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Composition Series and Jordan-Hölder Theorem Under Fuzzy Multigroups

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Abstract. The theory of fuzzy multigroups is an algebraic structure derivable from the application of fuzzy multisets to groups. Various concepts in group theory have been considered in fuzzy multigroup theory. However, simple groups, maximal normal subgroups, normal series, composition series, and the Jordan-Hölder Theorem are open problems in fuzzy multigroup theory. Hence, this paper defines simple fuzzy multigroups, maximal normal fuzzy submultigroups, normal series for fuzzy multigroups, and composition series for fuzzy multigroups with illustrations. In addition, the Jordan-Hölder Theorem in fuzzy multigroup theory is established. We show that each finite fuzzy multigroup of a finite group possesses a composition series, and we also prove that two composition series for a finite fuzzy multigroup of a finite group are comparable.

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Key Words and Phrases: Fuzzy multiset, fuzzy multigroup, order of fuzzy multigroup, simple fuzzy multigroup, maximal normal fuzzy submultigroup, normal series, composition series, Jordan–Hölder theorem

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1. Introduction

Zadeh [1] introduced the idea of fuzzy sets, which has enhanced the resolution of uncertainty. A fuzzy set is a set that includes degrees of membership of elements defined within $[0, 1]$ in its collection. Rosenfeld [2] utilized the idea of fuzzy sets to present the idea of fuzzy subgroups as an algebraic structure. Many scholars have applied various group's theoretic notions to fuzzy sets [3–5]. Using the idea of fuzzy sets and the concept of multiset which admit repeated elements [6], Yager [7] proposed a fuzzy multiset (FMS) as a set with repeated degrees of membership. Copious studies on FMSs have been presented with various real-world applications [8–13]. In [14], the notion of FMS theory was revisited to address formal inconsistencies in existing definitions, algebraic properties, and it was proved that the n-dimensional fuzzy sets are isomorphic to n-bounded fuzzy multisets.

Shinoj et al. [15] introduced fuzzy multigroup (FMG) as an application of groups under FMS framework and deduced some properties of the concept. The properties of fuzzy submultigroups (FSMGs) of a FMG were studied [16] and commutative FMGs were presented with a number of results [17, 18]. A number of group similar concepts in FMGs were presented [19–25]. The ideas of direct product of FMGs , solvable FMGs, $t-$norms over FMGs, and divisible FMGs have been explored [26–29]. In [30], some properties of FMG which are analogous of the classical group theory were explored, Mirvakili et al. [31] presented some FMGs obtained from fuzzy groups, and the concept of cosets was extended to FMGs [32], where count membership sequence is taken into consideration. Of a recent, the famous Jordan-Hölder Theorem has been instituted in both multigroups [33] and intuitionistic fuzzy groups [34], respectively.

Despite the fact that some group theory's views have been studied under FMGs, the concepts of simple FMGs, maximal normal FSMGs, normal series for FMGs, composition series for FMGs, and the Jordan-Hölder Theorem are yet to be established in FMG setting, which constitute the aim and motivations for this work. The outline of the work are as follows: as Section 1 has given the background and motivation of the study; Section 2 presents FMSs, FMGs, and their properties; Section 3 introduces the concepts of simple FMGs, maximal normal FSMGs, normal series for FMGs, composition series for FMGs, and the Jordan-Hölder Theorem under FMGs; and Section 4 concludes the paper.

2. Preliminaries

Let S and G represent a non-empty set and a group, respectively.

Definition 1 ([1])**.** A fuzzy subset denoted as $\mathfrak{X}$ in S is an object of the form

$$\mathfrak{X} = \{\langle s, \beta_{\mathfrak{X}}(s)\rangle \mid s \in S\}, \tag{1}$$

where $\beta_{\mathfrak{X}}\colon S \to [0, 1]$ is the membership grade of $s \in S$.

Definition 2 ([35])**.** A multiset $\mathcal{M}$ over S is a pair $\langle S, C_{\mathcal{M}}\rangle$, where

$$C_{\mathcal{M}} : S \to \mathbb{N} \tag{2}$$

is a function, such that $s \in S$ implies $\mathcal{M}(s) = C_{\mathcal{M}}(s) > 0$ and $C_{\mathcal{M}}(s)$ denoted the number of times an object s appears in $\mathcal{M}$. If $C_{\mathcal{M}}(s) = 0$, then $s \notin S$.

Definition 3 ([7])**.** A FMS denoted as $\mathfrak{W}$ drawn from S can be characterized by a function, count membership denoted by $CM_{\mathfrak{W}}$ such that

$$CM_{\mathfrak{W}} : S \to Q, \tag{3}$$

where Q is the set of all crisp multisets from the unit interval $I = [0, 1]$. Thus for any $s \in S$, $CM_{\mathfrak{W}}(s)$ is a crisp multiset drawn from $[0, 1]$.

According to [14], a FMS $\mathfrak{W}$ from S is represented by a multiplicity function

$$C_{\mathfrak{W}} : S \times [0, 1] \to \mathbb{N}, \tag{4}$$

where $C_{\mathfrak{W}}(s, \beta_{\mathfrak{W}}(s))$ denotes the number of times element $s \in S$ appears with membership value $\beta_{\mathfrak{W}}(s) \in [0, 1]$.

From [36], the membership sequence for $s \in S$ is defined to be the decreasingly-ordered sequence of the element in $CM_{\mathfrak{W}}(s)$, denoted by

$$CM_{\mathfrak{W}}(s) = \beta^1_{\mathfrak{W}}(s), \beta^2_{\mathfrak{W}}(s), \cdots, \beta^n_{\mathfrak{W}}(s), \tag{5}$$

where $\beta^1_{\mathfrak{W}}(s) \geq \beta^2_{\mathfrak{W}}(s) \geq \cdots \geq \beta^n_{\mathfrak{W}}(s)$.

A FMS $\mathfrak{W}$ can be represented by $\mathfrak{W} = \{\langle s, CM_{\mathfrak{W}}(s)\rangle | s \in S\}$ or $\mathfrak{W} = \{\langle \frac{CM_{\mathfrak{W}}(s)}{s}\rangle | s \in S\}$.

Definition 4 ([10, 36])**.** Suppose $\mathfrak{W}$ and $\mathfrak{Y}$ are FMSs of S, then

(i) $\mathfrak{W} = \mathfrak{Y} \iff CM_{\mathfrak{W}}(s) = CM_{\mathfrak{Y}}(s) \ \forall \ s \in S$,

(ii) $\mathfrak{W} \subseteq \mathfrak{Y} \iff CM_{\mathfrak{W}}(s) \leq CM_{\mathfrak{Y}}(s) \ \forall \ s \in S$,

(iii) $\mathfrak{W} \cap \mathfrak{Y} \implies CM_{\mathfrak{W}\cap\mathfrak{Y}}(s) = \min\{CM_{\mathfrak{W}}(s), CM_{\mathfrak{Y}}(s)\} \ \forall \ s \in S$,

(iv) $\mathfrak{W} \cup \mathfrak{Y} \implies CM_{\mathfrak{W}\cup\mathfrak{Y}}(s) = \max\{CM_{\mathfrak{W}}(s), CM_{\mathfrak{Y}}(s)\} \ \forall \ s \in S$,

(v) $\mathfrak{W} \oplus \mathfrak{Y} \implies CM_{\mathfrak{W}\oplus\mathfrak{Y}}(s) = \min\{CM_{\mathfrak{W}}(s) + CM_{\mathfrak{Y}}(s), 1\} \ \forall \ s \in S$,

(vi) $\mathfrak{W} \ominus \mathfrak{Y} \implies CM_{\mathfrak{W}\ominus\mathfrak{Y}}(s) = \max\{CM_{\mathfrak{W}}(s) - CM_{\mathfrak{Y}}(s), 0\} \ \forall \ s \in S$.

Definition 5 ([36])**.** Suppose $\mathfrak{W}$ is FMS of S, then the support/level set of $\mathfrak{W}$ denoted as $\mathfrak{W}_*$ is defined by

$$\mathfrak{W}_* = \{s \in S | CM_{\mathfrak{W}}(s) \neq 0 \text{ and } \max\{CM_{\mathfrak{W}}(s)\} > 0\},$$

where $CM_{\mathfrak{W}}(s)$ is the multiset of membership grades.

