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Bipolar Intuitionistic Fuzzy Ideals in BCK-Algebras

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Abstract. This study focuses on unifying theories of bipolar intuitionistic fuzzy sets in the field of BCK-algebras, leading to the development of a new framework for bipolar intuitionistic fuzzy ideals in BCK-algebras. The article introduces the concepts of bipolar intuitionistic fuzzy subalgebras and bipolar intuitionistic fuzzy ideals in BCK-algebras, with illustrative examples. Subsequently, we investigate various related characteristics. We also provide conditions under which a bipolar intuitionistic fuzzy set is considered a bipolar intuitionistic fuzzy ideal.

2020 Mathematics Subject Classifications: 03E72, 06F35, 03G25

Key Words and Phrases: Bipolar Fuzzy Set (BPFS), bipolar Intuitionistic Fuzzy Set (BPIFS), bipolar Intuitionistic Fuzzy Subalgebra (BPIFSA), bipolar Intuitionistic Fuzzy Ideal (BPIFI).

1. Introduction

In many real-world situations, decision-making and information management involve scenarios in which the clarity of data or outcomes is not well-defined. This lack of definiteness and accuracy is generally known as uncertainty. Uncertainty may arise from several causes, such as incomplete or missing data, measurement errors, errors during data collection, and inherent variability in complex systems. Properly handling uncertainty is vital for making sound decisions, particularly in fields like engineering, business, and AI. One

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of the most effective approaches to handling uncertainty is fuzzy set theory. The fuzzy set concept was initially introduced in the middle of the twentieth century by Zadeh [1]. As a mathematical framework, fuzzy set theory enables us to systematically handle uncertain and imprecise information. The study of BCK/BCI-algebras was initiated by Imai and Iséki in 1966, as shown in their research on propositional logic and set-theoretic difference [2, 3]. Several researchers, including Lee and Jun [4], Liu [5], and Jun [6, 7], have made significant contributions to the study of fuzzy logic in BCK/BCI-algebras. Various researchers (see [8–10]) have also made notable contributions to this field, exploring multiple branches of algebra from different perspectives.

Bipolar fuzzy sets, which extend the notion of fuzzy sets [11, 12], are formulated to address the situations where both positive and negative membership degrees must be considered. This expanded framework makes it possible to consider and differentiate among the roles and effects of positive and negative membership degrees. Unlike standard fuzzy sets, where an element's membership is either complete (membership value = 1) or partial (membership value between 0 and 1), bipolar fuzzy sets enable a more nuanced and expressive representation. Instead, in a BPFS, membership levels can be assigned to any value in $[-1, 1]$, indicating the degree of a positive or negative relationship of an element to the set. This extension provides a deeper understanding of ambiguous or changing data, which is particularly useful in situations where both positive and negative aspects must be considered. BPFS theory has been applied to study various aspects of BCK/BCI-algebras, such as a-ideals in BCI-algebras [4], subalgebras and ideals of BCK/BCI-algebras [13], and many others, as explained in [14–22]. Another study, in [23], looks at positive implicative and closed bipolar-valued fuzzy ideals in BCK-algebras. Many researchers have further advanced this field by examining multiple algebraic aspects from different perspectives [24–31].

Following the establishment of the fuzzy set concept, many scholars have concentrated on examining various extensions of this idea. By introducing Intuitionistic Fuzzy Sets in 1986 [32], Atanasov expanded on the fundamental concepts of fuzzy set theory. In this field, Ezhilmaran and Shankar [33] introduced a novel concept known as bipolar intuitionistic fuzzy sets (BPIFSs). This extension of fuzzy sets includes not only positive and negative membership but also positive and negative non-membership for every element in a given set.

The existence of these previous studies serves as an abundant source of ideas. To our knowledge, there is currently no literature on the bipolar-valued intuitionistic fuzzification of ideals in BCK/BCI-algebras. The lack of prior research motivated us to initiate theoretical work on this specific topic. We describe the process using a flowchart, as shown in Figure 1. Our goal is that this graphical representation will improve your understanding of the concept.

In this work, we introduce the concepts of bipolar intuitionistic fuzzy subalgebras (BPIFSAs) and bipolar intuitionistic fuzzy ideals (BPIFIs) in BCK-algebras, with illustrative examples. The cases where a BPIFS is a BPIFI are carefully studied.

