$$\hat{H} \;\;=\;\; -\frac{1}{2}[J_A(\hat{\sigma}_A^x \otimes I) + J_B(I \otimes \hat{\sigma}_B^x) - 2J_{AB}(\hat{\sigma}_A^z \otimes \hat{\sigma}_B^z)], \tag{2}$$

$J_k(k = A, B)$ represents the Josephson energies of the k-SCC-qubit, J_{AB} represents the mutual two superconducting-charge-qubits coupling energy. These two superconducting-charge-qubits Hamiltonians have potential applications in quantum gates [60] and quantum teleportation [61].

On a standard basis: $\{|S1\rangle = |00\rangle, |S2\rangle = |01\rangle, |S3\rangle = |10\rangle, |S4\rangle = |11\rangle\}$, the eigenstates $|V_i\rangle(i = 1, 2, 3, 4)$ and the eigenvalues V_i of the Hamiltonian of Eq. (2) are given by

$$\begin{aligned}
|V_1\rangle &= \frac{(J_A - J_B)[-|S_1\rangle + R_1|S_2\rangle - R_1|S_3\rangle + |S_4\rangle]}{\sqrt{2(J_A - J_B)^2 + 2(H^- + 2J_{AB})^2}}, \\
|V_2\rangle &= \frac{(J_A - J_B)[-|S_1\rangle - R_2|S_2\rangle + R_2|S_3\rangle + |S_4\rangle]}{\sqrt{2(J_A - J_B)^2 + 2(H^- - 2J_{AB})^2}}, \\
|V_3\rangle &= \frac{(J_A + J_B)[|S_1\rangle + R_3|S_2\rangle + R_3|S_3\rangle + |S_4\rangle]}{\sqrt{2(J_A + J_B)^2 + 2(H^+ + 2J_{AB})^2}}, \\
|V_4\rangle &= \frac{(J_A + J_B)[|S_1\rangle - R_4|S_2\rangle - R_4|S_3\rangle + |S_4\rangle]}{\sqrt{2(J_A + J_B)^2 + 2(H^+ - 2J_{AB})^2}}, \qquad (3)
\end{aligned}$$

with

$$\begin{aligned}
H^{\pm} &= \sqrt{(J_A \pm J_B)^2 + 4J_{AB}}, \\
R_{1,2} &= (H^- \pm 2J_{AB})/(J_A - JE_B), \\
R_{3,4} &= (H^+ \pm 2J_{AB})/(J_A + J_B).
\end{aligned}$$

The corresponding two superconducting-charge-qubits eigenvalues are,

$$\begin{aligned}
V_1 &= -\frac{1}{2}\sqrt{(J_A - J_B)^2 + 4J_{AB}}, \\
V_2 &= \frac{1}{2}\sqrt{(J_A - J_B)^2 + 4J_{AB}}, \\
V_3 &= -\frac{1}{2}\sqrt{(J_A + J_B)^2 + 4J_{AB}}, \\
V_4 &= \frac{1}{2}\sqrt{(J_A + J_B)^2 + 4J_{AB}}. \qquad (4)
\end{aligned}$$

Using the previous eigenvalues V_k $(k = 1, 2, 3, 4)$ and eigenstates $|V_k\rangle$, the thermal two superconducting-charge-qubits state behavior of Eq. (1) is

$$\hat{\rho}(T) = \frac{\sum_{i=1}^{4} \exp[\frac{-V_i}{T}]|V_i\rangle\langle V_i|}{\mathrm{Tr}\{\sum_{i=1}^{4} \exp[\frac{-V_i}{T}]|V_i\rangle\langle V_i|\}}. \qquad (5)$$

This density matrix is used to explore the ability of the two superconducting-charge-qubit interactions to generate thermal nonlocality, including Bell-inequality-violation (BIV) and uncertainty-induced (UI) quantum nonlocality, beyond logarithmic negativity entanglement (LN entanglement).