In fact, $\mathfrak{W}_*$ is a crisp set containing all elements $s \in S$ that have a non-zero membership sequence, meaning at least one membership degree is greater than zero.

Example 1. Suppose $S = \{a, b, c\}$, then the FMSs drawn from S can be

$$\mathfrak{W} = \{\langle \frac{0.6, 0.4, 0.2}{a} \rangle, \langle \frac{0.6, 0.6}{b} \rangle, \langle \frac{0.4}{c} \rangle\}$$

or

$$\mathfrak{Y} = \{\langle \frac{0.6, 0.4}{a} \rangle, \langle \frac{0.4, 0.4, 0.1}{c} \rangle\}.$$

Then, $\mathfrak{W}_* = \{a, b, c\} = S$ and $\mathfrak{Y}_* = \{a, c\} \subset S$. Thus every support of a FMS over S is contained in S.

Definition 6 ([36])**.** Suppose $\mathfrak{W}$ is a FMS of S, then the length of an element $s_i \in S$ in $\mathfrak{W}$ is denoted as $L(s_i; \mathfrak{W})$ and defined by

$$L(s_i; \mathfrak{W}) = \max\{i | \beta^i_{\mathfrak{W}}(s_i) \neq 0\}, \tag{6}$$

where $i = 1, 2 \cdots, n$.

The length of an element $s_i \in S$ in FMSs $\mathfrak{W}$ and $\mathfrak{Y}$ of S is denoted as $L(s_i; \mathfrak{W}, \mathfrak{Y})$ and defined by

$$L(s_i; \mathfrak{W}, \mathfrak{Y}) = \max\{L(s_i; \mathfrak{W}), L(s_i; \mathfrak{Y})\}, \tag{7}$$

where $i = 1, 2 \cdots, n$.

Using Example 1, we have

$$\begin{aligned} L(a; \mathfrak{W}) = 3, L(b; \mathfrak{W}) = 2, L(c; \mathfrak{W}) = 1, \\ L(a; \mathfrak{Y}) = 2, L(b; \mathfrak{Y}) = 0, L(c; \mathfrak{Y}) = 3, \\ L(a; \mathfrak{W}, \mathfrak{Y}) = 3, L(b; \mathfrak{W}, \mathfrak{Y}) = 2, L(c; \mathfrak{W}, \mathfrak{Y}) = 3. \end{aligned}$$

Definition 7. Suppose $\mathfrak{W}$ is a FMS of S, then the cardinality of $\mathfrak{W}$ denoted as $\mathrm{Card}(\mathfrak{W})$ is defined by

$$\mathrm{Card}(\mathfrak{W}) = \sum_{i=1}^{n} L(s_i, \mathfrak{W}). \tag{8}$$

Using Example 1, we have $\mathrm{Card}(\mathfrak{W}) = 6$ and $\mathrm{Card}(\mathfrak{Y}) = 5$.

Definition 8 ([15])**.** A FMS $\mathfrak{W}$ of G is called a FMG of G if:

(i) $CM_{\mathfrak{W}}(xy) \geq \min\{CM_{\mathfrak{W}}(x), CM_{\mathfrak{W}}(y)\} \ \forall \ x, y \in G$,

(ii) $CM_{\mathfrak{W}}(x^{-1}) = CM_{\mathfrak{W}}(x) \ \forall \ x \in G$.

It is worthy to note that, $CM_{\mathfrak{W}}(e) \geq CM_{\mathfrak{W}}(x) \ \forall \ x \in X$ since

$$CM_{\mathfrak{W}}(e) = CM_{\mathfrak{W}}(xx^{-1}) \geq \min\{CM_{\mathfrak{W}}(x), CM_{\mathfrak{W}}(x)\} = CM_{\mathfrak{W}}(x) \ \forall x \in G.$$

In addition, $\mathfrak{W}_*$ is a subgroup of G.

Theorem 1 ([16])**.** The support of every FMG of G is a subgroup of G.

Example 2. Let $G = \{1, a, a^2, a^3\}$ be a group where $a^4 = 1$, $a^{-1} = a^3$, $(a^2)^{-1} = a^2$, and $(a^3)^{-1} = a$, then $\mathfrak{W} = \{\langle\frac{0.9, 0.9, 0.9, 0.9}{1}\rangle, \langle\frac{0.6, 0.6, 0.6}{a^2}\rangle\}$ is a FMG of G. Certainly,

$$\begin{gathered} CM_{\mathfrak{W}}(1.1) = CM_{\mathfrak{W}}(1) \geq \min\{CM_{\mathfrak{W}}(1), CM_{\mathfrak{W}}(1)\} = CM_{\mathfrak{W}}(1), \\ CM_{\mathfrak{W}}(1.a^2) = CM_{\mathfrak{W}}(a^2) \geq \min\{CM_{\mathfrak{W}}(1), CM_{\mathfrak{W}}(a^2)\} = CM_{\mathfrak{W}}(a^2), \\ CM_{\mathfrak{W}}(a^2.a^2) = CM_{\mathfrak{W}}(1) \geq \min\{CM_{\mathfrak{W}}(a^2), CM_{\mathfrak{W}}(a^2)\} = CM_{\mathfrak{W}}(a^2), \\ CM_{\mathfrak{W}}((a^2)^{-1}) = CM_{\mathfrak{W}}(a^2). \end{gathered}$$

Also, $\mathfrak{Y} = \{\langle\frac{0.9, 0.9, 0.9, 0.9}{1}\rangle, \langle\frac{0.5, 0.5}{a}\rangle, \langle\frac{0.6, 0.6, 0.6}{a^2}\rangle, \langle\frac{0.5, 0.5}{a^3}\rangle\}$ is a FMG of G. Then, $\mathfrak{W}_* = \{1, a^2\}$ and $\mathfrak{Y}_* = \{1, a, a^2, a^3\}$ are subgroups of G (but $\mathfrak{Y}_*$ is a trivial subgroup since $\mathfrak{Y}_* = G$).

Definition 9 ([17])**.** A FMG $\mathfrak{W}$ of G is commutative if $CM_{\mathfrak{W}}(xy) = CM_{\mathfrak{W}}(yx)\ \forall\ x, y \in G$. If G is commutative, then a FMG $\mathfrak{W}$ of G is a commutative FMG.

Definition 10 ([16])**.** If $\mathfrak{W}$ and $\mathfrak{Y}$ are FMGs of G, then $\mathfrak{W}$ is a FSMG of $\mathfrak{Y}$ if $\mathfrak{W} \subseteq \mathfrak{Y}$. Again, $\mathfrak{W}$ is a proper FSMG of $\mathfrak{Y}$ if $\mathfrak{W} \subseteq \mathfrak{Y}$ and $\mathfrak{W} \neq \mathfrak{Y}$.

Definition 11 ([23])**.** Let $\mathfrak{W}$ and $\mathfrak{Y}$ be FMGs of G. Then, the product $\mathfrak{W} \circ \mathfrak{Y}$ is a FMS of G defined as follows:

$$CM_{\mathfrak{W}\circ\mathfrak{Y}}(x) = \begin{cases} \bigvee_{x=yz} \min\{CM_{\mathfrak{W}}(y), CM_{\mathfrak{Y}}(z)\}, & \text{if } \exists\, y, z \in G \text{ where } x = yz \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

Definition 12 ([21])**.** If $\mathfrak{W}$ is a FSMG of a FMG $\mathfrak{Y}$ of G, then $\mathfrak{W}$ is normal in $\mathfrak{Y}$ denoted by $\mathfrak{W} \triangleleft \mathfrak{Y}$ if $CM_{\mathfrak{W}}(xy) = CM_{\mathfrak{W}}(yx) \iff CM_{\mathfrak{W}}(y) = CM_{\mathfrak{W}}(x^{-1}yx)\ \forall\ x, y \in G$. Certainly, any normal FSMG is commutative and self-normal.

Definition 13 ([22])**.** Assume $\mathfrak{W}$ is a FSMG of a FMG $\mathfrak{Y}$ of G. Then, the fuzzy submultiset $y\mathfrak{W}$ of $\mathfrak{Y}$ for $y \in G$ defined by $CM_{y\mathfrak{W}}(x) = CM_{\mathfrak{W}}(y^{-1}x)\ \forall\ x \in G$ is a left fuzzy comultiset of $\mathfrak{W}$. Similarly, $\mathfrak{W}y$ of $\mathfrak{Y}$ such that $CM_{\mathfrak{W}y}(x) = CM_{\mathfrak{W}}(xy^{-1})\ \forall\ x \in G$ is a right fuzzy comultiset of $\mathfrak{W}$.

