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Iterative Learning Control for Nonlinear Pantograph Systems with Hilfer Fractional Derivatives

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Abstract. This paper investigates the trajectory tracking problem for nonlinear pantograph systems governed by the Hilfer fractional derivative. The presence of fractional memory and proportional delay significantly complicates the control design. To address this issue, P-type and D-type iterative learning control laws are developed to progressively improve the tracking performance over repeated iterations. Additionally, the $(1-\gamma,\lambda)$-weighted norm is used for the robust convergence of the suggested control techniques. Finally, a numerical example illustrates the effectiveness of the proposed control strategies.

2020 Mathematics Subject Classifications: 34A37, 93C15, 93C40

Key Words and Phrases: Hilfer fractional derivative, Pantograph equation, Iterative learning control

1. Introduction

The idea of fractional calculus (FC) has been playing a significant role in various fields of science and engineering. This theory is important because it provides the best description and shape for many real-world phenomena. The correspondence between L'hospital and Leibnitz in 1695 is where the concept of FC first emerged (Refn. [1]). Many researchers soon began experimenting in this area, with many focusing on defining new fractional operators (Refns. [2, 3]). Over time, a number of definitions emerged in this context and were further applied. Recently, new definitions of fractional integral and derivative operators were found by Khan and Khan (Refn. [4]).

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Hilfer fractional derivative (HFD), which was first introduced by Hilfer (Refn. [5]). It is defined as

$$({}^{H}D_{a^+}^{\alpha,\beta}f)(t) = (I_{a^+}^{\beta(1-\alpha)}D(I_{a^+}^{(1-\beta)(1-\alpha)}f))(t),$$

where $\alpha \in (0,1)$, and $\beta \in [0,1]$. According to this definition, the derivative depends on two parameters, α (called the order) and β (called the type). The parameter β determines the nature of the fractional derivative.

- When $\beta = 0$, the HFD corresponds to the Riemann-Liouville (R-L) derivative (Refn. [6]).
- When $\beta = 1$, it corresponds to the Liouville-Caputo (L-C) derivative (Refn. [6]).

Thus, the parameter β allows interpolation between the L-C types of fractional derivatives. Kamocki et al. (Refn. [7]) obtained an integral representation of HFD and showed the advantages. Tomovski et al. (Refn. [8]) analyzed the relationship between Mittag-Leffler function and HFD, Furati et al., (Refn. [9]) studied existence and uniqueness of Hilfer fractional Cauchy problems. Bhairat (Refn. [10]) studied global existence of a unique solution of nonlinear initial value problem with HFD (Refns. [11, 12]). There are many modeling studies on HFD (Refns. [13–15]).

One of the most significant types of delay differential equations (DDEs), the pantograph equation is crucial for the explanation of a number of phenomena. Electrodynamics is one of the many fields in which the pantograph DDEs appear. Numerous numerical techniques, including the variational iteration method (Refn. [16]), Bernoulli polynomials (Refn. [17]), and Chebyshev polynomials (Refn. [18]), have been used in recent years to solve pantograph DDEs of integer order. However, there aren't many publications that focus on numerically solving fractional-order pantograph DDEs. We can mention the Legendre multiwavelet collocation method (Refn. [19]), the spectral-collocation method (Refn. [20]), and the Hermite wavelet method (Refn. [21]) from these works. Vivek et al. (Refn. [22]) used fixed-point theorems to establish existence and uniqueness results for pantograph equations with a generalised fractional derivative. Abd-Elhameed et al. (Refn.[23]) proposed a spectral algorithm using generalized Chebyshev polynomials for fractional delay pantograph equations, achieving accurate and efficient results through improved convergence properties. Haniye Dehestani (Refn.[24]) developed a Ritz-Piecewise Gegenbauer approach for fractional pantograph models with both standard and piecewise derivatives, demonstrating enhanced numerical performance and stability.

Iterative learning control (ILC), is a relatively recent addition to the control algorithm toolbox. The performance of systems that perform repetitive tasks is the focus of this study, which includes examples like robot arm manipulators. The fundamental concept of this method is to use the knowledge gained from the last operation to make the controlled system perform increasingly better from one operation to the next. Since Arimoto et al. (Refn. [25]) presented it as a formal theory, this technique has been the focus of many researcher's attention for the past few decades (Refns. ([26–29]). Vivek et al. (Refn. [30]) investigated the convergence analysis of ILC for nonlinear pantograph equations with impulses using the HFD. Further, Vivek et al. (Refn. [31]) studied ILC for Hilfer-type fractional-order quaternion-valued impulsive systems and established significant convergence results. In another work, Vivek et al. (Refn. [32]) proposed a P-type ILC approach for impulsive pantograph equations involving the HFD. Later, Vivek et al. (Refn. [33]) analyzed both convergence and stability of iterative learning control in impulsive fractional systems governed by HFD. Moreover, Sunmitha et al. (Refn. [34]) presented an advanced ILC concept for switched systems subject to non-instantaneous impulses using HFD. In addition, Sunmitha et al. (Refn. [35]) examined P-type ILC for Hilfer-type fractional-order quaternion-valued systems with initial state deviation and highlighted applications in soft robot actuators. Recently, Sunmitha et al. (Refn. [36]) investigated ILC strategies for dynamic systems with time-varying disturbance rejection through HFD. Furthermore, Sunmitha et al. (Refn. [37]) studied the application of ILC in nonlinear systems with HFD to enhance precision in robotic manipulators.

An advantage of the ILC is that it generally enhances tracking performance through a self-tuning process that does not rely on the system dynamic model. After a few iterations, achieving high performance is the main goal of using ILC. The precision comes first, followed by the speed of convergence and the rejection of uncertainties and disturbances. It is remarked that only few researches have been conducted to compare two or three different ILC algorithms in terms of convergence speed and final tracking performance (Refns. [38, 39]). A traditional P-type ILC updates the control input signal as a function of the previous stored control input and the stored output errors, whereas a D-type ILC updates the control input as a function of the previous stored control input and the stored derivative of the output errors. One advantage of the P-type ILC is the solely measurement requirement of the system output that has thus attracted considerable attentions from control researchers.

The following papers discuss P-type and D-type ILC methods and highlight the research gaps addressed in each study:

In our paper, we focus on the analysis of P-type and D-type ILC strategies applied to fractional-order pantograph systems involving HFD. We investigate the dynamic behavior of the system under each learning law and provide a comparative study highlighting their convergence performance. Furthermore, we perform detailed error analysis across multiple iterations to demonstrate how the output trajectory approaches the desired Refnerence, confirming the effectiveness of both control types under fractional dynamics.

The structure of the paper is as follows: In Section 2, we present fundamental and relevant results that form the basis of our study. Section 3 focuses on the convergence of the P-type ILC for a fractional-order nonlinear pantograph equations (FO-NPEs) involving the HFD, analyzed using the $(1-\gamma,\lambda)$-weighted norm $\|\cdot\|_{1-\gamma,\lambda}$. Section 4 introduces the D-type convergence analysis for the proposed system with order $1<\alpha<2$ and type $0\le\beta\le1$, using the λ-weighted norm $\|\cdot\|_{\lambda}$. Finally, illustrative numerical simulations are provided to verify the theory.

Year	Authors	P-type Discussion	D-type Discussion
2005	Ratcliffe et al. (Refn. [40])	P-type ILC for systems with resonance.	D-type not discussed.
2007	Chien et al. (Refn. [41])	Robust tracking via P-type ILC.	D-type not discussed.
2013	Lan et al. (Refn. [42])	P-type not discussed.	D-type ILC for fractional delay systems.
2013	Huang et al. (Refn. [43])	P-type baseline in boundary control.	Anticipatory D-type ILC for heat equation.
2020	Hamidaoui et al. (Refn. [44])	P-type in partial differential equation background.	D-type ILC with open/closed-loop convergence.
2023	Chen et al. (Refn. [45])	P-type ILC with random shifts.	D-type not discussed.
2025	Vivek et al. (Refn. [31])	P-type ILC for quaternion impulsive systems.	D-type not discussed.
2025	Vivek et al. (Refn. [32])	P-type ILC for impulsive pantograph system.	D-type not discussed.
2025	Wei et al. (Refn. [46])	P-type ILC with varying Refnerence and trial.	D-type not discussed.

2. Primary Document

Let $J = [0, b]$, where $b > 0$. Denote by $\mathfrak{C}(J, \mathbb{R}^n)$ the Banach space of continuous vector-valued functions from J to $\mathbb{R}^n$, equipped with the norm

$$||y||_\infty = \sup_{t \in J} ||y(t)||.$$

Equivalently, for $\lambda > 0$, we define the weighted norm

$$||y||_\lambda = \sup_{t \in J} e^{-\lambda t} ||y(t)||.$$

Let $0 < \alpha < 1$ and $0 \leq \beta \leq 1$.
Define $\gamma = \alpha + \beta - \alpha\beta$.
We introduce the weighted function space (Refn. [47], Section 2, Page 851)

$$\mathfrak{C}_{1-\gamma}(J, \mathbb{R}^n) = \{y \in (0, b] \to \mathbb{R}^n : t^{1-\gamma} y(t) \in \mathfrak{C}(J, \mathbb{R}^n)\}.$$

On this space, we define the $(1 - \gamma, \lambda)$-weighted norm by

$$||y||_{1-\gamma,\lambda} = \max_{t \in J} t^{1-\gamma} e^{-\lambda t} ||y(t)||.$$

It is known (Refn. [48], Section 2, Page 3) that

$$\big(\mathfrak{C}_{1-\gamma}(J, \mathbb{R}^n), || \cdot ||_{1-\gamma,\lambda}\big),$$

is also a Banach space.