3. Thermal nonlocality quantifiers

- **Uncertainty-induced (UI) quantum nonlocality**
Here, the UI-quantum nonlocality is used as a measure of the quantum resources of two superconducting charge qubits beyond entanglement. The definition of the nonlocality function of the UI-quantum nonlocality depends on the maximization of the skew information quantity $I(\hat{\rho}(T), K)$ [28, 30] of the state of the two superconducting charge qubits $\hat{\rho}(T)$ and in general, the local observables of the two-qubit system K. The UI-quantum nonlocality closed function [28, 30] is given by

$$U(t) = \begin{cases} 1 - \lambda_{\min}(C), & \vec{r} = 0; \\ \\ 1 - \frac{1}{|\vec{r}|^2} \vec{r}\ C\ \vec{r}^T, & \vec{r} \neq 0, \end{cases} \tag{6}$$

where $\vec{r}$ represents the Bloch vector with the elements $x_i = Tr(\hat{\rho}(T)(\sigma_i \otimes I))(i = x, y, z)$. $\lambda_{\min}(C)$ denotes the smallest eigenvalue of matrix $C = [C_{ij}]$ $(i, j = 1, 2, 3)$, which depends on the square root of state $\hat{\rho}(T)$, and its elements are given by:

$$C_{ij} = Tr\big\{\sqrt{\hat{\rho}(T)}(\hat{\sigma}_i \otimes I)\sqrt{\hat{\rho}(T)}(\hat{\sigma}_j \otimes I)\big\}. \tag{7}$$

If $U(t) = 0$, the two superconducting-charge-qubit states have zero UI-quantum nonlocality. However, for the nonlocality values of UI-quantum $0 < U(t) < 1$ and $U(t) = 1$, the two superconducting-charge-qubit states have two partially and maximally entangled states, respectively.

- **Bell inequality violation (BIV nonlocality)**
Here, the two superconducting-charge-qubit nonlocality depends on the Bell inequality violation, that is, the two SCC-qubit states have BIV nonlocality when the maximum Bell-CHSH quantity $B_{\max}(t)$ allows the inequality [62, 63]:

$$B_{\max}(t) = 2\sqrt{m_1 + m_2} \geq 2,$$

where $m_{1,2}$ are the two largest eigenvalues of the combined matrix $U = T^{\dagger}T$ based on the correlation matrix [63] $T = [T_{ij}]$: with $T_{ij} = \mathrm{Tr}\{\hat{\rho}(T)\hat{\sigma}_i^A\hat{\sigma}_j^B\}$, $i, j = 1, 2, 3$. Therefore, the range of the BIV nonlocality can be defined by the following function:

$$V(t) = \max(0, B_{\max}(t) - 2). \tag{8}$$

Where $V(t) = 0$ indicates that the two thermal two-SCC-qubit states do not exhibit BIV nonlocality. For values of the function $0 < V(t) < 2(\sqrt{2} - 1)$, the thermal state exhibits partial BIV nonlocality. The state has a maximal BIV nonlocality if $V(t) = 2(\sqrt{2} - 1) \approx 0.8284$.

- **Log-negativity (LN) entanglement**
Here, the log-negativity of the density matrix $\hat{\rho}(T)$ was used to realize two superconducting-charge-qubit entanglements (LN entanglement). The closed form of logarithmic negativity is given by [15]

$$N(t) = \log_2[1 + 2\mu_T], \tag{9}$$

μ_T is the negativity of the two superconducting-charge-qubit density matrix, which is equal to the absolute sum of the negative eigenvalues of the partial transpose matrix $(\hat{\rho}(T))^{T_{A/B}} = [t_{ij,mn}]$ of the two superconducting-charge-qubit density $\hat{\rho}(T)$ with respect to qubit A/B. The elements of the matrix are given by: $t_{ij,mn} = \langle m, j|\hat{\rho}(T)|i, n\rangle$. For the disentangled states, the log-negativity function value was $N(t) = 0$. The log-negativity function values are $0 < N(t) < 1$ for partially entangled states and $N(t) = 1$ for two maximally entangled superconducting-charge qubit states.