Definition 14 ([22])**.** If $\mathfrak{W}$ is a FMG of G and $\mathfrak{W}$ is a normal FSMG in $\mathfrak{Y}$. Then, the set of right/left fuzzy comultisets of $\mathfrak{W}$ such that $CM_{x\mathfrak{W}\circ y\mathfrak{W}}(z) = CM_{xy\mathfrak{W}}(z)\ \forall\ x, y, z \in G$ is a quotient FMG of $\mathfrak{Y}$ by $\mathfrak{W}$, denoted by $\mathfrak{Y}/\mathfrak{W}$.

3. Main Results

Before we introduce normal series for FMGs, composition series for FMGs, and Jordan-Hölder therorem under FMGs, the notion of order in FMGs is defined.

Definition 15. Assume $\mathfrak{W}$ is a FMG of G, then the order of $\mathfrak{W}$ denoted as $\mathrm{O}(\mathfrak{W})$ or $|\mathfrak{W}|$ is defined by

$$\mathrm{O}(\mathfrak{W}) = \sum_{i=1}^{n} L(s_i, \mathfrak{W}), \tag{10}$$

where $L(s_i, \mathfrak{W})$ is the multiplicity of the count membership degrees for each of the elements in $\mathfrak{W}$.

Using Example 2, we have $L(1, \mathfrak{W}) = 4$, $L(a^2, \mathfrak{W}) = 3$, and thus, $\mathrm{O}(\mathfrak{W}) = 7$. For $\mathfrak{Y}$, we have $L(1, \mathfrak{W}) = 4$, $L(a, \mathfrak{W}) = 2$, $L(a^2, \mathfrak{W}) = 3$, and $L(a^3, \mathfrak{W}) = 2$. Thus, $\mathrm{O}(\mathfrak{Y}) = 11$. Hence, $\mathrm{O}(\mathfrak{W}) < \mathrm{O}(\mathfrak{W})$.

If $\mathrm{O}(\mathfrak{W}) < \mathbb{N}$, then $\mathfrak{W}$ is finite. Although the finiteness of G enhances the finiteness of $\mathfrak{W}$, the finiteness of G cannot solely decide the finiteness of $\mathfrak{W}$.

Now, we reiterate the notion of solvable FMG as presented in [27].

Definition 16. Let $\mathfrak{W}$ be a finite FMG of a finite G, then $\mathfrak{W}$ has a chain of consecutive FSMGs:

$$\mathfrak{W}_0 \subseteq \mathfrak{W}_1 \subseteq \cdots \subseteq \mathfrak{W}_n = \mathfrak{W}, \tag{11}$$

where the underlying elements of the consecutive FSMGs are the same as the elements of G. That is, $(\mathfrak{W}_0)_* = (\mathfrak{W}_1)_* = \cdots = (\mathfrak{W}_n)_* = \mathfrak{W}_* = G$.

Then (11) can be presented as:

$$CM_{\mathfrak{W}_0}(x) \leq CM_{\mathfrak{W}_1}(x) \leq \cdots \leq CM_{\mathfrak{W}_n}(x) = CM_{\mathfrak{W}}(x) \; \forall x \in G. \tag{12}$$

Definition 17. If $\mathfrak{W}$ is a FMG of G, then $\mathfrak{W}$ is solvable if it has a chain of consecutive FSMGs:

$$\mathfrak{W}_0 \subseteq \mathfrak{W}_1 \subseteq \cdots \subseteq \mathfrak{W}_n = \mathfrak{W}, \tag{13}$$

where $\mathfrak{W}_{i-1} \lhd \mathfrak{W}_i$ and $\mathfrak{W}_i/\mathfrak{W}_{i-1}$ is commutative $\forall\ 1 \leq i \leq n$.

The finite chain of the consecutive FSMGs of $\mathfrak{W}$ is a solvable series for $\mathfrak{W}$ represented by $\mathfrak{W}_i$ for $i = 1, 2, \cdots, n$, and it is:

$$\mathfrak{W}_0 \lhd \mathfrak{W}_1 \lhd \cdots \lhd \mathfrak{W}_n = \mathfrak{W}, \tag{14}$$

where $(\mathfrak{W}_0)_* = (\mathfrak{W}_1)_* = \cdots = (\mathfrak{W}_n)_* = \mathfrak{W}_* = G$.

Next, the concepts of maximal normal FSMG of a FMG and simple FMG are defined. Using Definition 12, we define a maximal normal FSMG of a FMG as follows:

Definition 18. A FSMG $\mathfrak{M}$ of a FMG $\mathfrak{W}$ in G is trivial if:

(i) $\mathfrak{M}$ contains only the count membership degrees of the identity element,

(ii) $\mathfrak{M}$ is a fuzzy group of G or the fuzzy group of the subgroups of G,

(iii) $\mathfrak{M} = \mathfrak{W}$.

Example 3. Suppose $G = \{1, a, a^2, a^3\}$ where $a^4 = 1$, $a^{-1} = a^3$, $(a^2)^{-1} = a^2$, and $(a^3)^{-1} = a$. Then, a FMG of G is:

$$\mathfrak{W} = \{\langle \frac{0.9, 0.9, 0.9, 0.9}{1} \rangle, \langle \frac{0.5, 0.5}{a} \rangle, \langle \frac{0.6, 0.6, 0.6}{a^2} \rangle, \langle \frac{0.5, 0.5}{a^3} \rangle\}.$$

Certainly, $\mathfrak{W}$ is commutative fuzzy multigroup. The FSMGs of $\mathfrak{W}$ can be presented in terms of:

(i) the properties of G without count membership degrees (since every fuzzy subgroup is a trivial FSMG), i.e. $\{\langle\frac{0.9}{1}\rangle\}$, $\{\langle\frac{0.9}{1}\rangle, \langle\frac{0.6}{a^2}\rangle\}$, $\{\langle\frac{0.9}{1}\rangle, \langle\frac{0.5}{a}\rangle, \langle\frac{0.6}{a^2}\rangle, \langle\frac{0.5}{a^3}\rangle\}$ are FSMGs of $\mathfrak{W}$. Here, all the FSMGs are trivial.

(ii) $\mathfrak{W}_*$ with count membership degrees, i.e. $\{\langle\frac{0.9,0.9,0.9}{1}\rangle, \langle\frac{0.5}{a}\rangle, \langle\frac{0.6,0.6}{a^2}\rangle, \langle\frac{0.5}{a^3}\rangle\}$, which is a non-trivial FSMG of $\mathfrak{W}$.

(iii) the properties of G with count membership degrees, i.e. $\{\langle\frac{0.9,0.9,0.9}{1}\rangle\}$ and $\{\langle\frac{0.9,0.9,0.9}{1}\rangle, \langle\frac{0.6,0.6}{a^2}\rangle\}$ are FSMGs of $\mathfrak{W}$. Here, $\{\langle\frac{0.9,0.9,0.9}{1}\rangle\}$ is trivial and $\{\langle\frac{0.9,0.9,0.9}{1}\rangle, \langle\frac{0.6,0.6}{a^2}\rangle\}$ is a non-trivial FSMG of $\mathfrak{W}$.

Also, $\mathfrak{W}$ is a trivial FSMG of itself.

Definition 19. Let $\mathfrak{M}$ and $\mathfrak{W}$ be FMGs of G with $\mathfrak{M} \triangleleft \mathfrak{W}$. Then

(i) $\mathfrak{M}$ is a maximal non-trivial normal FSMG if it is the largest proper non-trivial normal FSMG of $\mathfrak{W}$.

(ii) $\mathfrak{W}$ is simple if it has no proper non-trivial normal FSMG.

