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# Hankel Determinants of 2-Fold and 3-Fold Symmetric Functions and Zalcman Functional for Novel Subclasses of Analytic Functions

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**Abstract.** This study focuses on certain subclasses of analytic functions connected with strongly Janowski-type functions that transform the open unit disc onto different circular symmetric domains. Sharp estimates for Hankel determinants of both 2-fold and 3-fold symmetric functions and exact Zalcman functional results are established for certain subclasses of analytic functions, refining earlier results and providing valuable insights into their underlying structure. The outcomes enrich existing knowledge and refine earlier results.

**2020 Mathematics Subject Classifications**: 30C45, 30C50, 30C80
**Key Words and Phrases**: Analytic functions, symmetric functions, convolution, hankel determinant, zalcman functional, subordination, strongly janowski type function

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## 1. Introduction

Let $\mathfrak{A}$ represent the class of normalized analytic functions $f$ in an open unit disc $\mathscr{U} = \{z \in \mathbb{C} : |z| < 1\}$. Taylor series representation for class $\mathfrak{A}$ is given as

$$f(z) = z + \sum_{n=2}^{\infty} d_n z^n, \quad for\ all\ z \in \mathscr{U}. \tag{1}$$

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An analytic function is said to be univalent if it is one-to-one, that is $f(z_1) = f(z_2)$ implies $z_1 = z_2$ for all $z_1, z_2 \in \mathscr{U}$. Let $\mathcal{S}$ denote the subclass of $\mathfrak{A}$ that contains all univalent functions in $\mathscr{U}$. The widely recognized subclasses of $\mathcal{S}$ are the classes of convex, starlike and bounded turning functions represented as $\mathcal{C}$, $S^*$, and $\mathscr{R}$ respectively.
The concept of subordination between analytic functions was given by Lindelof [1]. Furthermore, Littlewood [2] and Rogosinski [3, 4] introduced fundamental results involving subordination. The $f$ is subordinate to $g$ denoted as $f \prec g$, if there exists a Schwarz function $\mathfrak{h}$ with $\mathfrak{h}(0) = 0$ and $|\mathfrak{h}(0)| < 1$ for all $z \in \mathscr{U}$ such that $f(z) = g(\mathfrak{h}(z))$. Furthermore, if $g(z)$ is univalent in $\mathscr{U}$, then $f(0) = g(0)$ and $f(\mathscr{U}) \subset g(\mathscr{U})$.
Caratheodory functions play a crucial role in Geometric Function Theory, enabling the exploration of various new subclasses of univalent functions. The Caratheodory class $\wp$ consist of functions $\mathfrak{h}(z)$ that map $\mathscr{U}$ on right half plane, satisfying $\mathfrak{h}(0) = 1$ and $\Re(h(z)) > 0$ with Taylor series representation

$$\mathfrak{h}(z) = 1 + \sum_{n=1}^{\infty} r_n z^n, \quad for\ all\ z \in \mathscr{U}. \tag{2}$$

The class of bounded turning functions $\mathscr{R}(\varphi(z))$ introduced in [5] is the subclass of analytic functions defined as

$$\mathscr{R}(\varphi(z)) = \left\{ f \in \mathcal{S} : \quad f'(z) \prec \varphi(z) \quad z \in \mathscr{U} \right\}. \tag{3}$$

The class of alpha-convex functions, denoted by $\mathscr{M}(\alpha, \varphi(z))$ where $\alpha \in \Re$, studied in [6] is defined as

$$\mathscr{M}(\alpha, \varphi(z)) = \left\{ f \in \mathcal{S} : \quad (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)} \prec \varphi(z) \quad for\ all\ z \in \mathscr{U} \right\}. \tag{4}$$

Several well-known classes are obtained from this class such that, for $\alpha = 0$ and $\alpha = 1$ the classes $\mathcal{C}$ and $S^*$ are obtained. [6].
Another interesting subclass is the class of starlike function with respect to symmetric points was given by Ravichandran in [7] and denoted as $S_s^*(\varphi)$. The class $S_s^*(\varphi)$ is defined as

$$S_s^*(\varphi) = \left\{ f \in \mathcal{S} : \quad \frac{2f'(z)}{f(z) - f(-z)} \prec \varphi(z), \quad for\ all\ z \in \mathscr{U} \right\}. \tag{5}$$

By choosing the suitable analytic function $\varphi(z)$, several well-known families are formed. For example, for $\varphi(z) = \sqrt{1+z}$, (3) we obtained the class of bounded turning associated with the lemniscate of Bernouli defined in [8]. Moreover, if we consider $\varphi(z) = \left(\frac{1+\mathscr{A}(z)}{1+\mathscr{B}(z)}\right)$ with $-1 \leq \mathscr{B} < \mathscr{A} \leq 1$ (1) and (5) we get the Janowski Mocanu type function and Janowski starlike function with respect to symmetric points respectively discussed in [9]. In similar way, many researchers obtained interesting geometric properties for subclasses of analytic functions by taking specific functions as $\varphi(z)$, for details, see [10–13]. Arif et al [14] introduced the Strongly Janowski type function $\varphi(z) = \left(\frac{1+\mathscr{A}(z)}{1+\mathscr{B}(z)}\right)^{\beta}$ where $-1 \leq \mathscr{B} <$

$\mathscr{A} \leq 1$ and $0 < \beta \leq 1$. This function was used to introduce certain important subclasses of analytic function studies by Kanwal et al in [15]. The graphical illustration of strongly Janowski type function for some particular values of $\beta, \mathscr{A}$ and $\mathscr{B}$ is as follows.

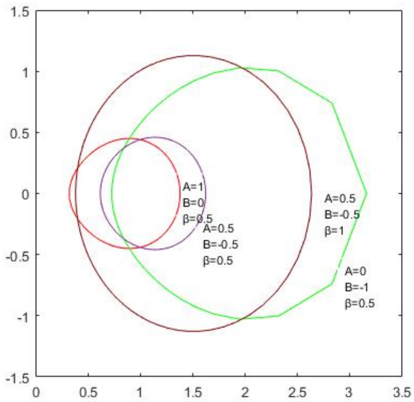

Figure 1: Graphical Illustration of Strongly Janowski type functions

By using strongly Janowski type functions in the classes defined in (3)-(5), we have defined the following subclasses of analytic functions.