4. Two-superconducting-qubits dynamics

Fig.1 shows the thermal two superconducting-charge-qubits nonlocalities of the BIV nonlocality, the UI-quantum nonlocality, and the LN entanglement against the nondimensional temperature T for the weak mutual two superconducting-charge-qubits coupling energy $J_{AB} = 0.5$ and a small Josephson energy difference $\Delta J_E = |J_A - J_B| = 0.05$ corresponding to the different values of Josephson A-qubit energy J_A. Energy is typically expressed in units of μeV or frequency (GHz), while temperature is given in mK [64]. However, in our work, all parameters shown in the figures are presented in dimensionless units. Fig.1a shows the thermal two superconducting-charge-qubits nonlocalities of the BIV nonlocality, the UI-quantum nonlocality, and the LN entanglement corresponding to the Josephson A-qubit energy $J_A = 0.5$ with the weak mutual two superconducting-charge-qubits coupling energy $J_{AB} = 0.5$ and a small Josephson energy difference $\Delta J_E = |J_A - J_B| = 0.05$. It was found that for the case where the weak mutual two superconducting-charge-qubit coupling energy and the Josephson SCC-qubit energies (the difference between the Josephson SCC-qubit energies is very small $\Delta J_E = |J_A - J_B| = 0.05$, the relatively weak two superconducting-charge-qubit interactions have a high ability to generate variations in the BIV nonlocality, the UI-quantum nonlocality, and the LN entanglement for the thermal two superconducting-charge-qubit states with temperature.

At low temperatures, the values of the BIV nonlocality, UI-quantum nonlocality, and LN entanglement remain constant at their maximum ($V(t) = 2\sqrt{2} - 2$ and $U(t) = N(t) = 1$). They are temperature-independent for a particular temperature interval. Subsequently, owing to the thermal fluctuations in the two superconducting charge qubit system, the BIV nonlocality, UI-quantum nonlocality, and LN entanglement decreased as the temperature increased.

At higher temperatures, they vanished and decayed exponentially. Beyond a certain temperature, the BIV nonlocality and LN entanglement suddenly vanish, which is known as the sudden death of the thermal two superconducting-charge-qubit phenomenon. BIV nonlocality sudden death (BNSD) occurs (e.g. $BIV < 10^{-8}$), whereas entanglement sudden death (ESD) occurs (e.g. $LN < 10^{-8}$). However, some states exhibit Bell-nonlocality sudden death (BNSD), in which nonlocality vanishes in finite time, even when state coherence persists [65]. Theoretically, the entanglement and Bell nonlocality of a bipartite entangled state under three types of decoherence are amplitude damping, phase damping,

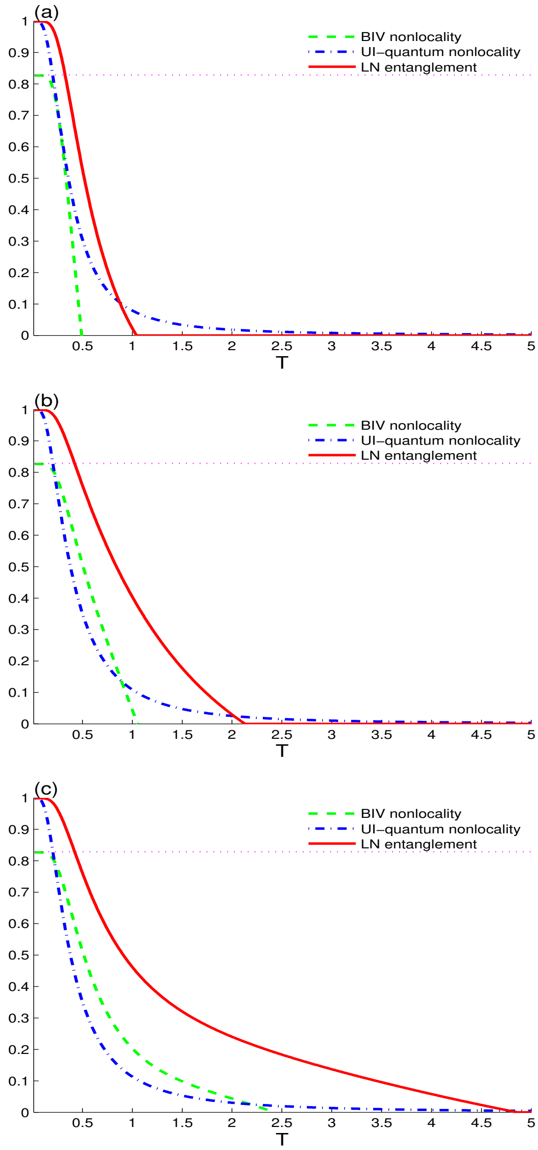

Figure 1: Thermal BIV nonlocality, UI-quantum nonlocality, and LN entanglement are shown against the nondimensional temperature T with the weak mutual two-SCC-qubits coupling energy $J_{AB} = 0.5$. A small Josephson energy difference $\Delta J_E = |J_A - J_B| = 0.05$ corresponding to the Josephson A-qubit energies J_A: $J_A = 0.5$ in (a), $J_A = 3$ in (b), and $J_A = 11$ in (c).

and depolarizing [66]. Tripartite Bell-inequality violations can be completely eliminated in finite time due to decoherence caused by coupling with the spin environment [67]. In addition, entanglement of a quantum state disappears over a finite time due to the decay of its parts [68–70]. This depends on the initial conditions [71], and has been observed in photon-atom pairs. Conversely, [72] predicts delayed entanglement, showing a phenomenon opposite to that of entanglement sudden death.