From Example 3, $\mathfrak{W}$ is commutative FMG. The FSMGs of $\mathfrak{W}$ based on Case (i) are as follows:

$$\begin{aligned}
\mathfrak{W}_0 &= \{\langle\frac{0.9}{1}\rangle\},\\
\mathfrak{W}_1 &= \{\langle\frac{0.9}{1}\rangle, \langle\frac{0.6}{a^2}\rangle\},\\
\mathfrak{W}_2 &= \{\langle\frac{0.9}{1}\rangle, \langle\frac{0.5}{a}\rangle, \langle\frac{0.6}{a^2}\rangle, \langle\frac{0.5}{a^3}\rangle\}.
\end{aligned}$$

Here, $\mathfrak{W}$ is not among the FSMGs because it contains count membership degrees and $\mathfrak{W}_0$, $\mathfrak{W}_1$, and $\mathfrak{W}_2$ are trivial FSMGs of $\mathfrak{W}$. Because there no proper non-trivial normal FSMG, $\mathfrak{W}$ is a simple FMG. In addition, $\mathfrak{W}$ has no maximal non-trivial normal FSMG.

The FSMGs of $\mathfrak{W}$ using Case (ii) can be seen as follows:

$$\begin{aligned}
\mathfrak{W}_2 &= \{\langle\frac{0.9}{1}\rangle, \langle\frac{0.5}{a}\rangle, \langle\frac{0.6}{a^2}\rangle, \langle\frac{0.5}{a^3}\rangle\},\\
\mathfrak{W}_3 &= \{\langle\frac{0.9, 0.9}{1}\rangle, \langle\frac{0.5}{a}\rangle, \langle\frac{0.6}{a^2}\rangle, \langle\frac{0.5}{a^3}\rangle\},\\
\mathfrak{W}_4 &= \{\langle\frac{0.9, 0.9}{1}\rangle, \langle\frac{0.5, 0.5}{a}\rangle, \langle\frac{0.6, 0.6}{a^2}\rangle, \langle\frac{0.5, 0.5}{a^3}\rangle\},\\
\mathfrak{W}_5 &= \{\langle\frac{0.9, 0.9, 0.9}{1}\rangle, \langle\frac{0.5}{a}\rangle, \langle\frac{0.6}{a^2}\rangle, \langle\frac{0.5}{a^3}\rangle\},\\
\mathfrak{W}_6 &= \{\langle\frac{0.9, 0.9, 0.9}{1}\rangle, \langle\frac{0.5, 0.5}{a}\rangle, \langle\frac{0.6, 0.6}{a^2}\rangle, \langle\frac{0.5, 0.5}{a^3}\rangle\},\\
\mathfrak{W}_7 &= \{\langle\frac{0.9, 0.9, 0.9}{1}\rangle, \langle\frac{0.5, 0.5}{a}\rangle, \langle\frac{0.6, 0.6, 0.6}{a^2}\rangle, \langle\frac{0.5, 0.5}{a^3}\rangle\},\\
\mathfrak{W}_8 &= \{\langle\frac{0.9, 0.9, 0.9, 0.9}{1}\rangle, \langle\frac{0.5}{a}\rangle, \langle\frac{0.6}{a^2}\rangle, \langle\frac{0.5}{a^3}\rangle\},
\end{aligned}$$

$$\begin{aligned}
\mathfrak{W}_9 &= \{\langle\frac{0.9,0.9,0.9,0.9}{1}\rangle,\langle\frac{0.5,0.5}{a}\rangle,\langle\frac{0.6,0.6}{a^2}\rangle,\langle\frac{0.5,0.5}{a^3}\rangle\},\\
\mathfrak{W}_{10} &= \{\langle\frac{0.9,0.9,0.9,0.9}{1}\rangle,\langle\frac{0.5,0.5}{a}\rangle,\langle\frac{0.6,0.6,0.6}{a^2}\rangle,\langle\frac{0.5,0.5}{a^3}\rangle\}.
\end{aligned}$$

Among the list, $\mathfrak{W}_2$ and $\mathfrak{W}_{10}$ are trivial FSMGs, $\mathfrak{W}_3$–$\mathfrak{W}_9$ are proper non-trivial normal FSMGs of $\mathfrak{W}$, and $\hat{\mathfrak{W}}_9$ is the maximal non-trivial normal FSMG of $\mathfrak{W}$ because it is the largest proper non-trivial normal FSMG of the FMG. Hence, $\mathfrak{W}$ is not a simple FMG.

The FSMGs of $\mathfrak{W}$ using Case (iii) can be seen as follows:

$$\begin{aligned}
\mathfrak{W}_0 &= \{\langle\frac{0.9}{1}\rangle\},\\
\mathfrak{W}_1 &= \{\langle\frac{0.9}{1}\rangle,\langle\frac{0.6}{a^2}\rangle\},\\
\mathfrak{W}_2 &= \{\langle\frac{0.9}{1}\rangle,\langle\frac{0.5}{a}\rangle,\langle\frac{0.6}{a^2}\rangle,\langle\frac{0.5}{a^3}\rangle\},\\
\mathfrak{W}_3 &= \{\langle\frac{0.9,0.9}{1}\rangle,\langle\frac{0.5}{a}\rangle,\langle\frac{0.6}{a^2}\rangle,\langle\frac{0.5}{a^3}\rangle\},\\
\mathfrak{W}_4 &= \{\langle\frac{0.9,0.9}{1}\rangle,\langle\frac{0.5,0.5}{a}\rangle,\langle\frac{0.6,0.6}{a^2}\rangle,\langle\frac{0.5,0.5}{a^3}\rangle\},\\
\mathfrak{W}_5 &= \{\langle\frac{0.9,0.9,0.9}{1}\rangle,\langle\frac{0.5}{a}\rangle,\langle\frac{0.6}{a^2}\rangle,\langle\frac{0.5}{a^3}\rangle\},\\
\mathfrak{W}_6 &= \{\langle\frac{0.9,0.9,0.9}{1}\rangle,\langle\frac{0.5,0.5}{a}\rangle,\langle\frac{0.6,0.6}{a^2}\rangle,\langle\frac{0.5,0.5}{a^3}\rangle\},\\
\mathfrak{W}_7 &= \{\langle\frac{0.9,0.9,0.9}{1}\rangle,\langle\frac{0.5,0.5}{a}\rangle,\langle\frac{0.6,0.6,0.6}{a^2}\rangle,\langle\frac{0.5,0.5}{a^3}\rangle\},\\
\mathfrak{W}_8 &= \{\langle\frac{0.9,0.9,0.9,0.9}{1}\rangle,\langle\frac{0.5}{a}\rangle,\langle\frac{0.6}{a^2}\rangle,\langle\frac{0.5}{a^3}\rangle\},\\
\mathfrak{W}_9 &= \{\langle\frac{0.9,0.9,0.9,0.9}{1}\rangle,\langle\frac{0.5,0.5}{a}\rangle,\langle\frac{0.6,0.6}{a^2}\rangle,\langle\frac{0.5,0.5}{a^3}\rangle\},\\
\mathfrak{W}_{10} &= \{\langle\frac{0.9,0.9,0.9,0.9}{1}\rangle,\langle\frac{0.5,0.5}{a}\rangle,\langle\frac{0.6,0.6,0.6}{a^2}\rangle,\langle\frac{0.5,0.5}{a^3}\rangle\},\\
\mathfrak{W}_{11} &= \{\langle\frac{0.9,0.9}{1}\rangle\},\\
\mathfrak{W}_{12} &= \{\langle\frac{0.9,0.9,0.9}{1}\rangle\},\\
\mathfrak{W}_{13} &= \{\langle\frac{0.9,0.9,0.9,0.9}{1}\rangle\},\\
\mathfrak{W}_{14} &= \{\langle\frac{0.9,0.9}{1}\rangle,\langle\frac{0.6}{a^2}\rangle\},\\
\mathfrak{W}_{15} &= \{\langle\frac{0.9,0.9}{1}\rangle,\langle\frac{0.6,0.6}{a^2}\rangle\},\\
\mathfrak{W}_{16} &= \{\langle\frac{0.9,0.9,0.9}{1}\rangle,\langle\frac{0.6}{a^2}\rangle\},\\
\mathfrak{W}_{17} &= \{\langle\frac{0.9,0.9,0.9}{1}\rangle,\langle\frac{0.6,0.6}{a^2}\rangle\},
\end{aligned}$$

$$\begin{aligned}
\mathfrak{W}_{18} &= \{\langle \frac{0.9, 0.9, 0.9}{1} \rangle, \langle \frac{0.6, 0.6, 0.6}{a^2} \rangle\}, \\
\mathfrak{W}_{19} &= \{\langle \frac{0.9, 0.9, 0.9, 0.9}{1} \rangle, \langle \frac{0.6}{a^2} \rangle\}, \\
\mathfrak{W}_{20} &= \{\langle \frac{0.9, 0.9, 0.9, 0.9}{1} \rangle, \langle \frac{0.6, 0.6}{a^2} \rangle\}, \\
\mathfrak{W}_{21} &= \{\langle \frac{0.9, 0.9, 0.9, 0.9}{1} \rangle, \langle \frac{0.6, 0.6, 0.6}{a^2} \rangle\}.
\end{aligned}$$

Among the list, $\mathfrak{W}_0$–$\mathfrak{W}_2$ and $\mathfrak{W}_{10}$–$\mathfrak{W}_{13}$ are trivial FSMGs of $\mathfrak{D}$. The FSMGs $\mathfrak{W}_3$–$\mathfrak{W}_9$ and $\mathfrak{W}_{14}$–$\mathfrak{W}_{21}$ are proper non-trivial normal FSMGs of $\mathfrak{W}$, and $\mathfrak{W}_9$ is the maximal non-trivial normal FSMG of $\mathfrak{W}$. Again, $\mathfrak{W}$ is not a simple FMG.