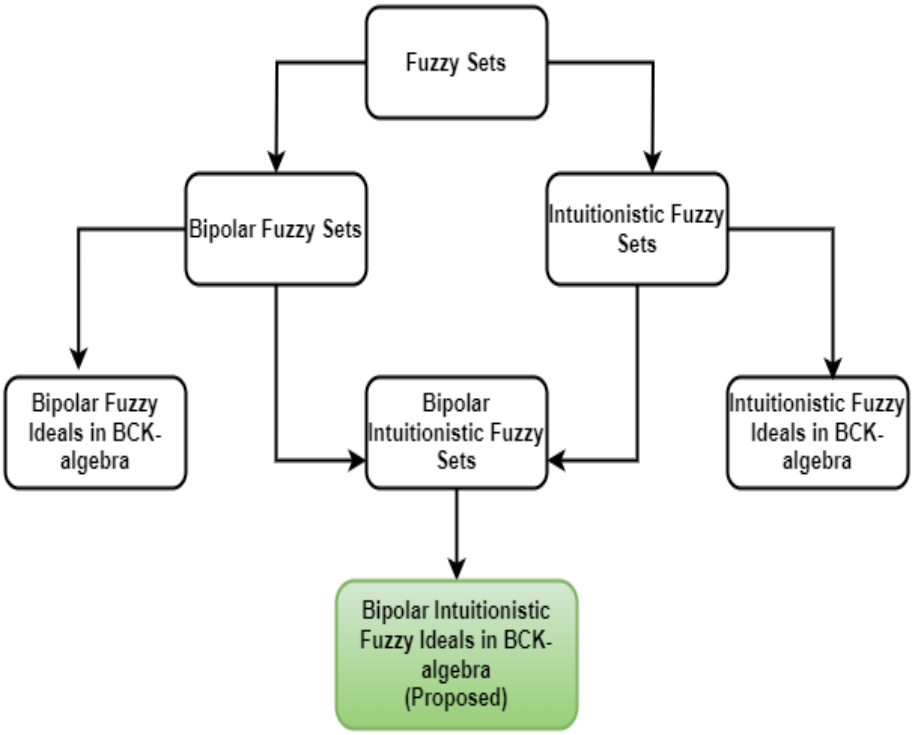


Figure 1: Flow chart of the proposed methodology.

2. Preliminaries

In this section, we present essential definitions and prior results that form the basis for our investigation. We begin by reviewing the structure of BCK-algebras, including subalgebras and ideals. We then recall the fundamental notions of fuzzy sets, bipolar fuzzy sets, and intuitionistic fuzzy sets, leading to the concept of bipolar intuitionistic fuzzy sets (BPIFSs). These notions provide the foundation for defining bipolar intuitionistic fuzzy subalgebras (BPIFSAs) and bipolar intuitionistic fuzzy ideals (BPIFIs), which are the central focus of our study.

Definition 1. *[2] A BCK-algebra* $(\mathrm{I}, 0)$ *is an algebra of type* $(2, 0)$ *if it follows the following axioms for all* $i_1, i_2, i_3 \in \mathrm{I}$,

(BCK-1) $((i_1 * i_2) * (i_1 * i_3)) * (i_3 * i_2) = 0$
(BCK-2) $(i_1 * (i_1 * i_2)) * i_2 = 0$
(BCK-3) $i_1 * i_1 = 0$
(BCK-4) $0 * i_1 = 0$
(BCK-5) $i_1 * i_2 = 0$ *and* $i_2 * i_1 = 0 \Rightarrow i_1 = i_2$.

We can define a binary operation $\leq$ on I by assuming $\mathrm{i}_1 \leq \mathrm{i}_2$ if and only if $\mathrm{i}_1 * \mathrm{i}_2 = 0$. In a BCK-algebra, the following properties hold.

$$\mathrm{i}_1 * 0 = \mathrm{i}_1 \tag{1}$$

$$\mathfrak{i}_1 * \mathfrak{i}_2 \leq \mathfrak{i}_1 \tag{2}$$

$$(\mathfrak{i}_1 * \mathfrak{i}_2) * \mathfrak{i}_3 = (\mathfrak{i}_1 * \mathfrak{i}_3) * \mathfrak{i}_2 \tag{3}$$

$$(\mathfrak{i}_1 * \mathfrak{i}_3) * (\mathfrak{i}_2 * \mathfrak{i}_3) \leq \mathfrak{i}_1 * \mathfrak{i}_2 \tag{4}$$

$$\mathfrak{i}_1 * (\mathfrak{i}_1 * (\mathfrak{i}_1 * \mathfrak{i}_2)) = \mathfrak{i}_1 * \mathfrak{i}_2 \tag{5}$$