**Definition 1.** *A function $f \in \mathfrak{A}$ is considered to belong to a class of bounded turning related to strongly Janowski type function, if and only if*

$$\mathscr{R}(\mathscr{A}, \mathscr{B}; \beta) = \left\{ f \in \mathfrak{A} : \ f'(z) \prec \left( \frac{1 + \mathscr{A} z}{1 + \mathscr{B} z} \right)^{\beta}, \ for \ 0 < \beta \leq 1 \ and \ z \in \mathscr{U} \right\}. \quad (6)$$

**Remark 1.** *In equation (6), for specific values of $\beta$, $\mathscr{A}$ and $\mathscr{B}$ the class $\mathscr{R}(\mathscr{A}, \mathscr{B}; \beta)$ reduces to the following form*
*(i) The class of bounded Janowski turning function is obtained when $\beta = 1$.*
*(ii) The class of bounded function function [16] is obtained when $\beta = 1$, $\mathscr{A} = 1$ and $\mathscr{B} = -1$.*
*(iii) The class of bounded turning function aswith theated with the Lemniscate of Bernoulli is obtained when $\beta = \frac{1}{2}$, $\mathscr{A} = 1$ and $\mathscr{B} = 0$.*

**Definition 2.** *The class of strongly Janowski Mocanu type function $\mathscr{M}(\mathscr{A}, \mathscr{B}; \beta, \alpha)$ for $f \in \mathfrak{A}$ is defined as*

$$\mathscr{M}(\mathscr{A}, \mathscr{B}; \beta, \alpha) = \left\{ f \in \mathfrak{A} : \ (1 - \alpha) \frac{z f'(z)}{f(z)} + \alpha \frac{(z f'(z))'}{f'(z)} \prec \left( \frac{1 + \mathscr{A} z}{1 + \mathscr{B} z} \right)^{\beta}, \quad for \ all \ z \in \mathscr{U} \right\}. \quad (7)$$

*where where $\alpha \in \Re$, $0 < \beta \leq 1$ and $\mathscr{M}(\alpha) = (1 - \alpha) \frac{z f'(z)}{f(z)} + \alpha \frac{(z f'(z))'}{f'(z)}$, is of beta convex function. For $\alpha = 0$, the class $\mathscr{M}(\alpha)$ reduces to strongly Janowski type starlike function, and for $\alpha = 1$, the class reduces to strongly Janowski type convex function.*

**Remark 2.** *In equation (7) by giving some particular values to $\beta$, $\mathscr{A}$ and $\mathscr{B}$ , the class $\mathscr{M}(\mathscr{A},\mathscr{B};\beta,\alpha)$ is reduced to following form:*
*(i) The class of Janowski Mocanu type function is obtained for $\beta = 1$.*
*(ii) Choosing $\beta = 1$, $\mathscr{A} = 1$ and $\mathscr{B} = -1$, the class of Mocanu type function is obtained.*
*(iii) The class of Mocanu type function associated with Lemniscate of Bernoulli function is obtained [8] if $\beta = \frac{1}{2}$, $\mathscr{A} = 1$ and $\mathscr{B} = 0$.*

**Definition 3.** *The class of strongly Janowski type starlike function with respect to symmetric points is defined as*

$$\mathscr{S}_s^*(\mathscr{A},\mathscr{B};\beta) = \left\{ f \in \mathfrak{A} : \frac{2zf'(z)}{f(z)-f(-z)} \prec \left(\frac{1+\mathscr{A}z}{1+\mathscr{B}z}\right)^{\beta}, \quad for\ all\ z \in \mathscr{U} \right\}. \tag{8}$$

**Remark 3.** *Giving some particular values to $\beta$, $\mathscr{A}$ and $\mathscr{B}$ in equation (8), the class $\mathscr{S}_s^*(\mathscr{A},\mathscr{B};\beta)$ is reduced to following form:*
*(i) For $\beta = 1$, we obtained the class of Janowski starlike function with respect to Symmetric point. [9] (ii) The class of Starlike function with respect to Symmetric points is obtained for $\beta = 1$, $A = 1$ and $B = -1$.*
*(iii) For $\beta = \frac{1}{2}$, $A = 1$ and $B = 0$, the obtained class of starlike functions with respect to the Symmetric point associated with Lemniscate of Bernoulli.*

The investigation of Hankel determinants for different classes of analytic functions started in the 1960s. Pommerenke [17, 18] introduced the Hankel Determinant $H_{\mathtt{q},n}(f)$ with the perimeters $\mathtt{q}$, *and* $n \in \aleph = \{1, 2, 3...\}$ for $f(z) \in \mathscr{S}$ is defined by

$$|H\mathtt{q},n(f)| = \begin{vmatrix} r_n & r_{n+1} & \cdots & r_{n+\mathtt{q}-1} \\ r_{n+1} & r_{n+2} & \cdots & r_{r+\mathtt{q}} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ r_{n+\mathtt{q}-1} & r_{n+\mathtt{q}} & \cdots & r_{n+2\mathtt{q}-2} \end{vmatrix},$$

where $n$ and $\mathtt{q}$ are fixed positive integers. Hankel determinant played an important role in studying the power series of integral coefficient [19]. Different Hankel determinants will be obtained when different values of $\mathtt{q}$ and $n$ are chosen. For $q = 2$ and $n = 1$, the determinant is

$$|H2,1(f)| = \begin{vmatrix} r_1 & r_2 \\ r_2 & r_3 \end{vmatrix} = |r_3 - r_2^2|, \ where\ r_1 = 1.$$

This is the special case of Fekete Szego functional. The maximum value of $|H2,1(f)|$ for various subclasses was studied by many researchers [20–23]. Jateng et al investigated the second Hankel determinant for $q$ and $n$,

$$H_{2,2}(f) = r_2r_4 - r_3^2.$$

Furthermore, for $\mathsf{q} = 3$ and $n = 1$

$$H_{3,1}f = r_1r_3r_5 + 2r_2r_3r_4 - r_3^3 - r_2^2r_5 - r_1r_4^2.$$

Babola [24] was the first to study the upper bounds of the third Hankel determinant. Many other researchers have also studied the third Hankel determinant for different subclasses of $\mathcal{S}$ for details, see [25–28].

Many authors have investigated the results of the Hankel determinant for 2-fold and 3-fold symmetric functions [29–31]. Hankel determinants of multi-fold symmetric functions are useful in image processing as they can be used to capture symmetries and periodic patterns, leading to improved accuracy and efficiency in pattern recognition tasks.