Example 4. In a symmetry group S_n for $n = 3$ (i.e., $S = \{1, 2, 3\}$), $A_3 = \{\rho_0, \rho_1, \rho_2\} \subseteq S_3$ is a simple group, where $\rho_1^{-1} = \rho_2$, $\rho_2^{-1} = \rho_1$, $\rho_1^2 = \rho_2$, and $\rho_2^2 = \rho_1$ for $\rho_0 = (1)$, $\rho_1 = (123)$ and $\rho_2 = (132)$. Then, a FMG denoted by $\mathfrak{Y}$ from A_3 is:

$$\mathfrak{Y} = \{\langle \frac{0.8, 0.8, 0.8}{\rho_0} \rangle, \langle \frac{0.6, 0.6}{\rho_1} \rangle, \langle \frac{0.6, 0.6}{\rho_2} \rangle\}.$$

Because $\rho_1.\rho_2 = \rho_2.\rho_1 = \rho_0$, then $\mathfrak{Y}$ is commutative. The FSMGs of $\mathfrak{Y}$ using Case (i) can be seen as follows:

$$\begin{aligned}
\mathfrak{Y}_0 &= \{\langle \frac{0.8}{\rho_0} \rangle\}, \\
\mathfrak{Y}_1 &= \{\langle \frac{0.8}{\rho_0} \rangle, \langle \frac{0.6}{\rho_1} \rangle, \langle \frac{0.6}{\rho_2} \rangle\}.
\end{aligned}$$

Because $\mathfrak{Y}_0$ and $\mathfrak{Y}_1$ are trivial, then $\mathfrak{Y}$ is a simple FMG (because it has no proper non-trivial normal FSMG and hence, it has no maximal non-trivial normal FSMG).

The FSMGs of $\mathfrak{Y}$ using Case (ii) can be seen as follows:

$$\begin{aligned}
\mathfrak{Y}_1 &= \{\langle \frac{0.8}{\rho_0} \rangle, \langle \frac{0.6}{\rho_1} \rangle, \langle \frac{0.6}{\rho_2} \rangle\}, \\
\mathfrak{Y}_2 &= \{\langle \frac{0.8, 0.8}{\rho_0} \rangle, \langle \frac{0.6}{\rho_1} \rangle, \langle \frac{0.6}{\rho_2} \rangle\}, \\
\mathfrak{Y}_3 &= \{\langle \frac{0.8, 0.8}{\rho_0} \rangle, \langle \frac{0.6, 0.6}{\rho_1} \rangle, \langle \frac{0.6, 0.6}{\rho_2} \rangle\}, \\
\mathfrak{Y}_4 &= \{\langle \frac{0.8, 0.8, 0.8}{\rho_0} \rangle, \langle \frac{0.6}{\rho_1} \rangle, \langle \frac{0.6}{\rho_2} \rangle\}, \\
\mathfrak{Y}_5 &= \{\langle \frac{0.8, 0.8, 0.8}{\rho_0} \rangle, \langle \frac{0.6, 0.6}{\rho_1} \rangle, \langle \frac{0.6, 0.6}{\rho_2} \rangle\}.
\end{aligned}$$

We see that $\mathfrak{Y}_1$ and $\mathfrak{Y}_5$ are trivial and $\mathfrak{Y}_2$–$\mathfrak{Y}_4$ are proper non-trivial normal FSMGs of $\mathfrak{Y}$ without a maximal non-trivial normal FSMG of $\mathfrak{Y}$ since either $\mathfrak{Y}_3 \nsubseteq \mathfrak{Y}_4$ or $\mathfrak{Y}_4 \nsubseteq \mathfrak{Y}_3$. Hence, $\mathfrak{Y}$ is not simple.

The FSMGs of $\mathfrak{Y}$ using Case (iii) can be seen as follows:

$$\begin{aligned}
\mathfrak{Y}_0 &= \{\langle\frac{0.8}{\rho_0}\rangle\},\\
\mathfrak{Y}_1 &= \{\langle\frac{0.8}{\rho_0}\rangle, \langle\frac{0.6}{\rho_1}\rangle, \langle\frac{0.6}{\rho_2}\rangle\},\\
\mathfrak{Y}_2 &= \{\langle\frac{0.8, 0.8}{\rho_0}\rangle, \langle\frac{0.6}{\rho_1}\rangle, \langle\frac{0.6}{\rho_2}\rangle\},\\
\mathfrak{Y}_3 &= \{\langle\frac{0.8, 0.8}{\rho_0}\rangle, \langle\frac{0.6, 0.6}{\rho_1}\rangle, \langle\frac{0.6, 0.6}{\rho_2}\rangle\},\\
\mathfrak{Y}_4 &= \{\langle\frac{0.8, 0.8, 0.8}{\rho_0}\rangle, \langle\frac{0.6}{\rho_1}\rangle, \langle\frac{0.6}{\rho_2}\rangle\},\\
\mathfrak{Y}_5 &= \{\langle\frac{0.8, 0.8, 0.8}{\rho_0}\rangle, \langle\frac{0.6, 0.6}{\rho_1}\rangle, \langle\frac{0.6, 0.6}{\rho_2}\rangle\},\\
\mathfrak{Y}_6 &= \{\langle\frac{0.8, 0.8}{\rho_0}\rangle\},\\
\mathfrak{Y}_7 &= \{\langle\frac{0.8, 0.8, 0.8}{\rho_0}\rangle\}.
\end{aligned}$$

Here, $\mathfrak{Y}_0, \mathfrak{Y}_1, \mathfrak{Y}_6, \mathfrak{Y}_7$ and $\mathfrak{Y}_5$ are trivial, and $\mathfrak{Y}_2$–$\mathfrak{Y}_4$ are proper non-trivial normal FSMGs of $\mathfrak{Y}$ without a maximal non-trivial normal FSMG of $\mathfrak{Y}$ since either $\mathfrak{Y}_3 \nsubseteq \mathfrak{Y}_4$ or $\mathfrak{Y}_4 \nsubseteq \mathfrak{Y}_3$. Hence, $\mathfrak{Y}$ is not a simple FMG.

Remark 1. From the considered examples, the following remarks follow:

(i) Every FMG whose FSMGs are trivial is a simple FMG. To demonstrate this, given a FMG $\mathfrak{Y} = \{\langle\frac{0.8,0.8}{\rho_0}\rangle, \langle\frac{0.6}{\rho_1}\rangle, \langle\frac{0.6}{\rho_2}\rangle\}$ defined over A_3. Then, the FSMGs of $\mathfrak{Y}$ are:

$$\begin{aligned}
\mathfrak{Y}_0 &= \{\langle\frac{0.8}{\rho_0}\rangle\},\\
\mathfrak{Y}_1 &= \{\langle\frac{0.8}{\rho_0}\rangle, \langle\frac{0.6}{\rho_1}\rangle, \langle\frac{0.6}{\rho_2}\rangle\},\\
\mathfrak{Y}_2 &= \{\langle\frac{0.8, 0.8}{\rho_0}\rangle, \langle\frac{0.6}{\rho_1}\rangle, \langle\frac{0.6}{\rho_2}\rangle\} = \mathfrak{Y}.
\end{aligned}$$

We observe that all these FSMGs of $\mathfrak{Y}$ are trivial and so, $\mathfrak{Y}$ is simple because it has no maximal non-trivial normal FSMG.