$$\mathfrak{i}_1 \leq \mathfrak{i}_2 \Rightarrow \mathfrak{i}_1 * \mathfrak{i}_3 \leq \mathfrak{i}_2 * \mathfrak{i}_3 \text{ and } \mathfrak{i}_3 * \mathfrak{i}_2 \leq \mathfrak{i}_3 * \mathfrak{i}_1 \tag{6}$$

$$\mathfrak{i}_1 * \mathfrak{i}_2 \leq \mathfrak{i}_3 \Rightarrow \mathfrak{i}_1 * \mathfrak{i}_3 \leq \mathfrak{i}_2 \tag{7}$$

for all $\mathfrak{i}_1, \mathfrak{i}_2, \mathfrak{i}_3 \in \mathrm{I}$.

Definition 2. *[34] A non-empty subset* $\mathfrak{I}$ *of a BCK-algebra* I *is termed a subalgebra of* I *if the following condition holds* $i_1 * i_2 \in \mathfrak{I}$ *for all* $i_1, i_2 \in \mathfrak{I}$.

Definition 3. *[34] A non-empty subset* $\mathfrak{I}$ *of a BCK-algebra* I *is termed an ideal of* I *if the following condition holds*
(I1) $0 \in \mathfrak{I}$
(I2) $i_1 * i_2 \in \mathfrak{I}, i_2 \in \mathfrak{I} \Rightarrow i_1 \in \mathfrak{I}$ *for all* $i_1, i_2 \in \mathrm{I}$.

Definition 4. *[1] Let* I *be a non-empty set. A fuzzy set in* I *is a mapping* $M_{\mathfrak{B}} : \mathrm{I} \to [0, 1]$.

Definition 5. *[1] The complement of the fuzzy set* $M_{\mathfrak{B}}$*, denoted by* $\overline{M}_{\mathfrak{B}}$*, is also a fuzzy set defined as* $\overline{M}_{\mathfrak{B}} = 1 - M_{\mathfrak{B}}$ *for all* $i_1 \in \mathrm{I}$. *Also,* $(\overline{\overline{M}_{\mathfrak{B}}}) = M_{\mathfrak{B}}$.

Definition 6. *[35] A bipolar fuzzy set* $\mathfrak{B}$ *of* I *is defined as*

$$\mathfrak{B} = \{(i_1, M^p_{\mathfrak{B}}(i_1), M^n_{\mathfrak{B}}(i_1)) \mid i_1 \in \mathrm{I}\} \tag{8}$$

where $M^p_{\mathfrak{B}} : \mathrm{I} \to [0, 1]$ *and* $M^n_{\mathfrak{B}} : \mathrm{I} \to [-1, 0]$ *are mappings.The positive membership degree* $M^p_{\mathfrak{B}}$ *denotes the level of satisfaction that the element of* I *to the property associated with the bipolar fuzzy set* $\mathfrak{B}$*, while the negative membership degree* $M^n_{\mathfrak{B}}$ *denotes the level of satisfaction that the element of* I *to some implicit counter property of* $\mathfrak{B}$.

We shall use the symbol $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}})$ to denote a bipolar fuzzy set (see Equation (8)).

Definition 7. *[36] A bipolar fuzzy set* $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}})$ *of* I *is said to be a bipolar fuzzy subalgebra if the following conditions are satisfied*
(BPFSA-1) $M^p_{\mathfrak{B}}(i_1 * i_2) \geq \min\{M^p_{\mathfrak{B}}(i_1), M^p_{\mathfrak{B}}(i_2)\}$,
(BPFSA-2) $M^n_{\mathfrak{B}}(i_1 * i_2) \leq \max\{M^n_{\mathfrak{B}}(i_1), M^n_{\mathfrak{B}}(i_2)\}$ *for all* $i_1, i_2 \in \mathrm{I}$.

Definition 8. *[36] A bipolar fuzzy set* $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}})$ *of* I *is said to be a bipolar fuzzy ideal if the following conditions are satisfied*
(BPFI-1) $M^p_{\mathfrak{B}}(0) \geq M^p_{\mathfrak{B}}(i_1)$ *and* $M^n_{\mathfrak{B}}(0) \leq M^n_{\mathfrak{B}}(i_1)$
(BPFI-2) $M^p_{\mathfrak{B}}(i_1) \geq \min\{M^p_{\mathfrak{B}}(i_1 * i_2), M^p_{\mathfrak{B}}(i_2)\}$
(BPFI-3) $M^n_{\mathfrak{B}}(i_1) \leq \max\{M^n_{\mathfrak{B}}(i_1 * i_2), M^n_{\mathfrak{B}}(i_2)\}$ *for all* $i_1, i_2 \in \mathrm{I}$.