The domain $\nabla$ is m-fold symmetric if a rotation of $\nabla$ about the origin through an angle $\frac{2\pi}{m}$ carries $\nabla$ on itself. The study of $m$ fold symmetric functions was initiated by Golusin, Noshiro [32]and Robertson. An analytic function is m-fold symmetric in $\mathscr{U}$, if

$$f(\exp^{\frac{2\pi i}{m}} z) = \exp^{\frac{2\pi i}{m}} f(z),$$

The set of m-fold symmetric univalent functions $\mathcal{S}$, denoted by $\mathcal{S}^m$, has the following Taylor series form:

$$f(z) = z + \sum_{n=1}^{\infty} d_{mn+1} z^{mn+1} \; z \in \mathscr{U}. \tag{9}$$

The class of Caratheodory functions $\wp$ in term of $m$-fold symmetric function is defined as

$$\wp^m = \{\mathfrak{h} \in \wp : \mathfrak{h}(z) = 1 + \sum_{n=1}^{\infty} r_{mn} z^{mn}\}. \tag{10}$$

## 2. A Set of Lemmas

**Lemma 1.** *[33] Let $\mathfrak{h} \in \wp$ and it has the form equation (2), then*

$$|c_k| \leq 2 \text{ for } k \geq 1, \tag{11}$$

$$|c_{n+k} - \mu r_n c_k| < 2 \text{ for } 0 \leq \mu \leq 1, \tag{12}$$

**Lemma 2.** *[7] Let $\mathsf{q}$, $s$, $\mathrm{m}$, and $\mathrm{n}$ satisfy the condition $0 < \mathrm{m} < 1$, $0 < s < 1$ and*

$$8s(1-s)\left[(\mathrm{mn} - 2\mathsf{q})^2 + (\mathrm{m}(s+\mathrm{m}) + \mathrm{n})^2\right] + \mathrm{m}(1 - \mathrm{m}(\mathrm{n} - 2s\mathrm{m})^2 \leq \quad 4\mathrm{m}^2(1-\mathrm{m})^2 s(1-s).$$

*If $\mathfrak{h} \in \wp$ and it has the form in equation (2), then*

$$|\mathsf{q}{r_1}^4 + sr_2^2 + 2\mathrm{m}r_1r_3 - \frac{3}{2}\mathrm{n}r_1^2r_2 - r_4| \leq 2. \tag{13}$$

## 3. Main Results

### 3.1. Bounds of $|H_{3,1}f(z)|$ for two and three fold symmetric functions

The third-order Hankel determinant, denoted $|H_{3,1}f(z)|$, plays a significant role in the study of analytic functions, particularly in geometric function theory; for details, see [34–38] . The bounds of $|H_{3,1}f(z)|$ of 2,3-fold symmetric become essential in understanding the growth and distortion properties of these functions. The exploration of such bounds offers insights into the complex structure of symmetric functions and their behavior in relation to the coefficients of their power series expansions, leading to sharper results in the analysis of univalent and multivalent functions.

**Theorem 1.** *If $f(z) \in \mathfrak{A}$ of the form in equation (9), then the following results hold*
*(i) If $f(z) \in \mathscr{R}^2(\mathscr{A}, \mathscr{B}, \beta)$; then*

$$|H_{3,1}f(z) \leq \frac{1}{15}(\beta|\mathscr{A} - \mathscr{B}|)^2.$$

*(ii) If $f(z) \in \mathscr{M}^2(\mathscr{A}, \mathscr{B}; \beta, \alpha)$; then*

$$|H_{3,1}f(z)| \leq \frac{1}{8(1+2\alpha)(1+4\alpha)}(\beta|\mathscr{A} - \mathscr{B}|)^2.$$

*(iii) If $f(z) \in \mathscr{S}_s^{2*}(\mathscr{A}, \mathscr{B}; \beta)$; then*

$$|H_{3,1}f(z) \leq \frac{1}{8}(\beta|\mathscr{A} - \mathscr{B}|)^2.$$

*Proof.* (i) Let $f \in \mathscr{R}(\mathscr{A}, \mathscr{B}; \beta)$. Then there exist a function $\mathfrak{h} \in \wp^2$, such that

$$f'(z) = \left(\frac{1 + \mathscr{A}\mathfrak{h}(z)}{1 + \mathscr{B}\mathfrak{h}(z)}\right)^{\beta}. \tag{14}$$

using the form of $f(z)$ for 2-fold symmetric functions, equation (9) gives

$$f'(z) = 1 + 3d_3z^2 + 5d_5z^4 + \dots . \tag{15}$$

Expanding the right hand side of eq 14, we get

$$\left(\frac{1 + \mathscr{A}\mathfrak{h}(z)}{1 + \mathscr{B}\mathfrak{h}(z)}\right)^{\beta} = 1 + \frac{r_2\Gamma_1}{2}z^2 + \left(\frac{r_4}{2} + \frac{r_2^2}{4}[\Gamma_2 - \Gamma_1]\right)z^4 + \dots . \tag{16}$$

Where,

$$\Gamma_1 = \beta(\mathscr{A} - \mathscr{B}),$$
$$\Gamma 2 = \frac{(\beta^4 - 6\beta^3 + 11\beta^2 - 6\beta)}{24}\mathscr{A}^4 + \frac{(\beta^4 + 6\beta^3 + 11\beta^2 + 6\beta)}{24}\mathscr{B}^4 -$$

$$\frac{(\beta^4-3\beta^3+2\beta^2)}{6}\mathscr{A}\mathscr{B}^3+\frac{(-\beta^4+3\beta^3-2\beta^2)}{6}\mathscr{A}^3\mathscr{B}+\frac{(\beta^4-\beta^2)}{4}\mathscr{A}^2\mathscr{B}^2.$$

Comparing equation (15) and equation (16), we obtain

$$d_3=\frac{1}{6}r_2\Gamma_1,$$
$$d_5=\frac{1}{10}r_4\Gamma_1+\frac{1r_2^2}{20}(\Gamma_2-\Gamma_1).$$

There for third Hankel determinant is:

$$H_{3,1}f(z)=d_3d_5-d_3^3=\frac{1}{60}r_2\Gamma_1^2\left(r_4+r_2^2(\frac{\Gamma_2-\Gamma_1}{2\Gamma_1^2}-\frac{5}{9}\Gamma_1)\right),$$
$$|H_{3,1}f(z)|\leq\frac{|r_2|}{60}|\Gamma_1^2||\left(r_4+r_2^2[\frac{\Gamma_2-\Gamma_1}{2\Gamma_1^2}-\frac{5}{9}\Gamma_1]\right)|.$$

Applying equation (11) and equation (12), using value of $\Gamma_1$ we obtain

$$|H_{3,1}f(z)|\leq\frac{1}{15}(\beta|\mathscr{A}-\mathscr{B}|)^2.$$

This completes the proof of part(i)
(ii) Let $f(z)\in\mathscr{M}^2(\mathscr{A},\mathscr{B};\beta,\alpha)$. Then there exists a function $\mathfrak{h}\in\wp^2$ such that

$$(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\frac{(zf'(z))'}{f'(z)}=\left(\frac{1+\mathscr{A}\mathfrak{h}(z)}{1+\mathscr{B}\mathfrak{h}(z)}\right)^\beta \tag{17}$$

Using the series from equation (9) and equation (10) for m=2, we have

$$(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\frac{(zf'(z))'}{f'(z)}=1+2d_3(1+2\alpha)z^2+4d_5(1+4\alpha)z^4-2d_3^2(1+8\alpha)z^4. \tag{18}$$