(ii) Every FMG with one non-trivial FSMG is simple. To verify this, given a FMG $\mathfrak{W} = \{\langle\frac{0.9,0.9}{1}\rangle, \langle\frac{0.5}{a}\rangle, \langle\frac{0.6}{a^2}\rangle, \langle\frac{0.5}{a^3}\rangle\}$ of $G = \{1, a, a^2, a^3\}$. Then, the FSMGs of $\mathfrak{W}$ are:

$$\begin{aligned}
\mathfrak{W}_0 &= \{\langle\frac{0.9}{1}\rangle\},\\
\mathfrak{W}_1 &= \{\langle\frac{0.9}{1}\rangle, \langle\frac{0.6}{a^2}\rangle\},
\end{aligned}$$

$$\begin{aligned}
\mathfrak{W}_2 &= \{\langle\frac{0.9,0.9}{1}\rangle,\langle\frac{0.6}{a^2}\rangle\},\\
\mathfrak{W}_3 &= \{\langle\frac{0.9}{1}\rangle,\langle\frac{0.5}{a}\rangle,\langle\frac{0.6}{a^2}\rangle,\langle\frac{0.5}{a^3}\rangle\},\\
\mathfrak{W}_4 &= \{\langle\frac{0.9,0.9}{1}\rangle,\langle\frac{0.5}{a}\rangle,\langle\frac{0.6}{a^2}\rangle,\langle\frac{0.5}{a^3}\rangle\} = \mathfrak{W}.
\end{aligned}$$

Because $\mathfrak{W}_2$ is the only non-trivial FSMG of $\mathfrak{W}$ and it is not maximal, then $\mathfrak{W}$ is a simple FMG of G.

(iii) Every FMG with at most one non-trivial FSMG is a simple FMG. This can be seen by combining (i) and (ii), respectively.

Definition 20. Let G be a finite group and $\mathfrak{W}$ be a FMG of G with a finite count membership degrees. Then, $\mathfrak{W}$ possesses a normal series if there exist:

$$CM_{\mathfrak{W}_0}(x) \leq CM_{\mathfrak{W}_1}(x) \leq \cdots \leq CM_{\mathfrak{W}_n}(x) = CM_{\mathfrak{W}}(x)\ \forall x \in G, \tag{15}$$

such that $(\mathfrak{W}_0)_* = (\mathfrak{W}_1)_* = \cdots = (\mathfrak{W}_n)_* = \mathfrak{W}_* = G$ and $\mathfrak{W}_i \lhd \mathfrak{W}_{i+1}\ \forall\ 0 \leq i \leq n-1$.

Example 5. Using the FSMGs of $\mathfrak{W}$ in Example 3 with $(\mathfrak{W}_0)_* = (\mathfrak{W}_1)_* = \cdots = (\mathfrak{W}_n)_* = \mathfrak{W}_* = G$, we have the following normal series:

$$\begin{gathered}
\mathfrak{W}_2 \subseteq \mathfrak{W}_3 \subseteq \mathfrak{W}_4 \subseteq \mathfrak{W}_6 \subseteq \mathfrak{W}_7 \subseteq \mathfrak{W}_9 \subseteq \mathfrak{W}_{10} = \mathfrak{W},\\
\mathfrak{W}_2 \subseteq \mathfrak{W}_3 \subseteq \mathfrak{W}_5 \subseteq \mathfrak{W}_6 \subseteq \mathfrak{W}_7 \subseteq \mathfrak{W}_9 \subseteq \mathfrak{W}_{10} = \mathfrak{W},\\
\mathfrak{W}_2 \subseteq \mathfrak{W}_3 \subseteq \mathfrak{W}_5 \subseteq \mathfrak{W}_8 \subseteq \mathfrak{W}_9 \subseteq \mathfrak{W}_{10} = \mathfrak{W}.
\end{gathered}$$

Certainly, $\mathfrak{W}_i \lhd \mathfrak{W}_{i+1}\ \forall\ 0 \leq i \leq n-1$.

Example 6. Using the FSMGs of $\mathfrak{Y}$ in Example 4, with $(\mathfrak{Y}_0)_* = (\mathfrak{Y}_1)_* = \cdots = (\mathfrak{Y}_n)_* = \mathfrak{Y}_* = A_3$, we have the following normal series:

$$\begin{gathered}
\mathfrak{Y}_1 \subseteq \mathfrak{Y}_2 \subseteq \mathfrak{Y}_3 \subseteq \mathfrak{Y}_5 = \mathfrak{Y},\\
\mathfrak{Y}_1 \subseteq \mathfrak{Y}_2 \subseteq \mathfrak{Y}_4 \subseteq \mathfrak{Y}_5 = \mathfrak{Y}
\end{gathered}$$

because $\mathfrak{Y}_i \lhd \mathfrak{Y}_{i+1}\ \forall\ 0 \leq i \leq n-1$.

Next, the concept of composition series for FMG is presented.

Definition 21. Let G be a finite group and $\mathfrak{W}$ be a FMG of G with a finite count membership degrees. Then, $\mathfrak{W}$ possesses a composition series if there exist::

$$CM_{\mathfrak{W}_0}(x) \leq CM_{\mathfrak{W}_1}(x) \leq \cdots \leq CM_{\mathfrak{W}_n}(x) = CM_{\mathfrak{W}}(x)\ \forall x \in G, \tag{16}$$

such that $(\mathfrak{W}_0)_* = (\mathfrak{W}_1)_* = \cdots = (\mathfrak{W}_n)_* = \mathfrak{W}_* = G$ with the properties

(i) $\mathfrak{W}_i \lhd \mathfrak{W}_{i+1}\ \forall\ 0 \leq i \leq n-1$,

(ii) $\mathfrak{W}_{i+1}/\mathfrak{W}_i$ is simple $\forall\ 0 \leq i \leq n-1$.

Using the FSMGs of $\mathfrak{W}$ in Example 3 with $(\mathfrak{W}_0)_* = (\mathfrak{W}_1)_* = \cdots = (\mathfrak{W}_n)_* = \mathfrak{W}_* = G$, the composition series for $\mathfrak{W}$ are:

$$\mathfrak{W}_2 \subseteq \mathfrak{W}_3 \subseteq \mathfrak{W}_4 \subseteq \mathfrak{W}_6 \subseteq \mathfrak{W}_7 \subseteq \mathfrak{W}_9 \subseteq \mathfrak{W}_{10} = \mathfrak{W},$$
$$\mathfrak{W}_2 \subseteq \mathfrak{W}_3 \subseteq \mathfrak{W}_5 \subseteq \mathfrak{W}_6 \subseteq \mathfrak{W}_7 \subseteq \mathfrak{W}_9 \subseteq \mathfrak{W}_{10} = \mathfrak{W},$$
$$\mathfrak{W}_2 \subseteq \mathfrak{W}_3 \subseteq \mathfrak{W}_5 \subseteq \mathfrak{W}_8 \subseteq \mathfrak{W}_9 \subseteq \mathfrak{W}_{10} = \mathfrak{W},$$

where $\mathfrak{W}_i \lhd \mathfrak{W}_{i+1}\ \forall\ 0 \leq i \leq n-1$ and $\mathfrak{W}_{i+1}/\mathfrak{W}_i$ is simple $\forall\ 0 \leq i \leq n-1$.

Using the FSMGs of $\mathfrak{Y}$ in Example 4, with $(\mathfrak{Y}_0)_* = (\mathfrak{Y}_1)_* = \cdots = (\mathfrak{Y}_n)_* = \mathfrak{Y}_* = A_3$, the composition series for $\mathfrak{Y}$ are:

$$\mathfrak{Y}_1 \subseteq \mathfrak{Y}_2 \subseteq \mathfrak{Y}_3 \subseteq \mathfrak{Y}_5 = \mathfrak{Y},$$
$$\mathfrak{Y}_1 \subseteq \mathfrak{Y}_2 \subseteq \mathfrak{Y}_4 \subseteq \mathfrak{Y}_5 = \mathfrak{Y}$$

because $\mathfrak{Y}_i \lhd \mathfrak{Y}_{i+1}\ \forall\ 0 \leq i \leq n-1$ and $\mathfrak{Y}_{i+1}/\mathfrak{Y}_i$ is simple $\forall\ 0 \leq i \leq n-1$.