Definition 9. *[32] An intuitionistic fuzzy set* $\mathfrak{B}$ *in a non-empty set* I *is an object having the form*

$$\mathfrak{B} = \{(i_1, M_{\mathfrak{B}}(i_1), N_{\mathfrak{B}}(i_1)) \mid i_1 \in \mathrm{I}\} \tag{9}$$

where $M_{\mathfrak{B}}(i_1)$ *and* $N_{\mathfrak{B}}(i_1)$ *represent the degree of belongingness and degree of non-belongingness, respectively, of* $i_1 \in \mathrm{I}$. *Also,* $0 \leq M_{\mathfrak{B}}(i_1) + N_{\mathfrak{B}}(i_1) \leq 1$ *for all* $i_1 \in \mathrm{I}$.

Definition 10. *[33] A bipolar intuitionistic fuzzy set (BPIFS)* $\mathfrak{B}$ *in a non-empty set* I *is an object having the form*

$$\mathfrak{B} = \{(i_1, M^p_{\mathfrak{B}}(i_1), M^n_{\mathfrak{B}}(i_1), N_{\mathfrak{B}}{}^p(i_1), N_{\mathfrak{B}}{}^n(i_1)) \mid i_1 \in \mathrm{I}\} \tag{10}$$

where $M^p_{\mathfrak{B}} : \mathrm{I} \to [0,1]$, $M^n_{\mathfrak{B}} : \mathrm{I} \to [-1,0]$, $N_{\mathfrak{B}}{}^p : \mathrm{I} \to [0,1]$, *and* $N_{\mathfrak{B}}{}^n : \mathrm{I} \to [-1,0]$ *are such that* $0 \leq M^p_{\mathfrak{B}}(i_1) + N_{\mathfrak{B}}{}^p(i_1) \leq 1$ *and* $-1 \leq M^n_{\mathfrak{B}}(i_1) + N_{\mathfrak{B}}{}^n(i_1) \leq 0$. *Here,* $M^p_{\mathfrak{B}}$ *represents the positive membership level, indicating the extent to which an element* I *satisfies a property for a BPIFS* $\mathfrak{B}$. *On the other hand,* $M^n_{\mathfrak{B}}$ *represents the negative membership level, indicating how well an element* I *satisfies the implicit counter property associated with the BPIFS. The terms* $N_{\mathfrak{B}}{}^p(i_1)$ *and* $N_{\mathfrak{B}}{}^n(i_1)$ *refer to the positive non-membership level and negative non-membership level, respectively. We calculate* $N_{\mathfrak{B}}{}^p(i_1)$ *and* $N_{\mathfrak{B}}{}^n(i_1)$ *as* $N_{\mathfrak{B}}{}^p(i_1) = 1 - M^p_{\mathfrak{B}}(i_1)$ *and* $N_{\mathfrak{B}}{}^n(i_1) = -1 - M^n_{\mathfrak{B}}(i_1)$.

We shall use the symbol $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ of I to denote a BPIFS (see Equation (10)).

3. Main Results

Building on the foundational concepts outlined in the previous section, this part of the paper introduces and develops the core theoretical contributions concerning bipolar intuitionistic fuzzy subalgebras (BPIFSAs) and bipolar intuitionistic fuzzy ideals (BPIFIs) within the framework of BCK-algebras. The main objective here is to rigorously define these structures and explore their algebraic properties through a series of definitions, illustrative examples, and theorems.

We aim to establish necessary and sufficient conditions under which a bipolar intuitionistic fuzzy set qualifies as a subalgebra or an ideal in the sense of BCK-algebra theory. Furthermore, we investigate the logical relationships between BPIFSAs and BPIFIs, examine closure properties, and characterize level subsets associated with these fuzzy structures. These results provide a formal basis for extending fuzzy-logic applications to algebraic systems under bipolar and intuitionistic uncertainty.

Definition 11. *A BPIFS* $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ *of* I *is a bipolar intuitionistic fuzzy subalgebra (BPIFSA) of* I *if it satisfies*

(BPIFSA-1) $M^p_{\mathfrak{B}}(i_1 * i_2) \geq \min\{M^p_{\mathfrak{B}}(i_1), M^p_{\mathfrak{B}}(i_2)\}$
(BPIFSA-2) $M^n_{\mathfrak{B}}(i_1 * i_2) \leq \max\{M^n_{\mathfrak{B}}(i_1), M^n_{\mathfrak{B}}(i_2)\}$
(BPIFSA-3) $N^p_{\mathfrak{B}}(i_1 * i_2) \leq \max\{N^p_{\mathfrak{B}}(i_1), N^p_{\mathfrak{B}}(i_2)\}$
(BPIFSA-4) $N^n_{\mathfrak{B}}(i_1 * i_2) \geq \min\{N^n_{\mathfrak{B}}(i_1), N^n_{\mathfrak{B}}(i_2)\}$ *for all* $i_1, i_2 \in \mathrm{I}$.