Fig.1b shows that as the temperature T increases, the nonlocality and entanglement of the thermal two-superconducting qubits decrease. However, this decrease can be reduced by increasing the Josephson energy $J_k(k = A, B)$ of the $k-$qubit. This adjustment improves the resilience of the nonlocalities and entanglement between the thermal two superconducting qubits against temperature. When comparing 1b and 1a, it was found that BIV nonlocality, UI-quantum nonlocality, and LN entanglement exhibited the same general behavior at the same temperature. It was also observed that BIV nonlocality sudden death (BNSD) (e.g. $BIV < 10^{-8}$) and entanglement sudden death (ESD) (e.g. $LN < 10^{-8}$) occurred at higher temperatures than in Fig.1a. This means that increasing the Josephson $k-$qubit energy delays the sudden death phenomenon. Fig.1c confirms the increase in the Josephson $k-$qubit energies $J_k(k = A, B)$, such that the difference between them is a slight difference ($\Delta J_E = |J_A - J_B| = 0.05$), which has a high ability to resist the degradation temperature. It is clear from Fig.1 that the BIV nonlocality temperature interval is always smaller than the LN entanglement, which is smaller than that of the UI-quantum nonlocality, which is in turn greater than that of the BIV nonlocality at the same temperature, i.e., the UI-quantum nonlocality is more robust against bath temperature than the BIV nonlocality and the LN entanglement. Furthermore, increasing J_k requires a higher temperature to decrease and subsequently causes the BIV nonlocality, UI-quantum nonlocality, and LN entanglement to vanish. Moreover, as the $k-$qubit Josephson energy J_k increases, the temperatures at which the BIV nonlocality, UI-quantum nonlocality, and LN entanglement vanish also increase and delay the sudden death phenomenon.

In Fig.2, the thermal nonlocalities of Fig.1 are plotted, but for a large Josephson-qubit energy difference $\Delta J_E = |J_A - J_B| = 3$ corresponded to $J_A = 4$ in (a), $J_A = 11$ in (b).

As shown in Fig.2a, when the Josephson A-qubit energy $J_A = 4$, as the temperature increases, the BIV nonlocality, UI-quantum nonlocality, and LN entanglement decrease, eventually disappearing at higher temperatures. The two nonlocal superconducting-charge-qubit correlations exhibited the same decay pattern. The BIV nonlocality and LN entanglement steadily decreased as thermal fluctuations increased. At a certain temperature, BIV nonlocality and LN entanglement suddenly vanish, indicating the occurrence of a sudden-death phenomenon. The temperature at which the BIV nonlocality sudden death (e.g. $BIV < 10^{-8}$) entanglement sudden death (e.g. $LN < 10^{-8}$) occurs depends on the Josephson-qubit $J_k(k = A, B)$ energies and the difference between them.

In Fig.2b, the impact of increasing the Josephson A-qubit energy J_A to $J_A = 11$ on the BIV nonlocality, UI-quantum nonlocality, and LN entanglement is examined. It is evident that the temperature interval of the existing UI-quantum nonlocality is always greater than that of LN entanglement, which is greater than that of BIV nonlocality at the same temperature. Furthermore, as the k-qubit Josephson energy J_k increases, the temperatures at which BIV nonlocality, UI-quantum nonlocality, and LN entanglement increase, and the delay in the occurrence of sudden death. As the temperature T and Josephson A-qubit energy J_A increase, the BIV nonlocality, UI-quantum nonlocality, and LN entanglement gradually decrease. The large Josephson-qubit energy difference $\Delta J_E = |J_A - J_B|$ improved the degradation temperature for the two nonlocal correlations of superconducting-charge qubits.