Comparing equation (16) and equation (18), we get

$$d_3=\frac{1}{4(1+2\alpha)}r_2\Gamma_1,\ and$$
$$d_5=\frac{1}{8(1+4\alpha)}r_4\Gamma_1+\frac{1}{16(1+4\alpha)}r_2^2\left(\Gamma_2-\Gamma_1+\frac{(1+8\alpha)}{2(1+2\alpha)^2}\Gamma_1^2\right).$$

*Therefore we have*

$$H_{3,1}f(z)=d_3d_5-d_3^3=\frac{1}{32(1+2\alpha)(1+4\alpha)}r_2r_4\Gamma_1^2+\frac{1}{64}r_2^3\left(\Gamma_1\Gamma_2-\Gamma_1^2+\frac{(1+8\alpha)}{2(1+2\alpha)^2}\Gamma_1^3\right)$$
$$-\frac{1}{64(1+2\alpha)^3}r_2^3\Gamma_1^3,$$
$$|H_{3,1}f(z)|\leq|r_2|\left|\frac{1}{32(1+2\alpha)(1+4\alpha)}\Gamma_1^2\right|\left|r_4-\frac{1}{2}r_2^2\left(-\frac{\Gamma_2}{\Gamma_1}+1-\frac{(1+8\alpha)}{2(1+2\alpha)^2}\Gamma_1+\frac{(1+4\alpha)}{(1+2\alpha)^2}\Gamma_1\right)\right|.$$

Applying equation (11) and equation (12), putting values of $\Gamma_1$ we get

$$|H_{3,1}f(z)| \leq \tfrac{1}{8(1+2\alpha)(1+4\alpha)}(\beta|\mathscr{A}-\mathscr{B}|)^2.$$

This completes the proof of part(ii)
(iii) Let $f(z) \in \mathscr{S}_s^{2*}(\mathscr{A},\mathscr{B};\beta)$. Then there exist a function $\mathfrak{h} \in \wp^2$ such that

$$\frac{2zf'(z)}{f(z)-f(-z)} = \left(\frac{1+\mathscr{A}\mathfrak{h}(z)}{1+\mathscr{B}\mathfrak{h}(z)}\right)^{\beta}. \tag{19}$$

Using the series from equation (9) and taking m=2 in equation (10), we get

$$\frac{2zf'(z)}{f(z)-f(-z)} = 1 + 2d_3z^2 + (4d_5 - 2d_3^2)z^4_{\cdots}. \tag{20}$$

Comparing equation (16) and equation (20), we get

$$d_3 = \frac{1}{4}r_2\Gamma_1, \ and \qquad d_5 = \frac{1}{8}r_4\Gamma_1 - r_2^2(-\Gamma_2 + \Gamma_1 - \frac{1}{2}\Gamma_1^2).$$

Therefore, we have

$$H_{3,1}f(z) = d_3d_5 - d_3^3 = \frac{1}{32}\Gamma_1^2 r_2\left(r_4 - \frac{1}{2}r_2^2[1 - \frac{\Gamma_2}{\Gamma_1} + \frac{1}{2}\Gamma_1]\right)$$

$$|H_{3,1}f(z)| \leq \frac{1}{32}|\Gamma_1^2||r_2|\left|\left(r_4 - \frac{1}{2}r_2^2[1 - \frac{\Gamma_2}{\Gamma_1} + \frac{1}{2}\Gamma_1]\right)\right|.$$

Applying equation (11) and equation (12), by applying value of $\Gamma_1$ we obtained our result.

$$|H_{3,1}f(z)| \leq \frac{1}{8}(\beta|\mathscr{A}-\mathscr{B}|)^2.$$

**Corollary 1.** *If $f(z) \in \mathfrak{A}$ is of the form of equation (1), then for particular values in theorem equation (1) the following results hold*
*(i) For $\beta = 1$, we get $|H_{3,1}f(z)|$ for $\mathscr{R}^2(\mathscr{A},\mathscr{B};1)$, $\mathscr{M}^2(\mathscr{A},\mathscr{B};1,\alpha)$ and $\mathscr{S}_s^{2*}(\mathscr{A},\mathscr{B};1)$ respectively.*
*(ii) For $\beta = 1$, $\mathscr{A} = 1$ and $\mathscr{B} = -1$, we obtain third Hankel determinant for $\mathscr{R}^2(1,-1;1)$, $\mathscr{M}^2(1,-1;1,\alpha)$ and $\mathscr{S}_s^{2*}(1,-1;1)$ respectively.*
*(iii) For $\beta = \frac{1}{2}$, $\mathscr{A} = 1$ and $\mathscr{B} = 0$, we get $|H_{3,1}f(z)|$ for $\mathscr{R}^2(1,0;\frac{1}{2})$, $\mathscr{M}^2(1,0;\frac{1}{2},\alpha)$ and $\mathscr{S}_s^{2*}(1,0;\frac{1}{2})$ respectively.*

**Theorem 2.** *If $f(z) \in \mathfrak{A}$ of the form in equation (9), then the following results hold*
*(i) If $f(z) \in \mathscr{R}^3(\mathscr{A},\mathscr{B};\beta)$; then*

$$|H_{3,1}f(z)| \leq \frac{1}{16}(\beta|\mathscr{A}-\mathscr{B}|)^2.$$

*(ii) If $f(z) \in \mathscr{M}^3(\mathscr{A},\mathscr{B};\beta,\alpha)$, then*

$$|H_{(3,1)}f(z)| \leq \frac{1}{9(1+3\alpha)^2}(\beta|\mathscr{A}-\mathscr{B}|)^2.$$

*(iii) If* $f(z) \in \mathscr{S}_s^{3*}(\mathscr{A}, \mathscr{B}; \beta)$*; then*

$$|H_{3,1}f(z)| \leq \frac{1}{16}(\beta|\mathscr{A} - \mathscr{B}|)^2.$$

*Proof.* (i) Let $f(z) \in \mathscr{R}^3(\mathscr{A}, \mathscr{B}; \beta)$ , therefore there exist a function $\mathfrak{h} \in \wp^3$, such that

$$f'(z) = \left(\frac{1 + \mathscr{A}\mathfrak{h}(z)}{1 + \mathscr{B}\mathfrak{h}(z)}\right)^{\beta}.$$

Taking $m = 3$ in equation (9), we get

$$1 + 4d_4z^3 + 7b_7z^6 +_{...} = 1 + \Gamma_1(\frac{r_3}{2}z^3 + ...) + \Gamma_2(\frac{r_3}{2}z^3 + ...)^2 +_{...} . \tag{21}$$

Comparing the coefficients of eq (21), we get

$$d_4 = \frac{r_3}{4}\Gamma_1,$$
$$|H_{3,1}f(z)| = |-d_4^2| = |\frac{r_3^2}{4}\Gamma_1^2| \leq \frac{|r_3|^2}{64}|\Gamma_1^2|.$$