Definition 1. *(Refn. [1], Chapter 2, Eqs 2.1.1 and 2.1.2, Page 69) Let* $\alpha > 0$ *and* $f \in L^1(a,b)$*. The R-L fractional integral of order* α *is defined by*

$$(I^{\alpha}_{a^+}f)(y) = \frac{1}{\Gamma(\alpha)}\int_a^y \frac{f(t)}{(y-t)^{1-\alpha}}dt, \quad y > a.$$

Let $n = \lceil\alpha\rceil$*. The R-L fractional derivative of order* α *is defined by*

$$(D^{\alpha}_{a^+}f)(y) = \Big(\frac{d}{dy}\Big)^n (I^{n-\alpha}_{a^+}f)(y),$$

that is,

$$(D^{\alpha}_{a^+}f)(y) = \frac{1}{\Gamma(n-\alpha)}\Big(\frac{d}{dy}\Big)^n \int_a^y \frac{f(t)}{(y-t)^{\alpha-n+1}}dt, \quad y > a.$$

Here $\Gamma(\cdot)$ *denotes the Gamma function.*

Definition 2. *(Refn. [49], Remark 6, Page 4) Let* $0 < \alpha < 1$ *and* $0 \leq \beta \leq 1$*. Let* f *be such that the fractional integrals involved below exist. (For example,* $f \in L^1(a,b)$*).*
The HFD of order α *and type* β *is defined by*

$$({}^H D^{\alpha,\beta}_{a^+}f)(y) = (I^{\beta(1-\alpha)}_{a^+}\frac{d}{dy}(I^{(1-\beta)(1-\alpha)}_{a^+}f))(y), \quad y > a.$$

Equivalently,

$${}^H D^{\alpha,\beta}_{a^+} = I^{\beta(1-\alpha)}_{a^+}\frac{d}{dy}I^{(1-\beta)(1-\alpha)}_{a^+}.$$

Let $\alpha \in (n-1,n)$, where $n = \lceil\alpha\rceil$, and let $0 \leq \beta \leq 1$.
The HFD of order α and type β is defined by (Refn. [47], Definition 2.3, Page 851)

$$({}^H D^{\alpha,\beta}_{a^+}f)(y) = (I^{\beta(n-\alpha)}_{a^+}\frac{d^n}{dy^n}(I^{(1-\beta)(n-\alpha)}_{a^+}f))(y), \quad y > a.$$

Let $\gamma = \alpha + \beta(n-\alpha)$. Since $(1-\beta)(n-\alpha) = n-\gamma$, the above definition can be rewritten as

$$({}^H D^{\alpha,\beta}_{a^+}f)(y) = \big(I^{\beta(n-\alpha)}_{a^+}\frac{d^n}{dy^n}(I^{(n-\gamma)}_{a^+}f)\big)(y).$$

Using the Definition 1, we obtain the equivalent representation

$${}^H D^{\alpha,\beta}_{a^+} = I^{\beta(n-\alpha)}_{a^+}D^{(\gamma)}_{a^+}.$$

Definition 3. *(Refn. [50], Eqs. 1.1, 1.2, Page 1) Let* $\alpha > 0$*,* $\beta \in \mathbb{R}$ *and* $z \in \mathbb{C}$*. The two-parameter Mittag-Leffler (M-L) function is defined by*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}.$$

In particular, when $\beta = 1$, we obtain the classic one-parameter M-L function

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}.$$

A further generalization, known as the three-parameter M-L function is defined by

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{k!\Gamma(\alpha k + \beta)}, \quad \alpha > 0,\ \beta \in \mathbb{R},\ \gamma > 0.$$

where the Pochhammer symbol $(\gamma)_k$ is given by

$$(\gamma)_k = \frac{\Gamma(\gamma + k)}{\Gamma(\gamma)}, \qquad (\gamma)_0 = 1.$$

Special cases

(i) If $\gamma = 1$, then

$$E^{1}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z),$$

the classical two-parameter M-L function.

(ii) If $\gamma = 1$ and $\beta = 1$, then

$$E^{1}_{\alpha,1}(z) = E_\alpha(z),$$

the classical one-parameter M-L function.

(iii) If $0 < \alpha \leq 1$ and $\beta, \gamma > 0$, then for $z \leq 0$,

$$0 < E^{\gamma}_{\alpha,\beta}(z) \leq \frac{1}{\Gamma(\beta)}.$$

More generally, for bounded $z \leq 0$,

$$E^{\gamma}_{\alpha,\beta}(z) \leq \frac{C}{\Gamma(\beta)},$$

for some constant $C > 0$ depending on α, γ.

Definition 4. *(Refn. [51], Lemma 2, Page 126) Let $\alpha > 0$, $\beta \in \mathbb{R}$, and let $m \in \mathbb{N}$. Assume that $\beta - m \notin \{0, -1, -2, \cdots\}$. Then the integer-order derivative of the M-L function satisfies*

$$\left(\frac{d}{dz}\right)^m [z^{\beta-1} E_{\alpha,\beta}(z^\alpha)] = z^{\beta-m-1} E_{\alpha,\beta-m}(z^\alpha), \quad z \in \mathbb{C}.$$

Lemma 1. *(Refn. [52], Theorem 1, Page 1075) Let $\beta > 0$. Assume $m(t) \in L^1_{loc}(J)$ is a nonnegative function, $n(t) \in C(J)$ is nonnegative and nondecreasing, $n(t) \leq M$ for some constant $M > 0$, and $x(t) \in L^1_{loc}(J)$ is nonnegative,*
and suppose

$$x(t) \leq m(t) + n(t) \int_0^t (t-s)^{\beta-1} x(s)\, ds, \quad t \in J.$$

Then, for $t \in J$,

$$x(t) \le m(t) + \int_0^t \Big[\sum_{j=1}^{\infty} \frac{(n(t)\Gamma(\beta))^j}{\Gamma(j\beta)} (t-s)^{j\beta-1} m(s) \Big] ds.$$

Equivalently,

$$x(t) \le m(t) + \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(n(t)(t-s)^{\beta}) m(s)\, ds.$$

Remark 1. *Under the assumption of Lemma 1, suppose in addition that $m(t)$ is nonnegative and nondecreasing on J.*

$$x(t) \le m(t) + \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(n(t)(t-s)^{\beta}) m(s)\, ds.$$

Since m is nondecreasing,

$$m(s) \le m(t), \quad s \le t.$$

Thus,

$$x(t) \le m(t) \Big[1 + \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(n(t)(t-s)^{\beta})\, ds \Big].$$

We obtain

$$x(t) \le m(t) E_{\beta}(n(t)\Gamma(\beta) t^{\beta}).$$

Lemma 2. *(Refn. [53], Lemma 3.1, Page 780) Let u be a continuous function on $t \in J$, let v be a continuous nonnegative function on J, and let $w(t)$ be a positive, continuous, and nondecreasing function on J.*
Assume

$$u(t) \le w(t) + \int_0^t v(t-\tau) u(\tau)\, d\tau, \quad t \in J.$$

Define

$$V(t) = \int_0^t v(s)\, ds.$$

Then, for $t \in J$,

$$u(t) \le w(t) \exp\big(V(t)\big).$$

That is,

$$u(t) \le w(t) \exp\Big(\int_0^t v(s)\, ds \Big).$$

We adopt the following hypotheses for the purposes of our theoretical discussion.
(H_1) The function $f : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is jointly continuous.
Furthermore, there exist continuous, nondecreasing, and nonnegative functions $L_1, L_2 : J \to [0, \infty)$ such that for all $t \in J$ and for all $x_k, \hat{x}_k, y_k, \hat{y}_k \in \mathbb{R}^n$, we have

$$\|f(t, y_k, \hat{y}_k) - f(t, x_k, \hat{x}_k)\| \leq L_1(t) \left(\|y_k - x_k\| + \|\hat{y}_k - \hat{x}_k\|\right), \quad t \in J.$$

If the control input appears explicitly in the system, we additionally assume the Lipschitz condition

$$\|u_k(t) - \hat{u}_k(t)\| \leq L_2(t)\|u_k - \hat{u}_k\|, \quad t \in J,$$

which in practice means that the control operator is locally Lipschitz.
(H_2) Assume that $g : J \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable with respect to its second, third and fourth arguments.
There exist positive constants $\eta_j > 0$, $(j = 1, 2, 3, 4)$, such that for all $t \in J$, $y, \hat{y} \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, the following holds:

$$0 < \eta_1 \leq \frac{\partial g(t, y_k, \hat{y}_k, u_k)}{\partial u_k} \leq \eta_2,$$

and

$$0 < \eta_3 \leq \frac{\partial g(t, y_k, \hat{y}_k, u_k)}{\partial y_k} \leq \eta_4.$$

That is, the partial derivatives of g with respect to the control and state variables are uniformly bounded and strictly positive.
(H_3) Define

$$M = \max\left\{\sup_{t\in J} \frac{L_1(t)}{t^{1-\alpha}},\ \sup_{t\in J} \frac{L_2(t)}{t^{1-\alpha}}\right\}.$$

Assume that

$$M < \infty.$$

3. Convergence properties of P-type algorithms

In this section, we consider FO-NPEs in the sense of HFD, given by

$$\begin{cases} ({}^H D^{\alpha,\beta}_{0^+} y_k)(t) = \mu y_k(t) + f(t, y_k(t), y_k(\nu t)) + u_k(t),\ t \in J, \quad \mu < 0, \\ (I^{1-\gamma}_{0^+} y_0)(0) = y_0,\ \gamma = \alpha + \beta - \alpha\beta, \end{cases} \tag{1}$$

where $0 < \alpha < 1$ and $0 \leq \beta \leq 1$.
The corresponding output equation is

$$z_k(t) = g(t, y_k(t), y_k(\nu t), u_k(t)). \tag{2}$$

The index $k \in \mathbb{N}$ denotes the k^{th} learning iteration. The nonlinear functions are defined as $f : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $g : J \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. The variables $y_k(t), u_k(t), z_k(t) \in \mathbb{R}^n$ represent the state, input and output of the system, respectively and the parameter ν satisfies $0 < \nu < 1$. Moreover, ${}^H D^{\alpha,\beta}_{0^+}$ denotes the HFD of order $0 < \alpha < 1$ and type $0 \leq \beta \leq 1$. The operator $I^{1-\gamma}_{0^+}$ denotes the R-L fractional integral of order $1 - \gamma$.

Obviously, the solution of system (1) is equivalent to the solution of following Volterra integral equation (Refn.[54], Section 4, Page 59)

$$\begin{aligned} y_k(t) &= y_0 t^{\gamma-1} E_{\alpha,\gamma}(\mu t^\alpha) \\ &\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\mu(t-s)^\alpha)[f(s, y_k(s), y_k(\nu s)) + u_k(s)]ds, \; t \in J. \end{aligned} \tag{3}$$

For the system (1), we consider the open-loop P-type ILC updating law with initial state learning given by

$$\begin{cases} y_{k+1}(0) = y_k(0) + Le_k(0), \\ u_{k+1}(t) = u_k(t) + \delta_1 e_k(t), \quad t \in J, \end{cases} \tag{4}$$

where $z_d(t)$ denotes the desired reference trajectory, $e_k(t) = z_d(t) - z_k(t)$ is the tracking error, $L \in \mathbb{R}$ and $\delta_1 \in \mathbb{R}$ are learning gains to be designed.
We consider the closed-loop P-type ILC updating law with initial state learning given by

$$\begin{cases} y_{k+1}(0) = y_k(0) + Le_k(0), \\ u_{k+1}(t) = u_k(t) + \delta_2 e_{k+1}(t), \quad t \in J, \end{cases} \tag{5}$$

where $L \in \mathbb{R}$ and $\delta_2 \in \mathbb{R}$ are learning gains to be designed.