Fig.3 illustrates the thermal BIV nonlocality, UI-quantum nonlocality, and LN entan-

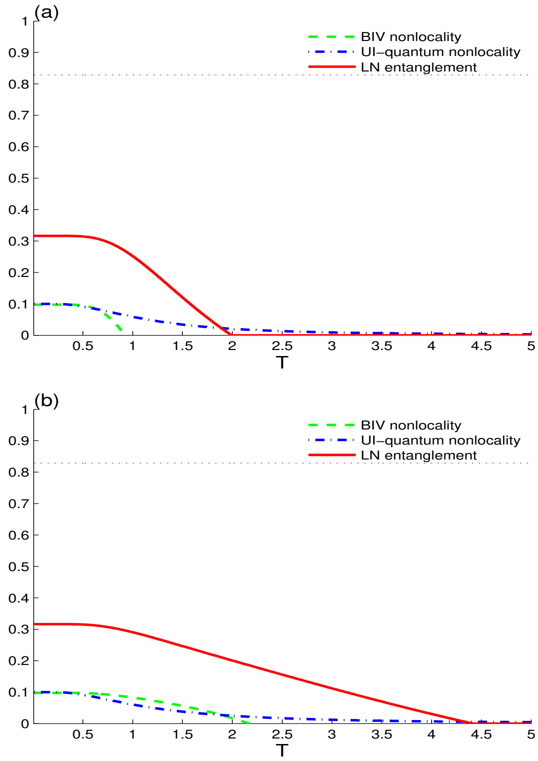

Figure 2: Thermal nonlocalities as Fig.1 are plotted but for a large Josephson energy difference $\Delta J_E = |J_A - J_B| = 3$ corresponded to $J_A = 4$ in (a), $J_A = 11$ in (b).

glement in Fig.1 for large mutual two-SCC-qubits couplings: $J_{AB} = 1$ in (a) and $J_{AB} = 8$ in (b). In Fig.3(a), the results show that the values of BIV nonlocality, UI-quantum nonlocality, and LN entanglement decreased as the temperature increased, eventually ceasing to exist at higher temperatures. It is important to note that all three values follow a similar exponential decay pattern. Furthermore, the sudden disappearance of BIV nonlocality and LN entanglement at a specific critical temperature indicate a sudden-death phenomenon.

By comparing the two-SCC-qubits nonlocalities in Fig.1a and Fig.3a, we find that the thermal interval of the existing maximal and partial BIV nonlocality, LN entanglement, and UI-quantum nonlocality are enhanced by increasing the mutual couplings of the two SCC qubits, which delays the higher temperatures at which BIV nonlocality sudden death (e.g. $BIV < 10^{-8}$) and entanglement sudden death (e.g. $LN < 10^{-8}$) occur.

Fig.3b illustrates how increasing the mutual coupling J_{AB} energy between two SCC qubits can improve their ability to maintain BIV nonlocality, UI-quantum nonlocality, and LN entanglement in the presence of thermal fluctuations. The results show that as the mutual coupling energy between the two-SCC-qubits increases, the amplitudes of BIV nonlocality, UI-quantum nonlocality, and LN entanglement also increase. This suggests that a stronger mutual coupling between qubits can enhance their ability to preserve these important quantum properties under thermal effects. When the mutual coupling energy

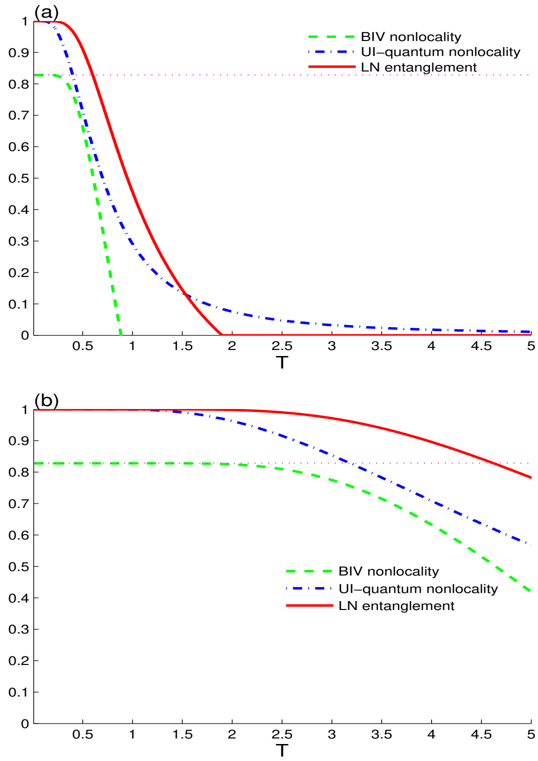

Figure 3: Thermal two-SCC-qubit nonlocalities as Fig.1a are plotted but for large mutual two-SCC-qubits couplings: $J_{AB} = 1$ in (a) and $J_{AB} = 8$ in (b).