Applying equation (11), we get

$$|H_{3,1}f(z)| \leq \tfrac{|\Gamma_1^2|}{16}$$

(ii) Let $f(z) \in \mathscr{M}^3(\mathscr{A}, \mathscr{B}; \beta, \alpha)$, therefore there exist function $\mathfrak{h} \in \wp^3$, such that

$$(1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{zf'(z))'}{f'(z)} = \left(\frac{1 + \mathscr{A}\mathfrak{h}(z)}{1 + \mathscr{B}\mathfrak{h}(z)}\right)^{\beta} \tag{22}$$

$$1 + 3d_4z^3(1 + 3\alpha) +_{...} = 1 + \Gamma_1(\frac{r_3}{2}z^3 + ...) + \Gamma_2(\frac{r_3}{2}z^3 + ...)^2 +_{...} . \tag{23}$$

Comparing the coefficients, we obtain

$$d_4 = \frac{1}{6(1 + 3\alpha)}\Gamma_1 r_3,$$
$$|H_{(3,1)}(f)| = |-d_4^2| \leq |\frac{1}{36(1 + 3\alpha)^2}r_3^2\Gamma_1^2| \leq |\frac{1}{36(1 + 3\alpha)}||r_3|^2|\Gamma_1^2|.$$

Applying equation (11), using value of $\Gamma_1$ we get our require result.
(iii) Let $f(z) \in \mathscr{S}_s^{3*}(\mathscr{A}, \mathscr{B}; \beta)$ then there exist function $\mathfrak{h} \in \wp^3$, put m=3 in equation (9 and 10), we get

$$\frac{2zf'(z)}{f(z) - f(-z)} = 4d_4z^3 + ... \tag{24}$$

Comparing coefficient; we get $d_4 = \frac{1}{8}r_3\Gamma_1$

$$|H_{(3,1)}f(z)| = -d_4^2 = \frac{1}{64}r_3^2\Gamma_1^2 \leq \frac{1}{64}|r_3^2||\Gamma_1^2|.$$

Applying Lemma(1), we get

$$|H_{(3,1)}f(z)| \leq \frac{1}{16}|\Gamma_1^2|.$$

**Corollary 2.** *If $f(z) \in \mathfrak{A}$ is of the form of equation (1), then for particular values in theorem equation (1) the following results hold*
*(i) For $\beta = 1$, we get $|H_{3,1}f(z)|$ for $\mathscr{R}^3(\mathscr{A},\mathscr{B};1)$, $\mathscr{M}^3(\mathscr{A},\mathscr{B};1,\alpha)$ and $\mathscr{S}_s^{3*}(\mathscr{A},\mathscr{B};1)$ respectively.*
*(ii) For $\beta = 1$, $\mathscr{A} = 1$ and $\mathscr{B} = -1$, we obtain third Hankel determinant for $\mathscr{R}^3(1,-1;1)$, $\mathscr{M}^3(1,-1;1,\alpha)$ and $\mathscr{S}_s^{3*}(1,-1;1)$ respectively.*
*(iii) For $\beta = \frac{1}{2}$, $\mathscr{A} = 1$ and $\mathscr{B} = 0$, we get $|H_{3,1}f(z)|$ for $\mathscr{R}^3(1,0;\frac{1}{2})$, $\mathscr{M}^3(1,0;\frac{1}{2},\alpha)$ and $\mathscr{S}_s^{3*}(1,0;\frac{1}{2})$ respectively.*

The hypergeometric function is frequently used to construct analytic and univalent functions and to represent extremal functions for various subclasses of analytic functions. The extremal function for 2-fold, 3-fold symmetry includes the hypergeometric function ${}_2F_1$ that is defined as follows

$${}_2F_1 = \sum_{n=0}^{\infty} \frac{(a_n)(n_n)}{(c_n)} \frac{z^n}{n!}$$

where $(a)_n, (b)_n, and\ (c)_n$ are Pochhammer symbols, (a,b,c) are parameters, and $z$ is a complex variable.
Extremal function for 2-fold symmetric function is

$$f(z) = z\ {}_2F_1\ \left(\frac{1}{2}, -\beta, \frac{3}{2}, -\beta z^2\right)\left(\frac{1+\mathscr{A}z^2}{1+\mathscr{B}z^2}\right)^{\beta}.$$

Extremal function for 3-fold symmetric function is

$$f(z) = z\ {}_2F_1\ \left(\frac{1}{2}, -\beta, \frac{4}{3}, -\beta z^2\right)\left(\frac{1+\mathscr{A}z^3}{1+\mathscr{B}z^3}\right)^{\beta}$$

### 3.2. Zalcman Functional

In 1960, Zalcman proposed one of the classical conjectures in the field of geometric function theory which states that the coefficients of functions in the class $\mathscr{S}$ satisfy

$$|d_n^2 - d_{2n-1}| \leq (n-1)^2. \tag{25}$$

This inequality attains equality only for the renowned Koebe function and its rotation. For $n = 2$ the inequality considered with the well known for Fekete Szego's inequality. Later the result has been proved by Kruskhal [39] for the entire class $\mathscr{S}$. He proved this for $n = 2, 3, 4, 5$ and 6. For more detail on the Zalman functional, see[40, 41] and [42].

**Theorem 3.** *If $f(z) \in \mathfrak{A}$ is of the form defined in equation (1), then the following results hold*
*(i) If $f(z) \in \mathscr{R}(\mathscr{A}, \mathscr{B}; \beta)$; then*

$$|d_3^2 - d_5| \leq \tfrac{1}{5}\beta|\mathscr{A} - \mathscr{B}|.$$

*(ii) If $f(z) \in \mathscr{M}(\mathscr{A}, \mathscr{B}; \beta, \alpha)$; then*

$$|d_3^2 - d_5| \leq \tfrac{1}{4(1+5\alpha)}\beta|\mathscr{A} - \mathscr{B}|.$$

*(iii)If $f(z) \in \mathscr{S}_s^*(\mathscr{A}, \mathscr{B}; \beta)$; then*

$$|d_3^2 - d_5| \leq \frac{1}{4}\beta|\mathscr{A} - \mathscr{B}|.$$

*Proof.* (i) Let $f(z) \in \mathscr{R}(\mathscr{A}, \mathscr{B}; \beta)$. Then there exist a Schwarz function $\mathfrak{h}(z)$ such that

$$f'(z) = \left(\frac{1 + \mathscr{A}\mathfrak{h}(z)}{1 + \mathscr{B}\mathfrak{h}(z)}\right)^{\beta}. \tag{26}$$

Let $\mathfrak{h} \in \wp$ then eq3 can be written in the form of Schwarz function as:

$$\mathfrak{h}(z) = \frac{1 + \mathfrak{h}(z)}{1 - \mathfrak{h}(z)} = 1 + r_1 z + r_2 z^2 + r_3 z^3 + \dots . \tag{27}$$