Remark 2. *(Refn. [55], Section 2, Page 3) The system* (1) *can be written in the following abstract form*

$$\begin{cases} ({}^H D^{\alpha,\beta}_{0^+} y_k)(t) = \mu y_k(t) + f(t, y_k(t), y_k(\nu t), u_k(t)), \quad t \in J, \quad \mu < 0, \\ (I^{1-\gamma}_{0^+} y_k)(0) = y_0, \end{cases}$$

where $f(t, y, y_v, u) = f(t, y, y_v) + u$.

Lemma 3. *Consider the FO-NPEs* (1)*, with a given reference trajectory* $z_d(t)$*.*
Define

$$\begin{aligned} \Delta u_k(t) &= u_{k+1}(t) - u_k(t), \\ \Delta y_k(t) &= y_{k+1}(t) - y_k(t). \end{aligned}$$

Assume that the learning gains L *and* δ_1 *satisfy*

$$\max\{|1 - \delta_1\eta_1 - L\eta_3|, |1 - \delta_1\eta_1 - L\eta_4|, |1 - \delta_1\eta_2 - L\eta_3|, |1 - \delta_1\eta_2 - L\eta_4|\} \leq \sigma_0 < 1.$$

Then, for any initial input $u_0(t)$ *and for all* $t \in J$*, the open-loop* P*-type ILC updating law* (4) *guarantees that*

$$\lim_{k\to\infty} ||e_k|| = 0.$$

Theorem 1. *Suppose that hypotheses $(H_1)-(H_3)$ hold. Consider system (1) with a desired trajectory $z_d(t)$. Assume that the conditions of Lemma 3 are satisfied and, in addition, that the learning gain δ_1 fulfills*

$$\max\{|1-\delta_1\eta_1|, |1-\delta_1\eta_2|\} \leq \sigma_1 < 1, \tag{6}$$

Then, for any initial input $u_0(t)$, the open-loop P-type ILC updating law (4) ensures that

$$\lim_{k\to\infty} ||z_k - z_d||_{(1-\gamma,\lambda)} = 0,$$

that is, $z_k \to z_d \in \mathfrak{C}(J,\mathbb{R})$ with respect to the $(1-\gamma,\lambda)$-weighted norm, provided that there exists $\lambda_0 > 0$ such that the convergence holds for all $\lambda > \lambda_0$.

Proof. It follow from the definition

$$e_k(t) = z_d(t) - z_k(t),$$

that

$$\begin{aligned} e_{k+1}(t) &= z_d(t) - z_{k+1}(t), \\ &= z_d(t) - z_k(t) + z_k(t) - z_{k+1}(t), \\ &= e_k(t) + z_k(t) - z_{k+1}(t). \end{aligned}$$

Since

$$z_k(t) = g(t, y_k(t), y_k(\nu t), u_k(t)),$$

the Mean Value Theorem (Refn. [56], Lemma 3.2, Page 3212) yields

$$z_{k+1}(t) - z_k(t) = g_y(\zeta)\Delta y_k(t) + g_\nu(\zeta)\Delta y_k(\nu t) + g_u(\zeta)\Delta u_k(t),$$

where

$$\Delta y_k(t) = y_{k+1}(t) - y_k(t), \quad \Delta u_k(t) = u_{k+1}(t) - u_k(t),$$

and the intermediate point $\zeta_1 = \big(t, y_k(t) + \theta\Delta y_k, y_k(\nu t) + \theta\Delta y_k(\nu t), u_k(t) + \theta\Delta u_k(t)\big), \theta \in (0,1)$.
Therefore,

$$e_{k+1}(t) = e_k(t) - g_y(\zeta_1)\Delta y_k(t) - g_\nu(\zeta_1)\Delta y_k(\nu t) - g_u(\zeta_1)\Delta u_k(t).$$

Using the open-loop updating law $\Delta u_k(t) = \delta_1 e_k(t)$, we obtain

$$e_{k+1}(t) = [1 - \delta_1 g_u(\zeta_1)]e_k(t) - g_y(\zeta_1)\Delta y_k(t) - g_\nu(\zeta_1)\Delta y_k(\nu t).$$

Taking absolute values yields

$$|e_{k+1}(t)| \leq |1 - \delta_1 g_u(\zeta_1)|\,|e_k(t)| + |g_y(\zeta_1)|\,|\Delta y_k(t)| + |g_\nu(\zeta_1)|\,|\Delta y_k(\nu t)|. \tag{7}$$

From integral representation Eq.(3)*, we have*

$$\Delta y_k(t) = \Delta y_k(0)t^{\gamma-1}E_{\alpha,\gamma}(\mu t^{\alpha}) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\mu(t-s)^{\alpha}) \\ \times [f(s, y_{k+1}(s), y_{k+1}(\nu s)) - f(s, y_k(s), y_k(\nu s))]ds \\ \times \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\mu(t-s)^{\alpha})[u_{k+1}(s) - u_k(s)]ds.$$

Using Definition 3, the bound for the M-L function, and the Lipschitz condition $(H1)$*, we obtain*

$$||\Delta y_k(t)|| \leq \frac{t^{\gamma-1}}{\Gamma(\gamma)}||\Delta y_k(0)|| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \\ \times L_1(s)\big(||\Delta y_k(s)|| + ||\Delta y_k(\nu s)||\big)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}L_2(s)||\Delta u_k(s)||ds.$$

Since $0 < \nu < 1$*, we have* $||\Delta y_k(\nu s)|| \leq ||\Delta y_k(s)||$*, and therefore*

$$||\Delta y_k(t)|| \leq \frac{t^{\gamma-1}}{\Gamma(\gamma)}||\Delta y_k(0)|| + \frac{2}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \\ \times L_1(s)||\Delta y_k(s)||ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}L_2(s)||\Delta u_k(s)||ds.$$

Multiplying both sides by $t^{1-\gamma}$*, we obtain*

$$t^{1-\gamma}||\Delta y_k(t)|| \leq \frac{1}{\Gamma(\gamma)}||\Delta y_k(0)|| + \frac{2t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}L_1(s)||\Delta y_k(s)||ds \\ + \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}L_2(s)||\Delta u_k(s)||ds.$$

Since $M = \max\left\{\sup_{s\in J}\frac{L_1(s)}{s^{1-\alpha}}, \sup_{s\in J}\frac{L_2(s)}{s^{1-\alpha}}\right\} < \infty$*, we write*

$$\frac{L_i(s)s^{1-\alpha}}{s^{1-\alpha}} \leq Ms^{1-\alpha}.$$

Hence,

$$t^{1-\gamma}||\Delta y_k(t)|| \leq \frac{1}{\Gamma(\gamma)}||\Delta y_k(0)|| + \frac{2M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{1-\alpha}s^{1-\gamma}||\Delta y_k(s)||ds \\ + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{1-\alpha}s^{1-\gamma}||\Delta u_k(s)||ds. \tag{8}$$

We now estimate

$$I(t) = \int_0^t (t-s)^{\alpha-1}s^{1-\gamma}||\Delta u_k(s)||ds.$$

Insert the weighted norm definition

$$||\Delta u_k||_{1-\gamma,\lambda} = \sup_{s\in J} s^{1-\gamma}e^{-\lambda s}||\Delta u_k(s)||.$$

Then,

$$\begin{aligned} I(t) &\leq \int_0^t (t-s)^{\alpha-1}s^{1-\gamma}e^{-\lambda s}||\Delta u_k(s)||e^{\lambda s}ds \\ &\leq ||\Delta u_k||_{1-\gamma,\lambda}\int_0^t (t-s)^{\alpha-1}e^{\lambda s}ds. \end{aligned}$$

Using the classical estimate

$$\int_0^t (t-s)^{\alpha-1}e^{\lambda s}ds = e^{\lambda t}\int_0^t r^{\alpha-1}e^{-\lambda r}dr \leq e^{\lambda t}\frac{\Gamma(\alpha)}{\lambda^\alpha},$$

we obtain

$$\int_0^t (t-s)^{\alpha-1}s^{1-\gamma}||\Delta u_k(s)||ds \leq ||\Delta u_k||_{1-\gamma,\lambda}\frac{\Gamma(\alpha)}{\lambda^\alpha}e^{\lambda t}. \tag{9}$$

From the inequality(8) *and estimate* (9)*, we obtain*

$$\begin{aligned} t^{1-\gamma}||\Delta y_k(t)|| \leq{}& \frac{1}{\Gamma(\gamma)}||\Delta y_k(0)|| + \frac{M}{\lambda^\alpha}e^{\lambda t}||\Delta u_k||_{1-\gamma,\lambda} \\ &+ \frac{2M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{1-\gamma}||\Delta y_k(s)||ds. \end{aligned}$$

Define

$$p(t) = \frac{1}{\Gamma(\gamma)}||\Delta y_k(0)|| + \frac{e^{\lambda t}M}{\lambda^\alpha}t^{1-\gamma}||\Delta u_k||_{1-\gamma,\lambda}.$$

Then, we obtain

$$t^{1-\gamma}||\Delta y_k(t)|| \leq p(t) + \frac{2t^{1-\gamma}M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{1-\gamma}||\Delta y_k(s)||ds, \tag{10}$$

Since $e^{\lambda t}$ *is increasing,* $p(t)$ *is nondecreasing.*
By Remark 1 (Grönwall inequality), we deduce

$$t^{1-\gamma}||\Delta y_k(t)|| \leq p(t)E_\alpha(2MT^{1+\alpha-\gamma}). \tag{11}$$

Since $t \leq b$*, we further obtain the uniform estimate*

$$t^{1-\gamma}||\Delta y_k(t)|| \leq p(t)E_\alpha(2Mb^{1+\alpha-\gamma}).$$

Multiplying both sides by $e^{-\lambda t}$*, we obtain*

$$t^{1-\gamma}e^{-\lambda t}||\Delta y_k(t)|| \leq \frac{e^{-\lambda t}}{\Gamma(\gamma)}||\Delta y_k(0)||E_\alpha(2Mb^{1+\alpha-\gamma}) + \frac{M}{\lambda^\alpha}E_\alpha(2Mb^{1+\alpha-\gamma})||\Delta u_k||_{1-\gamma,\lambda}.$$