J_{AB} increased, the BIV nonlocality, UI-quantum nonlocality, and the LN entanglement decreased slowly with increasing temperature. At low temperatures, the sudden death phenomenon disappeared, and BIV nonlocality, UI-quantum nonlocality, and LN entanglement reached their maximum values. We can deduce that mutual two-SCC-qubits coupling enhances the robustness (against thermal fluctuations), preservation, and thermal interval of the existing maximal and partial BIV nonlocality, LN entanglement, and UI-quantum nonlocality. Increasing the two-SCC-qubit coupling delays the temperatures at which BIV nonlocality sudden death (e.g. $BIV < 10^{-8}$) and entanglement sudden death (e.g. $LN < 10^{-8}$) occur.

Fig.4 illustrates the thermal BIV nonlocality, UI-quantum nonlocality, and LN entanglement against the increase in the Josephson A-qubit energy J_A at temperature $T = 0.5$ for different cases of (J_B, J_{AB}). Fig. 4a depicts the thermal evolution of the two-SCC-qubits nonlocalities for the weak Josephson energy and mutual energy coupling with specific values of $((J_B, J_{AB}) = (0.5, 0.5)$.

In Fig. 4(a), comparing the BIV nonlocality, UI-quantum nonlocality, and LN entanglement reveals striking differences in behavior.

When we explore the UI-quantum nonlocality and LN entanglement, we observe that both exhibit striking behavior. In particular, UI-quantum nonlocality and LN entanglement are maximized when the parameter J_A is zero, while BIV nonlocality reaches

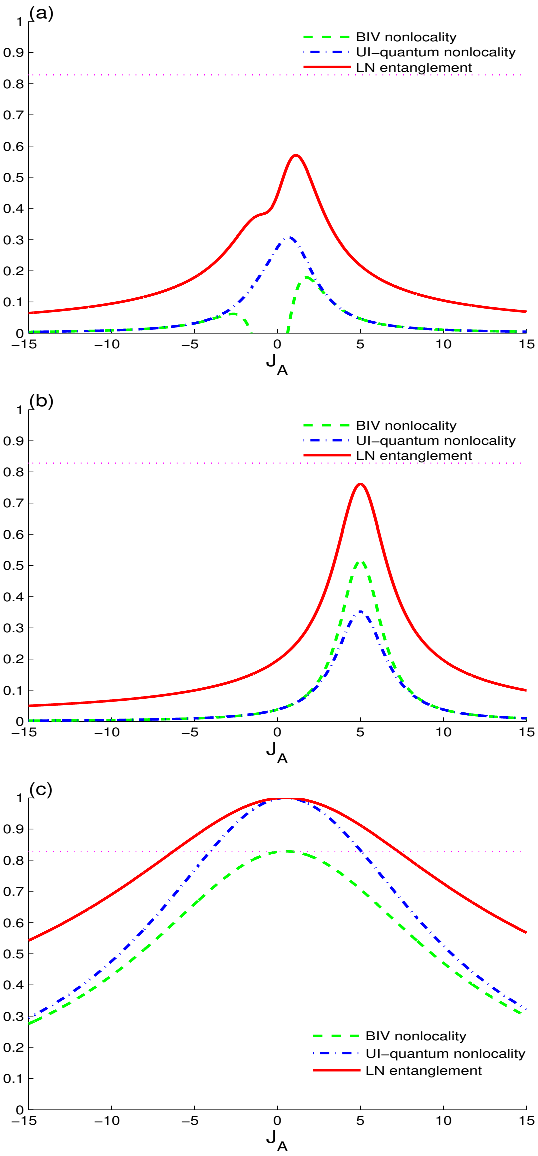

Figure 4: Thermal BIV nonlocality, UI-quantum nonlocality, and LN entanglement are shown against the increase of the Josephson A-qubit energy at the temperature $T = 0.5$ for different cases of (J_B, J_{AB}): $(J_B, J_{AB}) = (0.5, 0.5)$ in (a), $(J_B, J_{AB}) = (5, 0.5)$ in (b), and $(J_B, J_{AB}) = (0.5, 5)$ in (c).

its minimum value. Moreover, they both exhibit nearly symmetric patterns around this critical point. In addition, as J_A approaches infinity, BIV nonlocality, UI-quantum nonlocality, and LN entanglement decrease to zero. When the value of J_A increased while J_{AB} remained constant, as shown in Fig. 4 (b), we noticed a significant change in the behavior of the BIV nonlocality. The behavior of BIV nonlocality approaches that of UI-quantum nonlocality and LN entanglement, resembling a normal distribution. The point of symmetry shifted to the right at approximately $J_A = 5$.