Example 7. Let $\mathfrak{M} = \{\langle\frac{0.7,0.7,0.7}{0}\rangle, \langle\frac{0.5,0.5}{1}\rangle, \langle\frac{0.5,0.5}{2}\rangle, \langle\frac{0.5,0.5}{3}\rangle, \langle\frac{0.5,0.5}{4}\rangle, \langle\frac{0.5,0.5}{5}\rangle\}$ be a FMG of $\mathbb{Z}_6$. Then, the FSMGs of $\mathfrak{M}$ are:

$$\mathfrak{M}_0 = \{\langle\tfrac{0.7}{0}\rangle\}, \mathfrak{M}_1 = \{\langle\tfrac{0.7}{0}\rangle, \langle\tfrac{0.5}{3}\rangle\},$$
$$\mathfrak{M}_2 = \{\langle\tfrac{0.7}{0}\rangle, \langle\tfrac{0.5}{2}\rangle, \langle\tfrac{0.5}{4}\rangle\},$$
$$\mathfrak{M}_3 = \{\langle\tfrac{0.7}{0}\rangle, \langle\tfrac{0.5}{1}\rangle, \langle\tfrac{0.5}{2}\rangle, \langle\tfrac{0.5}{3}\rangle, \langle\tfrac{0.5}{4}\rangle, \langle\tfrac{0.5}{5}\rangle\},$$
$$\mathfrak{M}_4 = \{\langle\tfrac{0.7,0.7}{0}\rangle, \langle\tfrac{0.5}{1}\rangle, \langle\tfrac{0.5}{2}\rangle, \langle\tfrac{0.5}{3}\rangle, \langle\tfrac{0.5}{4}\rangle, \langle\tfrac{0.5}{5}\rangle\},$$
$$\mathfrak{M}_5 = \{\langle\tfrac{0.7,0.7}{0}\rangle, \langle\tfrac{0.5,0.5}{1}\rangle, \langle\tfrac{0.5,0.5}{2}\rangle, \langle\tfrac{0.5,0.5}{3}\rangle, \langle\tfrac{0.5,0.5}{4}\rangle, \langle\tfrac{0.5,0.5}{5}\rangle\},$$
$$\mathfrak{M}_6 = \{\langle\tfrac{0.7,0.7,0.7}{0}\rangle, \langle\tfrac{0.5}{1}\rangle, \langle\tfrac{0.5}{2}\rangle, \langle\tfrac{0.5}{3}\rangle, \langle\tfrac{0.5}{4}\rangle, \langle\tfrac{0.5}{5}\rangle\},$$
$$\mathfrak{M}_7 = \{\langle\tfrac{0.7,0.7}{0}\rangle\}, \mathfrak{M}_8 = \{\langle\tfrac{0.7,0.7,0.7}{0}\rangle\},$$
$$\mathfrak{M}_9 = \{\langle\tfrac{0.7,0.7}{0}\rangle, \langle\tfrac{0.5}{3}\rangle\}, \mathfrak{M}_{10} = \{\langle\tfrac{0.7,0.7}{0}\rangle, \langle\tfrac{0.5,0.5}{3}\rangle\},$$
$$\mathfrak{M}_{11} = \{\langle\tfrac{0.7,0.7,0.7}{0}\rangle, \langle\tfrac{0.5}{3}\rangle\},$$
$$\mathfrak{M}_{12} = \{\langle\tfrac{0.7,0.7,0.7}{0}\rangle, \langle\tfrac{0.5,0.5}{3}\rangle\},$$
$$\mathfrak{M}_{13} = \{\langle\tfrac{0.7,0.7}{0}\rangle, \langle\tfrac{0.5}{2}\rangle, \langle\tfrac{0.5}{4}\rangle\},$$
$$\mathfrak{M}_{14} = \{\langle\tfrac{0.7,0.7}{0}\rangle, \langle\tfrac{0.5,0.5}{2}\rangle, \langle\tfrac{0.5,0.5}{4}\rangle\},$$
$$\mathfrak{M}_{15} = \{\langle\tfrac{0.7,0.7,0.7}{0}\rangle, \langle\tfrac{0.5}{2}\rangle, \langle\tfrac{0.5}{4}\rangle\},$$
$$\mathfrak{M}_{16} = \{\langle\tfrac{0.7,0.7,0.7}{0}\rangle, \langle\tfrac{0.5,0.5}{2}\rangle, \langle\tfrac{0.5,0.5}{4}\rangle\},$$
$$\mathfrak{M}_{17} = \{\langle\tfrac{0.7,0.7,0.7}{0}\rangle, \langle\tfrac{0.5,0.5}{1}\rangle, \langle\tfrac{0.5,0.5}{2}\rangle, \langle\tfrac{0.5,0.5}{3}\rangle, \langle\tfrac{0.5,0.5}{4}\rangle, \langle\tfrac{0.5,0.5}{5}\rangle\} = \mathfrak{M}.$$

Among these FSMGs, the following have the same elements as $\mathbb{Z}_6$:

$$\mathfrak{M}_3 = \{\langle\tfrac{0.7}{0}\rangle, \langle\tfrac{0.5}{1}\rangle, \langle\tfrac{0.5}{2}\rangle, \langle\tfrac{0.5}{3}\rangle, \langle\tfrac{0.5}{4}\rangle, \langle\tfrac{0.5}{5}\rangle\},$$
$$\mathfrak{M}_4 = \{\langle\tfrac{0.7,0.7}{0}\rangle, \langle\tfrac{0.5}{1}\rangle, \langle\tfrac{0.5}{2}\rangle, \langle\tfrac{0.5}{3}\rangle, \langle\tfrac{0.5}{4}\rangle, \langle\tfrac{0.5}{5}\rangle\},$$

$$\mathfrak{M}_5 = \{\langle\frac{0.7,0.7}{0}\rangle, \langle\frac{0.5,0.5}{1}\rangle, \langle\frac{0.5,0.5}{2}\rangle, \langle\frac{0.5,0.5}{3}\rangle, \langle\frac{0.5,0.5}{4}\rangle, \langle\frac{0.5,0.5}{5}\rangle\},$$
$$\mathfrak{M}_6 = \{\langle\frac{0.7,0.7,0.7}{0}\rangle, \langle\frac{0.5}{1}\rangle, \langle\frac{0.5}{2}\rangle, \langle\frac{0.5}{3}\rangle, \langle\frac{0.5}{4}\rangle, \langle\frac{0.5}{5}\rangle\},$$
$$\mathfrak{M}_{17} = \{\langle\frac{0.7,0.7,0.7}{0}\rangle, \langle\frac{0.5,0.5}{1}\rangle, \langle\frac{0.5,0.5}{2}\rangle, \langle\frac{0.5,0.5}{3}\rangle, \langle\frac{0.5,0.5}{4}\rangle, \langle\frac{0.5,0.5}{5}\rangle\} = \mathfrak{M}.$$

Then, the composition series are:

$$\mathfrak{M}_3 \subseteq \mathfrak{M}_4 \subseteq \mathfrak{M}_5 \subseteq \mathfrak{M}_{17} = \mathfrak{M},$$
$$\mathfrak{M}_3 \subseteq \mathfrak{M}_4 \subseteq \mathfrak{M}_6 \subseteq \mathfrak{M}_{17} = \mathfrak{M},$$

because $\mathfrak{M}_i \lhd \mathfrak{M}_{i+1}$ $\forall\ 0 \leq i \leq n-1$ and $\mathfrak{M}_{i+1}/\mathfrak{M}_i$ is simple $\forall\ 0 \leq i \leq n-1$.

Theorem 2. *Every finite FMG of a finite group has a composition series.*

Proof. Let $\mathfrak{W}$ be a finite FMG of a finite group G. The proof is presented using the principle of induction. Assume every finite FMG of order less than $|\mathfrak{W}|$ has a composition series. If $\mathfrak{W}$ is simple, then it possesses a trivial composition series. Otherwise, if $\mathfrak{W}$ is not simple, then there is a non-trivial proper normal FSMG and a maximal non-trivial normal FSMG in $\mathfrak{W}$, since $\mathfrak{W}$ is finite. We denote such a maximal non-trivial proper normal FSMG in $\mathfrak{W}$ as $\mathfrak{M}$, and so $|\mathfrak{M}| < |\mathfrak{W}|$.