Taking supremum over $t \in J$, we obtain

$$||\Delta y_k(t)||_{1-\gamma,\lambda} \leq \frac{||\Delta y_k(0)||_\lambda}{\Gamma(\gamma)} E_\alpha(2Mb^{1+\alpha-\gamma}) + \frac{M}{\lambda^\alpha} E_\alpha(2Mb^{1+\alpha-\gamma})||\Delta u_k||_{1-\gamma,\lambda}. \quad (12)$$

Assume $||\Delta y_k||_{1-\gamma,\lambda} = \delta_1 ||\Delta e_k||_{1-\gamma,\lambda}$.
From Eq.(7),

$$||e_{k+1}||_{1-\gamma,\lambda} \leq |1 - \delta_1 g_u(\zeta_1)| ||e_k||_{1-\gamma,\lambda} + |g_y(\zeta_1)| ||\Delta y_k||_{1-\gamma,\lambda}.$$

Taking the weighted norm $|| \cdot ||_{1-\gamma,\lambda}$, *we obtain*

$$\begin{aligned} ||e_{k+1}||_{1-\gamma,\lambda} &\leq |1 - \delta_1 g_u(\zeta_1)| ||e_k||_{1-\gamma,\lambda} \\ &\quad + |g_y(\zeta_1)| \Big[\frac{||\Delta y_k(0)||_\lambda}{\Gamma(\gamma)} E_\alpha(2Mb^{1+\alpha-\gamma}) + \frac{\delta_1 M}{\lambda^\alpha} E_\alpha(2Mb^{1+\alpha-\gamma}) ||\Delta e_k||_{1-\gamma,\lambda} \Big]. \end{aligned}$$

If additionally the initial increment satisfies

$$\begin{aligned} ||e_{k+1}||_{1-\gamma,\lambda} &\leq \Big[|1 - \delta_1 g_u(\zeta_1)| + \frac{\delta_1 |g_y(\zeta_1)| M}{\lambda^\alpha} E_\alpha(2Mb^{1+\alpha-\gamma}) \Big] ||e_k||_{1-\gamma,\lambda} \\ &\quad + |g_y(\zeta)| \frac{E_\alpha(2Mb^{1+\alpha-\gamma})}{\Gamma(\gamma)} |L| \, ||e_k(0)||_\lambda. \end{aligned} \quad (13)$$

From the corrected contraction inequality (13), *define*

$$\rho(\lambda) = |1 - \delta_1 g_u(\zeta_1)| + \frac{\delta_1 |g_y(\zeta_1)| M}{\lambda^\alpha} E_\alpha(2Mb^{1+\alpha-\gamma}).$$

By Eq.(6), *we assume* $|1 - \delta_1 g_u(\zeta_1)| \leq \sigma_1 < 1$.
There exists as a sufficiently large $\lambda > 0$ *such that*

$$\rho(\lambda) = |1 - \delta_1 g_u(\zeta_1)| + \frac{\delta_1 M |g_y(\zeta_1)|}{\lambda^\alpha} E_\alpha(2Mb^{1+\alpha-\gamma}) \leq \sigma_1 + \frac{\delta_1 M |\eta_4|}{\lambda^\alpha} E_\alpha(2Mb^{1+\alpha-\gamma}) < 1.$$

Combining inequality (13) *with Lemma 3, we obtain*

$$\lim_{k \to \infty} ||e_k||_{1-\gamma,\lambda} \leq \rho(\lambda) ||e_k||_{1-\gamma,\lambda}, \quad 0 < \rho(\lambda) < 1.$$

Therefore,

$$\lim_{k \to \infty} ||e_k||_{1-\gamma,\lambda} = 0.$$

This completes the proof.

Remark 3. *For the system described by* (1), *assume that*

$$(I_{0^+}^{1-\gamma} y_k)(0) = y(0),$$

and that the Lipschitz function satisfy

$$L_1(t) = L_1 > 0, \quad L_2(t) = L_2 > 0, t \in J.$$

Then, the weighted hypothesis (H_3) *is no longer required, since no singular behavior appears near* $t = 0$.

In this case, we may work in the classical exponential norm, proceeding as in the proof Theorem 3 and applying the fractional Gröwall inequality we obtain

$$||e_{k+1}||_\lambda \leq \left[|1 - \delta_1 g_u(\zeta_1)| + \frac{|\delta_1| L_2 |\eta_4| E_\alpha(2L_1 b^{1+\alpha-\gamma})}{\lambda^\alpha} \right] ||e_k||_\lambda.$$

Moreover, if $|1 - \delta_1 g_u(\zeta)| \leq \sigma_1 < 1$, *then there exist a sufficiently large* $\lambda > 0$ *such that*

$$|1 - \delta_1 g_u(\zeta)| + \frac{|\delta_1| L_2 |\eta_4| E_\alpha(2L_1 b^{1+\alpha-\gamma})}{\lambda^\alpha} < 1.$$

Therefore, the open-loop P*-type ILC updating law* (4) *guarantees that*

$$\lim_{k \to \infty} ||e_k||_{1-\gamma,\lambda} = 0,$$

this is, z_k *converges to* z_d *in the exponential* λ*-norm.*

Lemma 4. *Consider the FO-NPEs described by* (1) *with the reference trajectory* $z_d(t)$, *defined on* J.

Assume that the closed-loop P*-type ILC updating law* (5) *satisfies*

$$\max \left\{ \left| \frac{1 - L\eta_3}{1 + \delta_2 \eta_1} \right|, \left| \frac{1 - L\eta_3}{1 + \delta_2 \eta_2} \right|, \left| \frac{1 - L\eta_4}{1 + \delta_2 \eta_1} \right|, \left| \frac{1 - L\eta_4}{1 + \delta_2 \eta_2} \right| \right\} \leq \sigma_2 < 1,$$

where σ_2 *is a positive constant independent of* k.

Then, for any initial input u_0, *the sequence of initial tracking errors generated by* (5) *satisfies*

$$\lim_{k \to \infty} ||e_k(0)||_{1-\gamma,\lambda} = 0.$$

Theorem 2. *Assume that hypotheses* $(H_1) - (H_3)$ *hold. Consider the FO-NPEs described in* (1) *with the reference trajectory* $z_d(t) \in \mathfrak{C}(J, \mathbb{R}^n)$. *Suppose that the conditions of Lemma 4 are satisfied, i.e.,*

$$\max \left\{ \left| \frac{1 - L\eta_3}{1 + \delta_2 \eta_1} \right|, \left| \frac{1 - L\eta_3}{1 + \delta_2 \eta_2} \right|, \left| \frac{1 - L\eta_4}{1 + \delta_2 \eta_1} \right|, \left| \frac{1 - L\eta_4}{1 + \delta_2 \eta_2} \right| \right\} \leq \sigma_2 < 1.$$

In addition,

$$\min \left\{ |1 + \delta_2 \eta_1|, |1 + \delta_2 \eta_2| \right\} \geq \sigma_3 > 1, \tag{14}$$

where σ_2, σ_3 are positive constants independent of k.
Then, for any initial input u_0, the closed-loop P-type ILC updating law (5) *guarantees that*

$$\lim_{k\to\infty} ||z_k - z_d||_{1-\gamma,\lambda} = 0,$$

provided that $\lambda > 0$ is chosen sufficiently large.

Proof. *The proof follows the same strategy as in Theorem 3, however, due to the closed-loop structure, additional algebraic manipulation required.*

From the closed-loop P-type ILC updating law (5)*, the tracking error satisfies*

$$e_{k+1}(t) = e_k(t) - g_y(\zeta_1)\Delta y_k(t) - g_u(\zeta_1)\delta_2 e_{k+1}(t),$$

where g_y and g_u denotes the partial derivative of the nonlinear function, and ζ_1 is given by the Mean Value Theorem (Refn. [56], Lemma 3.2, Page 3212) in the form ζ_1 as defined in previous theorem. Collecting the term involving $e_{k+1}(t)$, we obtain

$$(1 + g_u(\zeta_1)\delta_2)e_{k+1}(t) = e_k(t) - g_y(\zeta_1)\Delta y_k(t).$$

Provided that $1 + g_u(\zeta_1)\delta_2 \neq 0$, we can write

$$e_{k+1}(t) \leq \frac{1}{(1 + g_u(\zeta_1)\delta_2)}\big(e_k(t) + g_y(\zeta)\Delta y_k(t)\big).$$

For the rearranged error equation,

$$(1 + g_u(\zeta_1)\delta_2)e_{k+1}(t) = e_k(t) - g_y(\zeta_1)\Delta y_k(t),$$

taking norms gives

$$|1 + g_u(\zeta_1)\delta_2|\,||e_{k+1}(t)|| \leq ||e_k(t)|| + |g_y(\zeta_1)|\,||\Delta y_k(t)||,$$

Multiplying both sides by $t^{1-\gamma}e^{-\lambda t}$ and taking the supremum over $t \in J$, we obtain

$$|(1 + g_u(\zeta_1)\delta_2)|\,||e_{k+1}||_{1-\gamma,\lambda} \leq ||e_k||_{1-\gamma,\lambda} + |g_y(\zeta_1)|||\Delta y_k||_{1-\gamma,\lambda}.$$

Since condition (14) *ensures $|1 + g_u(\zeta)\delta_2| \geq \sigma_3 > 0$,*
we may divide both sides to obtain

$$||e_{k+1}||_{1-\gamma,\lambda} \leq \frac{1}{|(1 + g_u(\zeta_1)\delta_2)|}||e_k||_{1-\gamma,\lambda} + \frac{|g_y(\zeta_1)|}{|(1 + g_u(\zeta_1)\delta_2)|}||\Delta y_k||_{1-\gamma,\lambda}. \tag{15}$$

Using the estimate (12)

$$||e_{k+1}||_{1-\gamma,\lambda} \leq \frac{E_\alpha(2Mb^{1+\alpha-\gamma})}{\Gamma(\gamma)}||\Delta y_k(0)|| + \frac{M}{\Lambda^\alpha}E_\alpha(2Mb^{1+\alpha-\gamma})||\Delta u_k||_{1-\gamma,\lambda}.$$

Substituting into (15)*, we get*

$$||e_{k+1}||_{1-\gamma,\lambda} \leq \Big|\frac{1}{(1 + g_u(\zeta_1)\delta_2)}\Big|||e_k||_{1-\gamma,\lambda} + \Big|\frac{g_y(\zeta_1)}{(1 + g_u(\zeta_1)\gamma_2)}\Big|\frac{||\Delta y_k(0)||_\lambda}{\Gamma(\gamma)}E_\alpha(2Mb^{1+\alpha-\gamma})$$

$$+\left|\frac{g_y(\zeta_1)}{(1+g_u(\zeta_1)\delta_2)}\right|\frac{Mb^{1-\gamma}|\delta_2|}{\lambda^\alpha}E_\alpha(2Mb^{1+\alpha-\gamma})||e_{k+1}||_{1-\gamma,\lambda}.$$