As the value J_A approaches infinity, BIV nonlocality, UI-quantum nonlocality, and LN entanglement decrease towards zero. Fig. 4(c) shows the impact of increasing the parameter J_{AB} on the nonlocality of the BIV with the nonlocality of UI-quantum and LN entanglement. It was observed that the BIV nonlocality, UI-quantum nonlocality, and the entanglement of LN are symmetric around $J_A = 0$ and reach their maximum value at this point. As the magnitude of J_A increases, BIV nonlocality, UI-quantum nonlocality, and LN entanglement gradually decrease. It is evident that BIV nonlocality, UI-quantum nonlocality, and LN entanglement require a large value of $|J_A|$ to approach zero, which occurs as $|J_A|$ approaches infinity. Fig. 5 investigate The BIV nonlocality, UI-quantum nonlocality, and LN entanglement against the increase in the mutual two-SCC-qubits coupling energy J_{AB} at the temperature $T = 2$ for a small Josephson energy difference $\Delta J_E = 0.05$ corresponded to $J_A = 0.5$ in (a) and $J_A = 5$ in (b). However, in (c), for a large Josephson energy difference, $\Delta J_E = 3$ corresponds to $J_A = 8$. In Fig. 5a, we can see the behavior of BIV nonlocality, UI-quantum nonlocality, and LN entanglement at a weak Josephson A-qubit energy of $J_A = 0.5$. When $|J_{AB}| \geq 10$, all three measures reached a constant maximum value. BIV nonlocality achieves its highest possible value during this interval, while UI-quantum nonlocality and LN entanglement are equal and at their maximum. On the other hand, when $|J_{AB}| \leq 10$, there was a change in the behavior of these measures. They start to decrease from $J_{AB} \approx -10$, reaching their lowest value at $J_{AB} = 0$, before increasing again and reaching their maximum at $J_{AB} \approx 10$. There are symmetries around $J_{AB} = 0$, and a sudden death phenomenon occurs for both BIV nonlocality and LN entanglement, where the sudden death interval of the BIV nonlocality is greater than that of entanglement. With an increase in J_A to 5, as shown in Fig. 5b, it can be seen that during the interval $| J_{AB} | \geq 10$, BIV nonlocality, UI-quantum nonlocality, and LN entanglement have a constant value, which is the largest possible value, as well as symmetry can be observed around $J_{AB} = 0$, but during the interval $| J_{AB} | \leq 10$, which is the interval of change in the nonlocality behaviors and their behavior is similar to their behavior in Fig. 5a. It can also be observed that sudden death still occurs, but its interval of occurrence is shorter than that of 5a.

As J_A continued to increase, as shown in Fig. 5c, we observed that when the absolute value of J_{AB} was greater than or equal to 15, the values of BIV nonlocality, UI-quantum nonlocality, and LN entanglement remained constant and at their maximum value. During this interval of change ($| J_{AB} | \geq 15$), the behavior of BIV nonlocality, UI-quantum nonlocality, and LN entanglement is similar to their behavior as shown in Fig. 5b, and the disappearance of the phenomenon of sudden death. It is evident that the range of change in the behavior of Fig. 5b expands as J_A increases. Additionally, for large values of J_{AB}, there is a plateau in the values of the two SCC-qubits' nonlocalities at their maximum values.