By the principle of induction, $|\mathfrak{M}|$ has a composition series, which is:

$$CM_{\mathfrak{M}_0}(x) \leq CM_{\mathfrak{M}_1}(x) \leq \cdots \leq CM_{\mathfrak{M}_n}(x) = CM_{\mathfrak{M}}(x)\ \forall x \in X,$$

such that $(\mathfrak{M}_0)_* = (\mathfrak{M}_1)_* = \cdots = (\mathfrak{M}_n)_* = G$. But $\mathfrak{M} \lhd \mathfrak{W}$ since $\mathfrak{M}$ is maximal in $\mathfrak{W}$, and so

$$CM_{\mathfrak{M}_0}(x) \leq CM_{\mathfrak{M}_1}(x) \leq \cdots \leq CM_{\mathfrak{M}_n}(x) = CM_{\mathfrak{M}}(x) \leq CM_{\mathfrak{W}}(x)\ \forall x \in G,$$

which proves that $\mathfrak{W}$ has a composition series.

Theorem 3 (The Jordan-Hölder Theorem)**.** *Every finite FMG of a finite group has at least two equivalent composition series.*

Proof. Let $\mathfrak{W}$ be a finite FMG of a finite group G. Assume $\mathfrak{W}$ has two composition series:

$$CM_{\mathfrak{W}_0}(x) \leq CM_{\mathfrak{W}_1}(x) \leq \cdots \leq CM_{\mathfrak{W}_n}(x) = CM_{\mathfrak{W}}(x)\ \forall x \in G,$$
$$CM_{\mathfrak{M}_0}(x) \leq CM_{\mathfrak{M}_1}(x) \leq \cdots \leq CM_{\mathfrak{M}_m}(x) = CM_{\mathfrak{M}}(x)\ \forall x \in G,$$

such that $\mathfrak{W}_i \lhd \mathfrak{W}_{i+1}$ with $\mathfrak{W}_{i+1}/\mathfrak{W}_i$ simple $\forall\ 0 \leq i \leq n-1$ and $\mathfrak{M}_i \lhd \mathfrak{M}_{i+1}$ with $\mathfrak{M}_{i+1}/\mathfrak{M}_i$ simple $\forall\ 0 \leq i \leq m-1$. In each case, $(\mathfrak{W}_0)_* = (\mathfrak{W}_1)_* = \cdots = (\mathfrak{W}_n)_* = G$ and $(\mathfrak{M}_0)_* = (\mathfrak{M}_1)_* = \cdots = (\mathfrak{M}_m)_* = G$.

We need to verify that $n = m$ and the list of quotients

$$(\mathfrak{W}_1/\mathfrak{W}_0, \mathfrak{W}_2/\mathfrak{W}_1, \cdots, \mathfrak{W}_n/\mathfrak{W}_{n-1})$$

is a rearrangement "$\sim$" of the list of quotients

$$(\mathfrak{M}_1/\mathfrak{M}_0, \mathfrak{M}_2/\mathfrak{M}_1, \cdots, \mathfrak{M}_m/\mathfrak{M}_{m-1}).$$

We prove by induction on $|\mathfrak{W}|$. If $|\mathfrak{W}| = 1$, then the result is trivial. Assume $\mathfrak{W}_{n-1} = \mathfrak{M}_{m-1}$, then the result is established by induction. Now, assume $\mathfrak{W}_{n-1} \neq \mathfrak{M}_{m-1}$. Set $\Phi = \mathfrak{W}_{n-1}$, $\Psi = \mathfrak{M}_{m-1}$, and $\Omega = \Phi \cap \Psi$, where Ω is the maximal FSMG of $\mathfrak{W}_{n-1}$ and $\mathfrak{M}_{m-1}$.

Now, Ω has a composition series, $CM_{\Omega_0}(x) \leq CM_{\Omega_1}(x) \leq \cdots \leq CM_{\Omega_t}(x) = CM_{\Omega}(x)\ \forall x \in G$. Then,

$$CM_{\mathfrak{W}_0}(x) \leq CM_{\mathfrak{W}_1}(x) \leq \cdots \leq CM_{\mathfrak{W}_{n-1}}(x) = CM_{\Phi}(x)\ \forall x \in G \text{ and}$$
$$CM_{\Omega_0}(x) \leq CM_{\Omega_1}(x) \leq \cdots \leq CM_{\Omega_t}(x) = CM_{\Omega}(x) \leq CM_{\Phi}(x)\ \forall x \in G$$

are both composition series for Φ. By the principle of induction, we have $n-1 = t+1 \Rightarrow n-2 = t$, and

$$(\mathfrak{W}_1/\mathfrak{W}_0, \mathfrak{W}_2/\mathfrak{W}_1, \cdots, \mathfrak{W}_{n-1}/\mathfrak{W}_{n-2}) \sim (\Omega_1/\Omega_0, \Omega_2/\Omega_1, \cdots, \Omega_t/\Omega_{t-1}, \Phi/\Omega). \tag{17}$$

Likewise, we get

$$CM_{\mathfrak{M}_0}(x) \leq CM_{\mathfrak{M}_1}(x) \leq \cdots \leq CM_{\mathfrak{M}_{m-1}}(x) = CM_{\Psi}(x)\ \forall x \in G \text{ and}$$
$$CM_{\Omega_0}(x) \leq CM_{\Omega_1}(x) \leq \cdots \leq CM_{\Omega_t}(x) = CM_{\Omega}(x) \leq CM_{\Psi}(x)\ \forall x \in G$$

are both composition series for Ψ. Thus, $m-1 = t+1 \Rightarrow m-2 = t$, and

$$(\mathfrak{M}_1/\mathfrak{M}_0, \mathfrak{M}_2/\mathfrak{M}_1, \cdots, \mathfrak{M}_{m-1}/\mathfrak{M}_{m-2}) \sim (\Omega_1/\Omega_0, \Omega_2/\Omega_1, \cdots, \Omega_t/\Omega_{t-1}, \Psi/\Omega). \tag{18}$$

From $n-1 = t+1 = m-1$, we have $n = m$. By appending $\mathfrak{W}/\Phi$ to both sides of (17), we get

$$(\mathfrak{W}_1/\mathfrak{W}_0, \cdots, \mathfrak{W}_{n-1}/\mathfrak{W}_{n-2}, \mathfrak{W}/\mathfrak{W}_{n-1}) \sim (\Omega_1/\Omega_0, \cdots, \Omega_t/\Omega_{t-1}, \Phi/\Omega, \mathfrak{W}/\Phi). \tag{19}$$

Likewise, appending $\mathfrak{W}/\Psi$ to both sides of (18), we get

$$(\mathfrak{M}_1/\mathfrak{M}_0, \cdots, \mathfrak{M}_{m-1}/\mathfrak{M}_{m-2}, \mathfrak{W}/\mathfrak{M}_{m-1}) \sim (\Omega_1/\Omega_0, \cdots, \Omega_t/\Omega_{t-1}, \Psi/\Omega, \mathfrak{W}/\Psi). \tag{20}$$

The right hand side of (19) and (20) are the same except $(\Phi/\Omega, \mathfrak{W}/\Phi)$ and $(\Psi/\Omega, \mathfrak{W}/\Psi)$. Hence, $(\Phi/\Omega, \mathfrak{W}/\Phi) \sim (\Psi/\Omega, \mathfrak{W}/\Psi)$ and so

$$(\mathfrak{W}_1/\mathfrak{W}_0, \cdots, \mathfrak{W}_n/\mathfrak{W}_{n-1}) \sim (\mathfrak{M}_1/\mathfrak{M}_0, \cdots, \mathfrak{M}_m/\mathfrak{M}_{m-1}),$$

which complete the proof.

4. Conclusion

In this paper, the ideas of simple FMG, maximal normal FSMG, normal series for FMG, and composition series for FMG were defined as algebraic structures in FMS context and characterized with examples and some results. In addition, it was proven that every finite FMG defined over a finite group has a composition series. Finally, the Jordan-Hölder Theorem was presented in FMS context, and it was established that any two composition series of a finite FMG of a finite group are comparable. For further research, the concept of nilpotency under FMGs could be studied.

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