Since for closed-loop P-type ILC $\Delta u_k = \delta_2 e_{k+1}$, we have

$$||\Delta u_k||_{1-\gamma,\lambda} = |\delta_2|\,||e_{k+1}||_{1-\gamma,\lambda}.$$

Substituting again, we get

$$||e_{k+1}||_{1-\gamma,\lambda} \leq \frac{1}{|(1+g_u(\zeta_1)\delta_2)|}\Big|||e_k||_{1-\gamma,\lambda}+\left|\frac{g_y(\zeta_1)}{(1+g_u(\zeta_1)\gamma_2)}\right|\frac{||\Delta y_k(0)||_\lambda}{\Gamma(\gamma)}E_\alpha(2Mb^{1+\alpha-\gamma})$$
$$+\left|\frac{g_y(\zeta_1)}{(1+g_u(\zeta_1)\delta_2)}\right|\frac{Mb^{1-\gamma}|\delta_2|}{\lambda^\alpha}E_\alpha(2Mb^{1+\alpha-\gamma})||e_{k+1}||_{1-\gamma,\lambda}.$$

Define

$$C(\lambda) = \frac{|g_y(\zeta)|M|\delta_2|}{|1+g_u(\zeta_1)\delta_2|\lambda^\alpha}E_\alpha(2Mb^{1+\alpha-\gamma}).$$

Then

$$(1-C(\lambda))||e_{k+1}||_{1-\gamma,\lambda} \leq \frac{1}{|(1+g_u(\zeta_1)\delta_2)|}||e_k||_{1-\gamma,\lambda}+\left|\frac{g_y(\zeta_1)}{(1+g_u(\zeta)\gamma_2)}\right|\frac{||\Delta y_k(0)||_\lambda}{\Gamma(\gamma)}E_\alpha(2Mb^{1+\alpha-\gamma}).$$

Finally, we obtain

$$||e_{k+1}||_{1-\gamma,\lambda} \leq \frac{||e_k||_{1-\gamma,\lambda}}{|(1+g_u(\zeta_1)\delta_2)|-\frac{g_y(\zeta_1)q^*}{\lambda^\alpha}}+\frac{g_y(\zeta_1)E_\alpha(2Mb^{1+\alpha-\gamma})L||e_k||_\lambda}{\left[|(1+g_u(\zeta_1)\delta_2)|-\frac{g_y(\zeta_1)q^*}{\lambda^\alpha}\right]\Gamma(\alpha)}, \tag{16}$$

where $q^* = M\delta_2 E_\alpha(2Mb^{1+\alpha-\gamma})$.

Since λ can be sufficiently large, define $\rho := \frac{g_y(\zeta)q^*}{\lambda^\alpha} > 0$.
Then ρ can be made arbitrary small.
Assume from Eq.(14) that

$$\sigma_3 := |1+g_u(\zeta)\delta_2| > 1.$$

Then, for sufficiently large λ, $\sigma_3-\rho>1$.
Hence define

$$\epsilon := \frac{1}{|1+g_u(\zeta_1)\delta_2|-\rho}.$$

Because the denominator is strictly greater than 1, we have $0<\epsilon<1$.
Therefore,

$$||e_{k+1}||_{1-\gamma,\lambda} \leq \epsilon||e_k||_{1-\gamma,\lambda}+\epsilon C||e_k(0)||.$$

By Lemma 4, $||e_k(0)||\to 0$.
Hence, by the discrete contraction principle,

$$\lim_{k\to\infty}||e_k||_{1-\gamma,\lambda}=0.$$

The proof is completed.

Remark 4. *For the system given by* (3)*, assume that* $(I_{0^+}^{1-\gamma}y_k)(0) = y(0)$ *and that* $L_1(t) = L_1$ *and* $L_2(t) = L_2$ *are positive constants for* $t \in J$*. Then hypothesis* (H_3) *can be omitted. Moreover, it can be shown that the closed-loop* P*-type ILC updating law described in* (5) *guarantees that* z_k *converges to* z_d *in the* λ*-norm, provided* λ *is sufficiently large such that*

$$|1 - \delta_2 g_u(\zeta_1)| - \frac{|\delta_1| L_2 |\eta_4| E_\alpha(2L_1 b^{1+\alpha-\gamma})}{\lambda^\alpha} > 1.$$

4. Convergence properties of D-type algorithms

We analyze a Hilfer-type FO-NPEs described by

$$\begin{cases} ({}^H D_{0^+}^{\alpha,\beta} y_k)(t) = \mu y_k(t) + f(t, y_k(t), y_k(\nu t)) + u_k(t), \; t, \quad \mu < 0, \\ (I_{0^+}^{1-\gamma} y_k)(0) = y_k(0), \\ (I_{0^+}^{2-\gamma} y_k)(0) = \hat{y}_k(0), \end{cases} \tag{17}$$

where $0 < \alpha < 2, 0 \leq \beta \leq 1, \gamma = \alpha + \beta(2 - \alpha)$. The system output is given by

$$z_k(t) = c y_k(t) + d \int_0^t u_k(s) ds, \quad c, d \in \mathbb{R}, \tag{18}$$

where k denotes the k^{th} learning iteration and $b > 0$ is the prescribed iteration interval length. The nonlinear function $f : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is given and continuous. The variables $y_k(t), u_k(t), z_k(t) \in \mathbb{R}^n$ are denoted by state, input and output respectively. The operators $(I_{0^+}^{1-\gamma} y_k)(t)$, $(I_{0^+}^{2-\gamma} y_k)(t)$ denote the R-L fractional integrals, and $\nu \in (0, 1)$.
The solution of system (17) is equivalent to the solution of following integral equation (Refn. [54], Section 4, Page 15 & Refn. [57], Section 1, Page 18)

$$\begin{aligned} y_k(t) &= y_k(0) t^{\gamma-1} E_{\alpha,\gamma}(\mu t^\alpha) + \hat{y}_k(0) t^{\gamma-2} E_{\alpha,\gamma-1}(\mu t^\alpha) \\ &\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\mu(t-s)^\alpha)[f(s, y_k(s), y_k(\nu s)) + u_k(s)] ds. \end{aligned} \tag{19}$$

For the system (17), define the tracking error

$$e_k(t) = z_d(t) - z_k(t),$$

where $z_d(t)$ is the desired output trajectory.

We consider the open-loop D-type ILC updating law

$$\begin{cases} y_{k+1}(0) = y_k(0) + L\dot{e_k}(0) \\ u_{k+1}(t) = u_k(t) + \delta_1 \dot{e_k}(t), \end{cases} \tag{20}$$

where

$$\dot{e}_k(t) = \frac{d}{dt} e_k(t),$$

and $L, \delta_1 \in \mathbb{R}$ are learning gains to be determined.

The closed-loop D-type ILC updating law is given by

$$\begin{cases} y_{k+1}(0) = y_k(0) + L\dot{e_k}(0), \\ u_{k+1}(t) = u_k(t) + \delta_2 \dot{e_{k+1}}(t), \end{cases} \tag{21}$$

where L and δ_2 are learning gains to be designed.
We first analyze the convergence properties of the open-loop updating law (20).

Theorem 3. *Assuming that hypotheses $(H_1) - (H_3)$ are satisfied with $L_1(t) = L_1$ and $L_2(t) = L_2$ for $t \in J$. Consider the FO-NPEs in* (17) *with the initial conditions satisfying $e_k(0) = 0$, $\dot{e}_k(0) = 0$, and $\hat{y}_k(0) = \hat{y}(0)$, $\forall k \geq 0$. If the learning gain δ_1 satisfies $|1 - d\delta_1| < 1$, then for any initial input $u_0(t)$, the open-loop D-type ILC updating law* (20) *guarantees that*

$$z_k \to z_d \in \mathfrak{C}(J, \mathbb{R})$$

with respect to the λ-norm, provided λ is sufficiently large.

Proof. Note that

$$\begin{aligned} \dot{e}_{k+1}(t) &= \dot{z}_d(t) - \dot{z}_{k+1}(t) \\ &= \dot{z}_d(t) - \dot{z}_k(t) + \dot{z}_k(t) - \dot{z}_{k+1}(t) \\ &= \dot{e}_k(t) + (\dot{z}_k(t) - \dot{z}_{k+1}(t)). \end{aligned}$$

Since $\dot{z}_k(t) = c\dot{y}_k(t) + du_k(t)$, we obtain

$$\dot{z}_k(t) - \dot{z}_{k+1}(t) = c(\dot{y}_k(t) - \dot{y}_{k+1}(t)) + d(u_k(t) - u_{k+1}(t)).$$

Nothing that $\Delta u_k(t) = u_{k+1}(t) - u_k(t) = \delta_1 \dot{e}_k(t)$.
Therefore,

$$\dot{z}_k(t) - \dot{z}_{k+1}(t) = c(\dot{y}_k(t) - \dot{y}_{k+1}(t)) - d\delta_1 \dot{e}_k(t).$$

Substituting back yields

$$\dot{e}_{k+1}(t) = (1 - d\delta_1)\dot{e}_k(t) + c\big(\dot{y}_k(t) - \dot{y}_{k+1}(t)\big).$$

Using the integral representation and differentiating, we obtain

$$\begin{aligned} \dot{e}_{k+1}(t) &= (1 - d\delta_1)\dot{e}_k(t) \\ &\quad + c\int_0^t (t-s)^{\alpha-2} E_{\alpha,\alpha-1}(\mu(t-s)^\alpha)[f(s, y_k(s), y_k(\nu s)) - f(s, y_{k+1}(s), y_{k+1}(\nu s))]ds \\ &\quad + c\int_0^t (t-s)^{\alpha-2} E_{\alpha,\alpha-1}(\mu(t-s)^\alpha)[u_k(s) - u_{k+1}(s)]ds. \end{aligned}$$

Taking absolute values

$$|\dot{e}_{k+1}(t)| \leq |1 - d\gamma_1|\,|\dot{e}_k(t)|$$

$$+ |c| \int_0^t (t-s)^{\alpha-2} |E_{\alpha,\alpha-1}(\mu(t-s)^\alpha)| |f(s, y_k(s), y_k(\nu s)) - f(s, y_{k+1}(s), y_{k+1}(\nu s))| ds$$
$$+ |c| \int_0^t (t-s)^{\alpha-2} |E_{\alpha,\alpha-1}(\mu(t-s)^\alpha)| \, |u_k(s) - u_{k+1}(s)| ds.$$