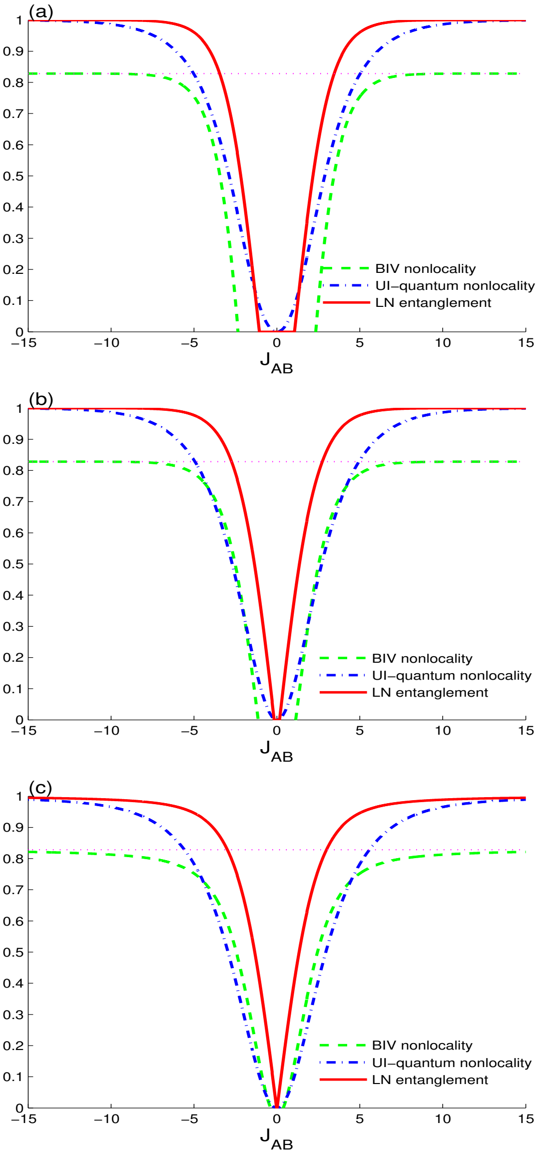

Figure 5: Thermal BIV nonlocality, UI-quantum nonlocality, and LN entanglement are shown against the increase of the mutual two-SCC-qubits coupling energy J_{AB} at the temperature $T = 2$ for a small Josephson energy difference $\Delta J_E = 0.05$ corresponded to $J_A = 0.5$ in (a) and $J_A = 5$ in (b). But in (c), for a large Josephson energy difference, $\Delta J_E = 3$ corresponded to $J_A = 8$.

5. Conclusion

This study explores the emergence of couplings between thermal equilibrium and two superconducting charge qubits and their effects on the behavior of two-qubit inequalities (BIV nonlocality and UI-quantum nonlocality) in thermal equilibrium and in their presence. We can prevent noise-induced losses in BIV nonlocality, UI-quantum nonlocality, and LN entanglement. The robustness of the generated nonlocalities against the bath temperature is stronger and more persistent for the strong mutual two-SCC-qubit coupling energy of the Josephson energy difference. The UI-quantum nonlocality is more robust against bath temperature than the BIV nonlocality and the LN entanglement.

The temperature interval with the BIV nonlocality is less than that of the LN entanglement, which is smaller than the UI-quantum nonlocality.

The increase in Josephson-qubit energies provides strong resistance to degradation at elevated temperatures. Increasing Josephson-qubit energies requires a higher temperature to decrease and subsequently causes both the two-superconducting-qubit nonlocalities to vanish.

Increasing the temperature T and Josephson-qubit energies, which have a large difference, gradually decreases the two superconducting-qubit nonlocalities. Higher temperatures cause delays in the temperature at which the BIV nonlocality sudden death (e.g. $BIV < 10^{-8}$) and entanglement sudden death (e.g. $LN < 10^{-8}$) occur. The large Josephson-qubit energy difference enhanced the degradation temperature for the two-superconducting-qubit nonlocal correlations. Increasing the mutual two-SCC-qubit couplings enhances the thermal interval of the existing maximal and partial two-superconducting-qubit nonlocalities, which delays the higher temperatures at which the BIV nonlocality sudden death (e.g. $BIV < 10^{-8}$) and entanglement sudden death (e.g. $LN < 10^{-8}$) occur. The symmetric dependence of the thermal two-SCC-qubits nonlocality on a Josephson-qubit energy can be enhanced by increasing the other Josephson-qubit energy and the mutual two-SCC-qubits coupling. However, the symmetrical dependence of the thermal nonlocality on the mutual two-SCC-qubit coupling can be controlled by the Josephson-qubit energies and their difference.

The dependence of the thermal two-SCC-qubits nonlocality on the Josephson-qubit energy and the mutual coupling highlights the BIV nonlocality sudden death and the entanglement sudden death.

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