Now from equation (27), we have

$$\begin{aligned}
\mathfrak{h}(z) = \frac{\mathfrak{h}(z) - 1}{\mathfrak{h}(z) + 1} &= \frac{1}{2}r_1 z + \left(\frac{1}{2}r_2 - \frac{1}{4}r_1^2\right) z^2 + \left(\frac{1}{2}r_3 - \frac{1}{2}r_1 r_2 + \frac{1}{8}r_1^3\right) z^3 \\
&+ \left(\frac{1}{2}r_4 - \frac{1}{4}r_2^2 + \frac{3}{8}r_1^2 r_2 - \frac{1}{2}r_1 r_3 - \frac{1}{16}r_1^4\right) z^4 \dots, \\
f'(z) &= 1 + 2d_2 z + 3d_3 z^2 + 4d_4 z^3 + 5d_5 z^4.
\end{aligned} \tag{28}$$

$$\begin{aligned}
&\left(\frac{1 + \mathscr{A}\mathfrak{h}(z)}{1 + \mathscr{B}\mathfrak{h}(z)}\right)^{\beta} = 1 + (\beta\mathscr{A} - \beta\mathscr{B})\mathfrak{h}(z) + \left(\frac{\beta^2 - \beta}{2}\mathscr{A}^2 + \frac{\beta^2 + \beta}{2}\mathscr{B}^2 - \beta^2\mathscr{A}\mathscr{B}\right)(\mathfrak{h}(z))^2 \\
&+ \left(\frac{\beta^3 - 3\beta^2 + 2\beta}{6}\mathscr{A}^3 - \frac{-\beta^3 + \beta^2}{2}\mathscr{A}^2\mathscr{B} + \frac{\beta^3 + \beta^2}{2}\mathscr{A}B^2\right)(\mathfrak{h}(z))^3 \\
&+ \left(\frac{\beta^4 - 6\beta^3 + 11\beta^2 - 6\beta}{24}A^4 + \frac{\beta^4 + 6\beta^3 + 11\beta^2 + 6\beta}{24}B^4 - \frac{\beta^4 + 3\beta^3 + 2\beta^2}{6}AB^3 \right. \\
&\left. + \frac{-\beta^4 + 3\beta^3 - 2\beta^2}{6}A^3 B + \frac{\beta^4 - \beta^2}{4}A^2 B^2\right)(\mathfrak{h}(z))^4 \\
&= 1 + \Gamma_1\mathfrak{h}(z) + \Gamma_2(\mathfrak{h}(z))^2 + \Gamma_3(\mathfrak{h}(z))^3 + \Gamma_4(\mathfrak{h}(z))^4,
\end{aligned}$$

where $\Gamma_1,\ \Gamma_2,\ \Gamma_3$ *and* $\Gamma_4$ are coefficients with $\mathfrak{h}(z),\ (\mathfrak{h}(z))^2,\ (\mathfrak{h}(z))^3$ *and* $(\mathfrak{h}(z)^4)$. Putting value of $\mathfrak{h}(z)$, we get

$$\begin{aligned}\left(\frac{1+\mathscr{A}\mathfrak{h}(z)}{1+\mathscr{B}\mathfrak{h}(z)}\right)^{\beta} &= 1+\frac{1}{2}r_1\Gamma_1 z+\left(\frac{1}{2}r_2\Gamma_1-\frac{1}{4}r_1^2[\Gamma_1-\Gamma_2]\right)z^2+\left(\frac{1}{4}r_1^3[\frac{1}{2}\Gamma_1-\Gamma_2+\frac{1}{2}\Gamma_3]\right.\\ &\left.+\frac{1}{2}r_1r_2[\Gamma_2-\Gamma_1]+\frac{1}{2}r_3\Gamma_1\right)z^3+\left(\frac{1}{16}r_1^4[-\Gamma_1+3\Gamma_2-3\Gamma_3+\Gamma_4]+\frac{1}{2}r_1r_3[\Gamma_2-\Gamma_1]\right.\\ &\left.+\frac{3}{8}r_1^2r_2[\Gamma_1-2\Gamma_2+\Gamma_3]+\frac{1}{4}r_2^2[\Gamma_2-\Gamma_1]+\frac{1}{2}r_4\Gamma_1\right)z^4_{\cdots}.\end{aligned} \tag{29}$$

By comparing equation (29) and eq(28), we get

$$d_2=\frac{1}{4}r_1\Gamma_1, \tag{30}$$

$$d_3=\frac{1}{6}r_2\Gamma_1-\frac{1}{12}r_1^2[\Gamma_1-\Gamma_2], \tag{31}$$

$$d_4=\frac{1}{16}r_1^3[\frac{1}{2}\Gamma_1-\Gamma_2+\frac{1}{2}\Gamma_3]+\frac{1}{8}r_1r_2[\Gamma_2-\Gamma_1]+\frac{1}{8}r_3\Gamma_1, \tag{32}$$

$$\begin{aligned}d_5&=\frac{1}{80}r_1^4[-\Gamma_1+3\Gamma_2-3\Gamma_3+\Gamma_4]+\frac{1}{10}r_1r_3[\Gamma_2-\Gamma_1]+\frac{3}{40}r_1^2r_2[\Gamma_1-2\Gamma_2+\Gamma_3]\\ &+\frac{1}{20}r_2^2[\Gamma_2-\Gamma_1]+\frac{1}{10}r_4\Gamma_1.\end{aligned} \tag{33}$$

Therefore, equation (31) and (33) yields.

$$\begin{aligned}|d_3^2-d_5| &= |\frac{1}{144}r_1^4\left(\Gamma_1^2+\Gamma_2^2-2\Gamma_1\Gamma_2+\frac{9}{5}\Gamma_1-\frac{27}{5}\Gamma_2+\frac{27}{5}\Gamma_3-\frac{9}{5}\Gamma_4\right)+\frac{1}{36}r_2^2\left(\Gamma_1^2-\frac{9}{5}\Gamma_2+\frac{9}{5}\Gamma_1\right)\\ &-\frac{1}{12}r_1^2r_2\left(\frac{1}{3}\Gamma_1^2-\frac{1}{3}\Gamma_1\Gamma_2-\frac{9}{10}\Gamma_1+\frac{9}{5}\Gamma_2-\frac{9}{10}\Gamma_3\right)+\frac{1}{10}r_1r_3(\Gamma_1-\Gamma_1)-\frac{1}{10}r_4\Gamma_1|,\\ &\le\frac{1}{10}|\Gamma_1||\frac{5}{72\Gamma_1}r_1^4\left(\Gamma_1^2+\Gamma_2^2-2\Gamma_1\Gamma_2+\frac{9}{5}\Gamma_1-\frac{27}{5}\Gamma_2+\frac{27}{5}\Gamma_3-\frac{9}{5}\Gamma_4\right)+\frac{5}{18}\Gamma_1r_2^2\left(\Gamma_1^2-\frac{9}{5}\Gamma_2\right.\\ &\left.+\frac{9}{5}\Gamma_1\right)-\frac{3}{2}\frac{5}{9\Gamma_1}r_1^2r_2\left(\frac{1}{3}\Gamma_1^2-\frac{1}{3}\Gamma_1\Gamma_2-\frac{9}{10}\Gamma_1+\frac{9}{5}\Gamma_2-\frac{9}{10}\Gamma_3\right)+\frac{1}{\Gamma_1}r_1r_3(\Gamma_1-\Gamma_1)-r_4|,\end{aligned}$$

Applying the Lemma (13), and by substituting the value of $\Gamma_1$ we obtained our desired result.