Using M-L estimate (since $\mu < 0$)

$$|E_{\alpha,\alpha-1}(\mu(t-s)^\alpha)| \leq \frac{1}{\Gamma(\alpha-1)}.$$

Thus,

$$\begin{aligned} ||\dot{e}_{k+1}(t)|| \leq{}& ||1 - d\gamma_1|| \, ||\dot{e}_k(t)|| \\ &+ \frac{|c|}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} ||f(s, y_k(s), y_k(\nu s)) - f(s, y_{k+1}(s), y_{k+1}(\nu s))|| ds \\ &+ \frac{|c|}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \, ||u_k(s) - u_{k+1}(s)|| ds. \end{aligned}$$

Taking norms in the previous inequality and using the Lipschitz condition, we obtain

$$\begin{aligned} ||\dot{e}_{k+1}(t)|| \leq{}& ||1 - d\delta_1|| \, ||\dot{e}_k(t)|| \\ &+ \frac{|c|}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} (2L_1) ||\Delta y_k(s)|| ds \\ &+ \frac{|c|}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} L_2 ||\Delta u_k|| ds, \end{aligned} \tag{22}$$

where

$$\Delta y_k(s) = y_k(s) - y_{k+1}(s), \quad \Delta u_k(s) = u_k(s) - u_{k+1}(s).$$

Since $\Delta y_k(0) = L\dot{e}_k(0) = 0$, we obtain from the previous estimate

$$\begin{aligned} ||\Delta y_k(s)|| \leq{}& \frac{2L_1}{\Gamma(\alpha-1)} \int_0^s (s-\tau)^{\alpha-1} ||\Delta y_k(\tau)|| d\tau \\ &+ \frac{L_2}{\Gamma(\alpha-1)} \int_0^s (s-\tau)^{\alpha-1} ||\Delta u_k(\tau)|| d\tau, \end{aligned}$$

According to Lemma 2, we have

$$\begin{aligned} ||\Delta y_k(s)|| &\leq \frac{L_2}{\Gamma(\alpha-1)} \int_0^s (s-\tau)^{\alpha-1} e^{\lambda\tau} ||\Delta u_k(\tau)||_\lambda d\tau e^{\int_0^s \frac{L_2}{\Gamma(\alpha-1)} (s-\tau)^{\alpha-1} d\tau} \\ &\leq \frac{hL_2}{\alpha\Gamma(\alpha-1)} e^{\lambda s} s^\alpha ||\Delta u_k||_\lambda, \end{aligned}$$

where $\quad h = e^{\frac{L_2 b^\alpha}{\alpha\Gamma(\alpha-1)}}$.
From this,

$$e^{-\lambda s} ||\Delta y_k(s)|| \leq \frac{hL_2}{\alpha\Gamma(\alpha-1)} s^\alpha ||\Delta u_k||_\lambda,$$

which means that

$$||\Delta y_k(s)||_\lambda \leq \frac{hL_2}{\alpha\Gamma(\alpha-1)} b^\alpha ||\Delta u_k||_\lambda. \tag{23}$$

And then, we have

$$\begin{aligned} 2L_1 \int_0^t (t-s)^{\alpha-2} ||\Delta y_k(s)|| ds &= 2L_1 \int_0^t (t-s)^{\alpha-2} e^{\lambda s} e^{-\lambda s} ||\Delta y_k(s)|| ds \\ &\leq 2L_1 \frac{e^{\lambda t}}{\lambda^{\alpha-1}} \Gamma(\alpha-1) ||\Delta y_k(s)||_\lambda. \end{aligned} \tag{24}$$

Similarly, we have

$$L_2 \int_0^t (t-s)^{\alpha-2} ||\Delta u_k(s)|| ds \leq L_2 \frac{e^{\lambda t}}{\lambda^{\alpha-1}} \Gamma(\alpha-1) ||\Delta u_k(s)||_\lambda. \tag{25}$$

Taking Eq.(24) and Eq.(25) into (22), we get

$$||\dot{e}_{k+1}(t)|| \leq |1 - d\delta_1| ||\dot{e}_k(t)|| + \frac{e^{\lambda t} 2L_1|c|}{\lambda^{\alpha-1}} ||\Delta y_k||_\lambda + \frac{e^{\lambda t} L_2|c|}{\lambda^{\alpha-1}} ||\Delta u_k||_\lambda.$$

Taking λ-norm, we obtain

$$||\dot{e}_{k+1}||_\lambda \leq |1 - d\delta_1| ||\dot{e}_k||_\lambda + \frac{2L_1|c|}{\lambda^{\alpha-1}} ||\Delta y_k||_\lambda + \frac{L_2|c|}{\lambda^{\alpha-1}} ||\Delta u_k||_\lambda. \tag{26}$$

Taking Eq.(23) into Eq.(26), we get

$$\begin{aligned} ||\dot{e}_{k+1}||_\lambda &\leq |1 - d\delta_1| ||\dot{e}_k||_\lambda + \frac{h2L_1L_2b^\alpha|c|}{\alpha\lambda^{\alpha-1}\Gamma(\alpha-1)} ||\Delta u_k||_\lambda + \frac{L_2|c|}{\lambda^{\alpha-1}} ||\Delta u_k||_\lambda \\ &\leq |1 - d\delta_1| ||\dot{e}_k||_\lambda + \Big[\frac{h2L_1L_2b^\alpha|c|}{\alpha\lambda^{\alpha-1}\Gamma(\alpha-1)} + \frac{L_2|c|}{\lambda^{\alpha-1}}\Big] ||\Delta u_k||_\lambda. \end{aligned}$$

Since we can choose λ large enough such that $|1 - d\delta_1| < 1$, we can get that

$$\lim_{k\to\infty} ||\dot{e}_k||_\lambda = 0.$$

Since $e_k(0) = 0, k = 0, 1, 2, \cdots$, then we obtain

$$\lim_{k\to\infty} ||e_k||_\lambda = 0.$$

The proof is complete.

Theorem 4. *Assume that hypotheses* (H_1) *and* (H_2) *are hold with* $L_1(t) = L_1$ *and* $L_2(t) = L_2$. *Consider the FO-NPEs in* (17) *under the initial condition* $e_k(0) = 0$, $\dot{e}_k(0) = 0$, *and assume that* $\hat{y}_k(0)$ *is constant for all* $k = 0, 1, 2, \ldots$. *Then, for any initial input* u_0, *the closed-loop D-type ILC updating law given in* (21) *guarantees that the output sequence* z_k *converges to desired trajectory* $z_d \in \mathfrak{C}(J, \mathbb{R})$ *in the* λ*-norm, provided that*

$$\Big|\frac{1}{1 + d\delta_2}\Big| < 1,$$

and $\lambda > 0$ *is chosen sufficiently large.*

Proof. We start from the error dynamics

$$\dot{e}_{k+1}(t) = \dot{e}_k(t) + c\big(\dot{y}_k(t) - \dot{y}_{k+1}(t)\big) - d\Delta u_k(t).$$

Using the closed-loop updating law

$$u_{k+1}(t) = u_k(t) + \delta_2 \dot{e}_{k+1}(t),$$

we have $\Delta u_k(t) = \delta_2 \dot{e}_{k+1}(t)$.
Substituting into the error equation gives

$$\dot{e_{k+1}}(t) = \dot{e_k}(t) + c\big(\dot{y}_k(t) - \dot{y}_{k+1}(t)\big) - d\delta_2 \dot{e}_{k+1}(t).$$

Move the term involving $\dot{e}_{k+1}$ to the left

$$(1 + d\delta_2)\dot{e}_{k+1}(t) = \dot{e}_k(t) + c\big(\dot{y}_k(t) - \dot{y}_{k+1}(t)\big).$$

Assume $1 - d\delta_2 \neq 0$, we obtain

$$\dot{e}_{k+1}(t) = \frac{1}{1 + d\delta_2}\dot{e}_k(t) + \frac{c}{1 + d\delta_2}(\dot{y}_k(t) - \dot{y}_{k+1}(t)).$$

Then

$$\begin{aligned} ||\dot{e}_{k+1}(t)|| \leq & \left|\frac{1}{1 + d\delta_2}\right| ||\dot{e}_k(t)|| \\ & + \left|\frac{c}{(1 + d\delta_2)\Gamma(\alpha - 1)}\right| \int_0^t (t-s)^{\alpha-2} 2L_1 ||\Delta y_k(s)|| ds \\ & + \left|\frac{c}{(1 + d\delta_2)\Gamma(\alpha - 1)}\right| \int_0^t (t-s)^{\alpha-2} L_2 ||\Delta u_k(s)|| ds. \end{aligned} \tag{27}$$

By adopting the same method used in the proof of Theorem 3 to deal with Eq. (27), we obtain

$$\begin{aligned} ||\dot{e}_{k+1}|| \leq & \left|\frac{1}{1 + d\delta_2}\right| ||\dot{e}_k|| + \left|\frac{c}{(1 + d\delta_2)\Gamma(\alpha - 1)}\right| ||\Delta y_k||_\lambda \frac{e^{\lambda t}}{\lambda^{\alpha-1}}\Gamma(\alpha - 1) \\ & + \left|\frac{c}{(1 + d\delta_2)\Gamma(\alpha - 1)}\right| ||\Delta u_k||_\lambda \frac{e^{\lambda t}}{\lambda^{\alpha-1}}\Gamma(\alpha - 1). \end{aligned}$$

Hence

$$||\dot{e}_{k+1}||_\lambda \leq \left|\frac{1}{1 + d\delta_2}\right| ||\dot{e}_k||_\lambda + \left|\frac{c}{1 + d\delta_2}\right| \frac{1}{\lambda^{\alpha-1}} ||\Delta y_k||_\lambda + \left|\frac{c}{1 + d\delta_2}\right| \frac{1}{\lambda^{\alpha-1}} ||\Delta u_k||_\lambda.$$

Using the estimate obtained previously for Δy_k,

$$||\Delta y_k||_\lambda \leq \frac{2L_1 L_2 h}{\alpha \Gamma(\alpha - 1)} ||\Delta u_k||_\lambda,$$

we obtain

$$||\dot{e}_{k+1}||_\lambda \le \Big|\frac{1}{1+d\delta_2}\Big|\,||\dot{e}_k||_\lambda + \Big|\frac{c}{1+d\delta_2}\Big|\frac{2L_1L_2h}{\alpha\Gamma(\alpha-1)}\frac{1}{\lambda^{\alpha-1}}||\Delta u_k||_\lambda + \Big|\frac{c}{1+d\delta_2}\Big|\frac{1}{\lambda^{\alpha-1}}||\Delta u_k||_\lambda.$$