(ii) Let $f(z)\in\mathscr{M}(\mathscr{A},\mathscr{B};\beta,\alpha)$ then there exist a Schwarz function $\mathfrak{h}(z)$, such that

$$(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\frac{(zf'(z))'}{f'(z)}=\left(\frac{1+\mathscr{A}\mathfrak{h}(z)}{1+\mathscr{B}\mathfrak{h}(z)}\right)^{\beta}. \tag{34}$$

This implies that

$$(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\frac{(zf'(z))'}{f'(z)}=1+(d_2+\alpha d_2)z+(2d_3-d_2^2+4\alpha d_3-3\alpha d_2^2)z$$

$$+ (3d_4 - 3d_2d_3 + d_2^3 + 9\alpha d_4 - 15\alpha d_2d_3 - 7\alpha d_2^3)z^3$$
$$+ (4d_5 - 2d_3^2 - 4d_2d_4 + 4d_2^2d_3 - d_2^4 + 16\alpha d_5 - 16\alpha d_3^2$$
$$- 28\alpha d_2d_4 + 44\alpha d_2^2d_3 - 15\alpha d_2^4)z^4 + ... . \tag{35}$$

Comparing the equations (32, 35), we get

$$d_2 = \frac{1}{2(1+\alpha)} r_1\Gamma_1, \tag{36}$$

$$d_3 = \frac{1}{4(2\alpha+1)} r_2\Gamma_1 + \frac{1}{8(2\alpha+1)} r_1^2 \left(\Gamma_2 - \Gamma_1 + \frac{3\alpha+1}{(1+\alpha)^2}\Gamma_1^2\right), \tag{37}$$

$$d_4 = \frac{1}{12(1+3\alpha)} r_1^3 \left(\frac{1}{2}\Gamma_1 - \Gamma_2 + \frac{1}{2}\Gamma_3 + \frac{3(1+5\alpha)}{4(2\alpha+1)(1+\alpha)}\left(\Gamma_2\Gamma_1 - \Gamma_1^2 + \frac{3\alpha+1}{(1+\alpha)^2}\Gamma_1^3\right)\right.$$
$$\left. - \frac{(1+7\alpha)}{2(1+\alpha)^3}\Gamma_1^3\right) + \frac{1}{6(1+3\alpha)} r_1r_2 \left(\Gamma_2 - \Gamma_1 + \frac{3(1+5\alpha)}{4(1+\alpha)(1+2\alpha)}\Gamma_1^2\right) + \frac{1}{6(1+3\alpha)} r_3\Gamma_1, \tag{38}$$

$$d_5 = \frac{1}{64(1+4\alpha)} r_1^4 \left(-\Gamma_1 + 3\Gamma_2 - 3\Gamma_3 + \Gamma_4 + \frac{1+8\alpha}{2(1+\alpha)^2}[\Gamma_2 - \Gamma_1 + \frac{1+3\alpha}{(1+\alpha)^2}]^2\right.$$
$$+ \frac{8(1+7\alpha)}{3(1+\alpha)(1+3\alpha)} \left(\frac{1}{2}\Gamma_1^2 - \Gamma_1\Gamma_2 + \frac{1}{2}\Gamma_3\Gamma_1 + \frac{3(1+5\alpha)}{4(1+2\alpha)(1+\alpha)}[\Gamma_1^2\Gamma_2 - \Gamma_1^3]\right.$$
$$\left.\left. + \frac{1+3\alpha}{(1+\alpha)^2}\Gamma_1^4 - \frac{1+7\alpha}{2(1+\alpha)^3}\right) - \frac{2(1+11\alpha)}{(1+2\alpha)(1+\alpha)^2}[\Gamma_1^2\Gamma_2 - \Gamma_1^3 + \frac{1+3\alpha}{(1+\alpha)^2}\Gamma_1^4] + \frac{1+15\alpha}{(1+\alpha)^4}\Gamma_1^4\right)$$
$$+ \frac{1}{8(1+4\alpha)} r_1r_3 \left(\Gamma_2 - \Gamma_1 + \frac{1+7\alpha}{3(1+3\alpha)(1+\alpha)}\Gamma_1^2\right) + \frac{3}{32(1+4\alpha)} r_1^2 (-\Gamma_1 + 2\Gamma_2 + \Gamma_3 +$$
$$\frac{1+8\alpha}{3(1+\alpha)^2}\left(\Gamma_1\Gamma_2 - \Gamma_1^2 + \frac{3(1+5\alpha)}{4(1\alpha)(1+2\alpha)}\Gamma_1^3\right) - \frac{1+11\alpha}{(1+2\alpha)(1+\alpha)^2}\Bigg) + \frac{1}{16(1+4\alpha)} r_2^2 (\Gamma_2$$
$$-\Gamma_1 + \frac{1+8\alpha}{2(1+2\alpha)^2}\Gamma_1^2\Bigg) + \frac{1}{8(1+4\alpha)} r_4\Gamma_1. \tag{39}$$