Since the closed-loop updating law satisfies

$$\Delta u_k(t) = \delta_2 \dot{e}_{k+1}(t),$$

we have

$$||\Delta u_k||_\lambda = |\delta_2|\,||\dot{e}_{k+1}||_\lambda.$$

Substituting this relation into the previous inequality yields

$$\begin{aligned} ||\dot{e}_{k+1}||_\lambda \le{}& \Big|\frac{1}{1+d\delta_2}\Big|\,||\dot{e}_k||_\lambda \\ &+ \Big[\Big|\frac{c}{1+d\delta_2}\Big|\frac{2L_1L_2h}{\alpha\Gamma(\alpha-1)}\frac{1}{\lambda^{\alpha-1}} + \Big|\frac{c}{1+d\delta_2}\Big|\frac{1}{\lambda^{\alpha-1}}\Big]|\delta_2|\,||\dot{e}_{k+1}||_\lambda. \end{aligned}$$

Hence

$$(1-A)\,||\dot{e}_{k+1}||_\lambda \le \Big|\frac{1}{1+d\delta_2}\Big|\,||\dot{e}_k||_\lambda,$$

where

$$A = \Big[\Big|\frac{c}{1+d\delta_2}\Big|\frac{2L_1L_2h}{\alpha\Gamma(\alpha-1)}\frac{1}{\lambda^{\alpha-1}} + \Big|\frac{c}{1+d\delta_2}\Big|\frac{1}{\lambda^{\alpha-1}}\Big]|\delta_2|.$$

Therefore,

$$||\dot{e}_{k+1}||_\lambda \le \frac{1}{1-A}\Big|\frac{1}{1+d\delta_2}\Big|||\dot{e}_k||_\lambda.$$

Since λ can be chosen sufficiently large and

$$\Big|\frac{1}{1+d\delta_2}\Big| < 1,$$

we obtain

$$\lim_{k\to\infty} ||\dot{e}_k||_\lambda = 0.$$

It follows that

$$\lim_{k\to\infty} ||e_k||_\lambda = 0.$$

The proof is complete.

Applications

In recent years, high-speed train systems have attracted considerable attention due to their complex dynamics and strict operational requirements. In particular, ILC has been widely applied to improve trajectory tracking performance in such systems, since train operations are typically repetitive in nature. Huang et al. (Refn.[58]) investigated an ILC scheme for high-speed train systems subject to velocity and displacement constraints. Their work focused on enhancing the tracking performance of the train while ensuring that the system satisfies practical operational limits. More recently, Yang et al. (Refn.[59]) developed an adaptive iterative learning unified operation control strategy for high-speed trains by incorporating the electrical structure model together with temperature compensation. Their approach addressed nonlinear characteristics and environmental influences in the train system, leading to improved operational accuracy and robustness.

Example 1. *Consider a high-speed electric train system in which continuous and stable electrical contact between the pantograph and the overhead catenary wire is required for efficient power transmission. During train operation, the pantograph repeatedly passes along the same section of the catenary line. Because of this repetitive behavior, the system can be effectively controlled using iterative learning control strategies.*

Let $y_k(t)$ denote the vertical displacement of the pantograph head at time $t \in J' := [0,1]$ during the k-th iteration. Owing to the viscoelastic characteristics of mechanical components and structural damping, the pantograph system exhibits memory and hereditary properties. These effects can be appropriately described using the HFD ${}^{H}D^{\alpha,\beta}_{0^+}$, where the order satisfies $0 < \alpha < 1$ and the type parameter satisfies $0 \leq \beta \leq 1$. For the numerical illustration, choose $\alpha = 0.8$ and $\beta = 0.5$. The associated parameter is defined by $\gamma = \alpha + \beta(1-\alpha)$, which yields $\gamma = 0.9$.

The pantograph dynamics are modeled by the fractional-order pantograph equation

$$({}^{H}D^{0.8,0.5}_{0^+} y_k)(t) = -y_k(t) + 0.3\, y_k(0.5t) + u_k(t), \qquad t \in J',$$

where the term $y_k(0.5t)$ represents delayed structural feedback caused by the flexibility of the pantograph arm, and $u_k(t)$ denotes the actuator force applied to regulate the pantograph displacement.
Initial condition is required, namely

$$(I^{1-\gamma}_{0^+} y_k)(0) = y_k(0).$$

Since $\gamma = 0.9$, the above condition becomes

$$(I^{0.1}_{0^+} y_k)(0) = y_k(0),$$

which represents the influence of past states on the system dynamics.

The measurable output of the system is defined by

$$z_k(t) = 1.3\, y_k(t) + 0.6 \int_0^t u_k(s)\, ds,$$

where the coefficient 1.3 denotes the sensitivity gain of the displacement sensor and the integral term reflects the accumulated actuator effort.

The control objective is to ensure that the output $z_k(t)$ tracks a desired reference trajectory $r(t)$ representing the optimal pantograph position required to maintain stable contact with the catenary wire. The reference trajectory is selected as

$$r(t) = 2t^{1.5}(1 - t^3).$$

To reduce the tracking error over successive iterations, a P-type open-loop ILC law is employed

$$u_{k+1}(t) = u_k(t) + L\, e_k(t),$$

with the initial condition updated by

$$y_{k+1}(0) = y_k(0) + \delta_1 e_k(0).$$

Choose the learning gains $L = 0.8$ and $\delta_1 = 0.5$. According to Theorem 1, the convergence condition is $|1 - d\delta_1| < 1$. Taking $d = 0.96$, we obtain

$$|1 - d\delta_1| = |1 - 0.96 \times 0.5| = 0.52 < 1,$$

which shows that the condition of Theorem 1 is satisfied. Consequently, the tracking error converges to zero as the iteration number increases.

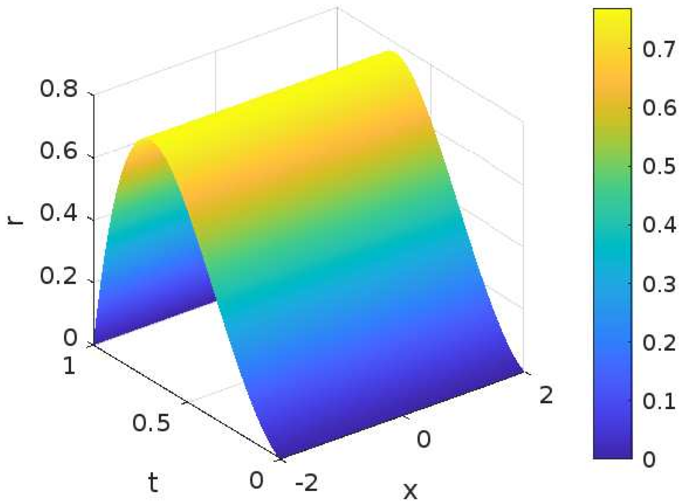

Figure 1: Reference trajectory

The simulation results illustrate the effectiveness of the proposed control strategy. Figure 1 presents the reference trajectory, Figure 2 shows the system output for several iterations, and Figure 3 depicts the tracking error. From Figure 2, it can be observed that the system output progressively approaches the desired reference trajectory as the iteration number increases. Meanwhile, Figure 3 shows that the tracking error decreases

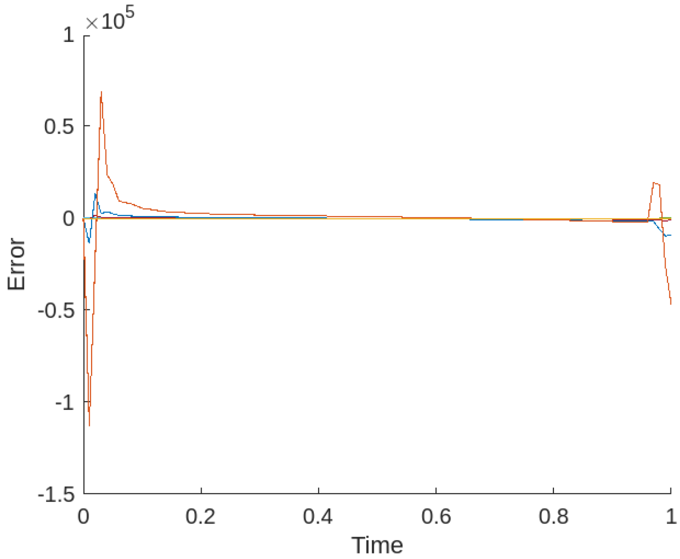

Figure 2: The system output

gradually and converges towards zero. These results verify the converges of the proposed ILC law and demonstrate its effectiveness for the considered fractional-order pantograph system.

Example 2. *Consider a high-speed railway system in which the interaction between the pantograph and the catenary must be controlled to ensure stable current collection. Let* $y_k(t)$ *denote the vertical displacement of the pantograph head at time* t *during the* k*-th iteration (i.e., the* k*-th train pass). Due to the viscoelastic nature of the mechanical structure, memory and hereditary effects arise in the system dynamics. These effects are effectively modeled using the HFD of order* $1 < \alpha < 2$.