Using the values of $d_3$ and $d_5$ obtained in equations (37) and (39), we get

$$|d_3^2 - d_5| = |\left(\frac{1}{4(2\alpha+1)} r_2\Gamma_1 + \frac{1}{8(2\alpha+1)} r_1^2 \left(\Gamma_2 - \Gamma_1 + \frac{3\alpha+1}{(1+\alpha)^2}\Gamma_1^2\right)\right) - (\frac{1}{64(1+4\alpha)} r_1^4 (-\Gamma_1$$
$$+3\Gamma_2 - 3\Gamma_3 + \Gamma_4 + \frac{1+8\alpha}{2(1+\alpha)^2}[\Gamma_2 - \Gamma_1 + \frac{1+3\alpha}{(1+\alpha)^2}]^2 + \frac{8(1+7\alpha)}{3(1+\alpha)(1+3\alpha)} \left(\frac{1}{2}\Gamma_1^2 - \Gamma_1\Gamma_2\right.$$
$$\left. + \frac{1}{2}\Gamma_3\Gamma_1 + \frac{3(1+5\alpha)}{4(1+2\alpha)(1+\alpha)}[\Gamma_1^2\Gamma_2 - \Gamma_1^3] + \frac{1+3\alpha}{(1+\alpha)^2}\Gamma_1^4 - \frac{1+7\alpha}{2(1+\alpha)^3}\right) - \frac{2(1+11\alpha)}{(1+2\alpha)(1+\alpha)^2}$$
$$[\Gamma_1^2\Gamma_2 - \Gamma_1^3 + \frac{1+3\alpha}{(1+\alpha)^2}\Gamma_1^4] + \frac{1+15\alpha}{(1+\alpha)^4}\Gamma_1^4\Bigg) + \frac{1}{8(1+4\alpha)} r_1r_3 \left(\Gamma_2 - \Gamma_1 + \frac{1+7\alpha}{3(1+3\alpha)(1+\alpha)}\Gamma_1^2\right)$$

$$+\frac{3}{32(1+4\alpha)}r_1^2r_2\left(-\Gamma_1+2\Gamma_2+\Gamma_3+\frac{1+8\alpha}{3(1+\alpha)^2}\left(\Gamma_1\Gamma_2-\Gamma_1^2+\frac{3(1+5\alpha)}{4(1\alpha)(1+2\alpha)}\Gamma_1^3\right)\right.$$
$$\left.-\frac{1+11\alpha}{(1+2\alpha)(1+\alpha)^2}\right)+\frac{1}{16(1+4\alpha)}r_2^2\left(\Gamma_2-\Gamma_1+\frac{1+8\alpha}{2(1+2\alpha)^2}\Gamma_1^2\right)+\frac{1}{8(1+4\alpha)}r_4\Gamma_1).$$

Applying Lemma equation (13), we get

$$|d_3^2-d_5|\le\frac{1}{4|1+2\alpha|}\Gamma_1.$$

Thus, the proof of (ii) is completed.
(iii) Let $f(z)\in\mathscr{S}_s^*(\mathscr{A},\mathscr{B};\beta)$ then there exist a Schwarz function $\mathfrak{h}(z)$, such that

$$\frac{2zf'(z)}{f(z)-f(-z)}=1+2b_2z+2d_3z^2+(4d_4-2d_2d_3)z^3+(4d_5-2d_3^2)z^4. \tag{40}$$

Comparing equations (32) and (40), we obtain

$$d_2=\frac{1}{4}r_1\Gamma_1, \tag{41}$$
$$d_3=\frac{1}{4}r_2\Gamma_1-\frac{1}{8}r_1^2[\Gamma_1-\Gamma_2], \tag{42}$$
$$d_4=\frac{1}{16}r_1^3\left(\frac{1}{2}\Gamma_1-\Gamma_2+\frac{1}{2}\Gamma_3-\frac{1}{4}\Gamma_1^2+\frac{1}{4}\Gamma_1\Gamma_2\right)-\frac{1}{8}r_1r_2\left(\Gamma_1-\Gamma_2-\frac{1}{4}\Gamma_1^2\right)$$
$$+\frac{1}{8}r_3\Gamma_1, \tag{43}$$

*and* (44)

$$d_5=\frac{1}{64}r_1^4\left(-\Gamma_1+3\Gamma_2+3\Gamma_3+\Gamma_4+\frac{1}{2}\Gamma_1^2+\frac{1}{2}\Gamma_2^2-\Gamma_1\Gamma_2\right)+\frac{1}{8}r_1r_3(\Gamma_2-\Gamma_1)$$
$$+\frac{3}{32}r_1^2r_2\left(-\Gamma_1+2\Gamma_2-\Gamma_3-\frac{1}{3}\Gamma_1^2+\frac{1}{3}\Gamma_1\Gamma_2\right)+\frac{1}{16}r_2^2\left(\Gamma_2-\Gamma_1+\frac{1}{2}\Gamma_1^2\right)+\frac{1}{8}r_4\Gamma_1. \tag{45}$$

Using the values from equation (42) and equation (45), we get

$$|d_3^2-d_5|=\frac{1}{64}r_1^4\left(\Gamma_1^2+\Gamma_2^2-\Gamma_1\Gamma_2+\Gamma_1-3\Gamma_2+3\Gamma_3-\Gamma_4\right)+\frac{1}{8}r_1r_3(\Gamma_1-\Gamma_2)$$
$$+\frac{1}{16}r_2^2\left(\frac{1}{2}\Gamma_1^2-\Gamma_2+\Gamma_1\right)-\frac{3}{16}\left(\frac{1}{2}\Gamma_1^2-\frac{1}{6}\Gamma_1\Gamma_2+\frac{1}{2}\Gamma_1-\Gamma_2+\frac{1}{2}\Gamma_3\right)-\frac{1}{8}r_4\Gamma_1$$

Applying Lemma (2) equation (13), we get

$$|d_3^2-d_5|\le\frac{1}{4}|\Gamma_1|$$

This completes the proof of the theorem.

**Corollary 3.** *If $f(z) \in \mathfrak{A}$ is of the form of equation (1), then for particular values in theorem equation (3) the following results hold*
*(i) For $\beta = 1$, we get $|d_3^2 - d_5|$ for $\mathscr{R}(\mathscr{A}, \mathscr{B}; 1)$, $\mathscr{M}(\mathscr{A}, \mathscr{B}; \beta, \alpha)$ and $\mathscr{S}_s^*(\mathscr{A}, \mathscr{B}; \beta)$ respectively.*
*(ii) For $\beta = 1$, $\mathscr{A} = 1$ and $\mathscr{B} = -1$, we obtain $|d_3^2 - d_5|$ for $\mathscr{R}(1, -1; 1)$, $\mathscr{M}(1, -1; 1, \alpha)$ and $\mathscr{S}_s^*(1, -1; 1)$ respectively.*
*(iii) For $\beta = \frac{1}{2}$, $\mathscr{A} = 1$ and $\mathscr{B} = 0$, we get $|d_3^2 - d_5|$ for $\mathscr{R}(1, 0; \frac{1}{2})$, $\mathscr{M}(1, 0; \frac{1}{2}, \alpha)$ and $\mathscr{S}_s^*(1, 0; \frac{1}{2})$ respectively.*

## 4. Conclusions

In this study, considering the subordination technique, we attain the results for m-fold symmetric functions and the Zalcman functional are obtained for some subclasses of analytic function associated with strongly Janowski type functions. Our calculation is based on the relationship between the coefficients of the functions in the considered classes and the coefficients of the functions in corresponding Caratheodory class $\wp$. Many fascinating results are obtained using subordination technique like sharp third Hankel determinant for 2, 3-fold symmetric functions, and the sharp Zalcman functional. For specific values of parameters, we obtain certain well-known results already present in the literature.

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