Let $\alpha = 1.7$ and $\beta = 0.3$, and consider the time interval $t \in [0, 1]$. The dynamics of the system are described by the fractional-order pantograph equation

$$\begin{cases} ({}^H D_{0^+}^{\alpha,\beta} y_k)(t) = -0.5 y_k(t) + 0.3 y_k(0.8t) + (t+1) u_k(t), & t \in [0,1], \\ \left(I_{0^+}^{1-\gamma} y_k \right)(0^+) = y_{k,0}, \\ \left(I_{0^+}^{2-\gamma} y_k \right)(0^+) = v_{k,0}, \end{cases}$$

where $y_k(0.8t)$ represents the delayed displacement feedback arising from the structural damping of the pantograph-catenary system, $u_k(t)$ denotes the actuator control input applied during the k-th iteration, and the factor $(t+1)$ reflects the increasing influence of the actuator force over time.
Define the nonlinear function

$$f(t, y, z) = -0.5y + 0.3z.$$

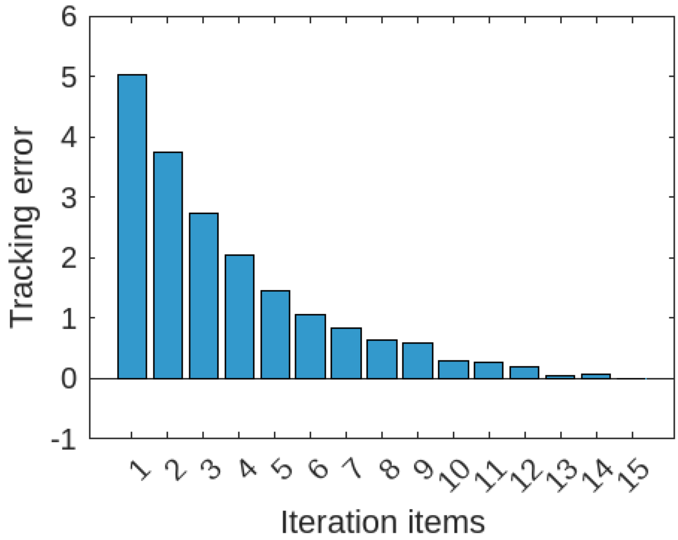

Figure 3: The tracking error

Then,

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| = | - 0.5(y_1 - y_2) + 0.3(z_1 - z_2)| \leq 0.5|y_1 - y_2| + 0.3|z_1 - z_2|.$$

Hence, the function f satisfies the Lipschitz condition with Lipschitz constant $L_1 = 0.5$. For the HFD with $1 < \alpha < 2$, define $\gamma = \alpha + \beta(2 - \alpha)$. Substituting $\alpha = 1.7$ and $\beta = 0.3$, we obtain $\gamma = 1.79$.
These conditions reflect the memory dependent initialization inherent in the HFD.
The measurable output of the system is defined as

$$z_k(t) = y_k(t) + 1.5u_k(t),$$

where the constant $d = 1.5$ represents the gain describing how strongly the control input influences the observed output.
Suppose the desired reference trajectory is

$$r(t) = 2t^2 + \cos(\pi t), \qquad t \in [0, 1].$$

To achieve output tracking, we employ a D-type ILC update law given by

$$y_{k+1}(0) = y_k(0) + Le_k(0), u_{k+1}(t) = u_k(t) + \delta_1 e_k(t),$$

where the learning gain is chosen as $\delta_1 = 0.6$. The initial control input is taken as

$$u_k(0) = 0, \qquad t \in [0, 1].$$

According to the convergence condition stated in Theorem 3, the learning process converges if

$$|1 - d\delta_1| < 1.$$

Substituting $d = 1.5$ and $\delta_1 = 0.6$, we obtain

$$|1 - d\delta_1| = |1 - 1.5 \times 0.6| = 0.1 < 1.$$

Hence, the contraction condition is satisfied. Consequently, all the assumptions of Theorem 3 hold and the tracking error satisfies

$$\lim_{k\to\infty} \|e_k\|_\lambda = 0.$$

To illustrate the convergence behavior, the numerical values of the tracking error $e_k(t)$ at selected time points are listed in Table 1. From Table 1, it can be observed that

Table 1: Tracking error values for different iterations

Iteration	$t = 0.00$	$t = 0.01$	$t = 0.02$	$t = 0.03$	$t = 0.04$	$t = 0.05$
$k = 1$	1.0000	0.9997	0.9988	0.9973	0.9952	0.9925
$k = 2$	1.0000	0.1999	0.1998	0.1995	0.1990	0.1985
$k = 3$	1.0000	0.0076	0.0073	0.0069	0.0066	0.0063
$k = 4$	1.0000	0.0268	0.0273	0.0275	0.0276	0.0277
$k = 5$	1.0000	0.0196	0.0204	0.0206	0.0208	0.0209

the tracking error decreases significantly during the initial iterations. Although slight oscillatory behavior appears in later iterations, which is typical for D-type ILC schemes, the overall magnitude of the error remains considerably smaller than that of the first iteration. Therefore, the weighted norm of the error sequence exhibits contraction behavior consistent with the theoretical convergence condition.

Figure 4 illustrates the desired reference trajectory, Figure 5 shows the system output at the 10^{th} iteration together with the reference trajectory, and Figure 6 depicts the tracking error evolution.

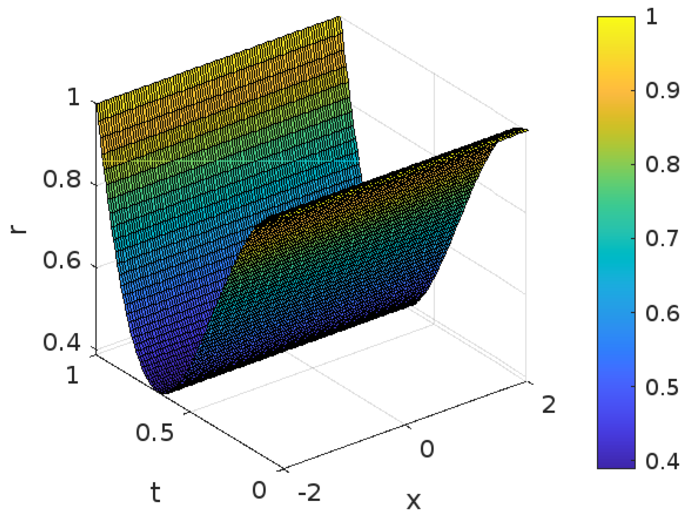

Figure 4: Reference trajectory

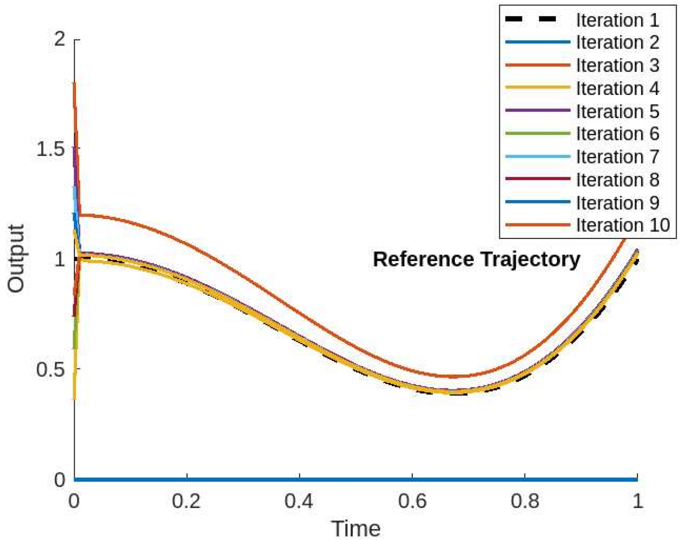

Figure 5: System output at the 10^{th} iteration

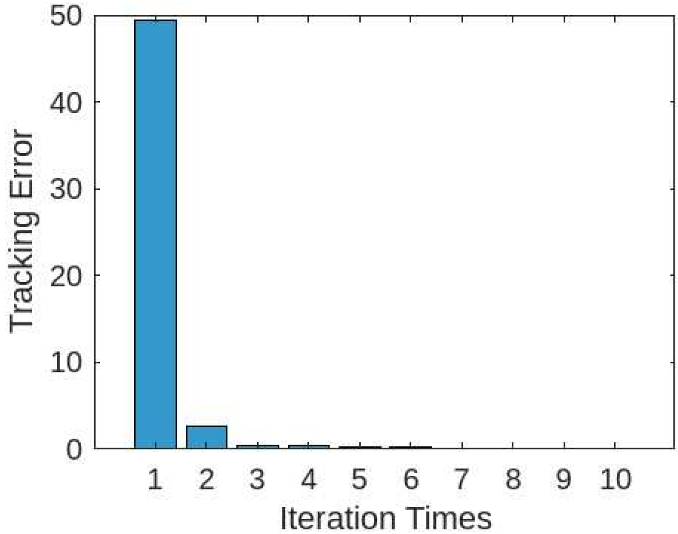

Figure 6: Tracking error evolution

Therefore, the numerical results validate the theoretical analysis the D-type iterative learning controller ensures asymptotic convergence of the tracking error despite the presence of fractional-order memory effect and delayed state feed back.

Remark 5. *(Refn. [60], Page 701) In our forthcoming paper, we investigate neutral FO-NPEs in the sense of the HFD. The system is formulated as*

$$({}^{H}D_{0^+}^{\alpha,\beta}y_k)(t) = \mu y_k(t) + f(t, y_k(t), y_k(\nu t), ({}^{H}D_{0^+}^{\alpha,\beta}y_k)(\nu t), u_k(t)), \quad t \in J, \mu < 0, \tag{28}$$

where

$$1 < \alpha < 2, \quad 0 \le \beta \le 1, \quad 0 < \nu < 1.$$

The term $y_k(\nu t)$ *denotes the pantograph-type delayed state, while* $({}^H D_{0^+}^{\alpha,\beta} y_k)(\nu t)$ *represents the neutral delayed fractional derivative.*

Since $1 < \alpha < 2$*, two R-L type initial conditions are required. Let* $\gamma = \alpha + \beta(1-\alpha)$*. Then*

$$\left(I_{0^+}^{(1-\beta)(2-\alpha)} y_k\right)(0) = y_{k,0},$$
$$\left(I_{0^+}^{(1-\beta)(2-\alpha)} \tilde{y}_k\right)(0) = \tilde{y}_{k,0},$$

Here, $y_{k,0}$ *and* $\tilde{y}_{k,0}$ *represent generalized initial displacement and velocity, respectively.*

For this system (28)*, we design the* PD*-type ILC updating law in both open-loop and closed-loop configurations. Under appropriate contraction conditions, the proposed learning scheme ensures*

$$\lim_{k \to \infty} ||e_k(t)|| = 0,$$

which implies asymptotic convergence of the tracking error.

Conclusion

This paper addressed the trajectory tracking problem for a class of nonlinear fractional pantograph systems governed by the HFD within the concept of ILC. Both P-type and D-type ILC updating laws were developed in open-loop and closed-loop configurations to enhance tracking performance over repeated iterations. By employing a suitable weighted norm and contraction analysis, sufficient conditions were established to guarantee the asymptotic convergence of the tracking error. A numerical example motivated by a high-speed electric train pantograph system illustrated the effectiveness of the proposed control schemes. The simulation results confirmed that the system output approached the desired reference trajectory while the tracking error decreased as the iteration number increased. Future work was suggested toward extending the analysis to neutral fractional-order nonlinear pantograph equations in the Hilfer sense using more advanced learning strategies such as PD-type ILC laws.

Authors Contribution

D. Vivek: Writing - original draft; conceptualization; methodology; validation; writing - review and editing; supervision. S. Sunmitha: Writing - original draft; validation; conceptualization; methodology. A. A. Alghamdi, M. M El-Dessoky: review and editing; validation, methodology. E. M. Elsayed: Writing - original draft review; validation; guidance.

Conflict of Interest Statement

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data Availability

The data utilized in this study are accessible from the corresponding author upon reasonable request.

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