Example 1. *Define a binary operation* $*$ *on* $\mathrm{I} = \{0, 1, 2, 3\}$ *as given below. Then* I *is a BCK-algebra.*

$$0 * i_1 = 0 \quad \forall i_1 \in \mathrm{I}$$

$$1 * i_1 = \begin{cases} 0, & \text{if } \ i_1 \in \{1, 2\} \\ 1, & \text{if } \ i_1 \in \{0, 3\} \end{cases}$$

$$2 * i_1 = \begin{cases} 0, & \text{if } \ i_1 = 2 \\ 1, & \text{if } \ i_1 = 1 \\ 2, & \text{if } \ i_1 \in \{0, 3\} \end{cases}$$

$$3 * i_1 = \begin{cases} 0, & \text{if } \ i_1 = 3 \\ 3, & \text{if } \ i_1 \in \{0, 1, 2\} \end{cases}$$

Let $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ *be a BPIFS in* I*, defined as shown in Table 1.*

Table 1: Bipolar Intuitionistic Fuzzy Set.

I	$M^p_{\mathfrak{B}}(\mathfrak{i}_1)$	$M^n_{\mathfrak{B}}(\mathfrak{i}_1)$	$N^p_{\mathfrak{B}}(\mathfrak{i}_1)$	$N^n_{\mathfrak{B}}(\mathfrak{i}_1)$
0	0.75	−0.63	0.19	−0.27
1	0.75	−0.63	0.19	−0.27
2	0.25	−0.23	0.43	−0.77
3	0.75	−0.63	0.19	−0.27

It is easy to verify that $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ *is a BPIFSA of* I.

Theorem 1. *Every BPIFSA* $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ *of* I *satisfies the inequalities* $M^p_{\mathfrak{B}}(0) \geq M^p_{\mathfrak{B}}(i_1)$, $M^n_{\mathfrak{B}}(0) \leq M^n_{\mathfrak{B}}(i_1)$, $N^p_{\mathfrak{B}}(0) \leq N^p_{\mathfrak{B}}(i_1)$, *and* $N^n_{\mathfrak{B}}(0) \geq N^n_{\mathfrak{B}}(i_1)$ *for all* $i_1 \in \mathrm{I}$.

Proof. For all $\mathfrak{i}_1 \in \mathrm{I}$,

$$\begin{aligned} M^p_{\mathfrak{B}}(0) = M^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_1) &\geq \min\{M^p_{\mathfrak{B}}(\mathfrak{i}_1), M^p_{\mathfrak{B}}(\mathfrak{i}_2)\} = M^p_{\mathfrak{B}}(\mathfrak{i}_1) \\ &\Rightarrow M^p_{\mathfrak{B}}(0) \geq M^p_{\mathfrak{B}}(\mathfrak{i}_1), \\ M^n_{\mathfrak{B}}(0) = M^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_1) &\leq \max\{M^n_{\mathfrak{B}}(\mathfrak{i}_1), M^n_{\mathfrak{B}}(\mathfrak{i}_1)\} = M^n_{\mathfrak{B}}(\mathfrak{i}_1) \\ &\Rightarrow M^n_{\mathfrak{B}}(0) \leq M^n_{\mathfrak{B}}(\mathfrak{i}_1), \\ N^p_{\mathfrak{B}}(0) = N^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_1) &\leq \max\{N^p_{\mathfrak{B}}(\mathfrak{i}_1), N^p_{\mathfrak{B}}(\mathfrak{i}_1)\} = N^p_{\mathfrak{B}}(\mathfrak{i}_1) \\ &\Rightarrow N^p_{\mathfrak{B}}(0) \leq N^p_{\mathfrak{B}}(\mathfrak{i}_1), \\ N^n_{\mathfrak{B}}(0) = N^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_1) &\geq \min\{N^n_{\mathfrak{B}}(\mathfrak{i}_1), N^n_{\mathfrak{B}}(\mathfrak{i}_1)\} = N^n_{\mathfrak{B}}(\mathfrak{i}_1) \\ &\Rightarrow N^n_{\mathfrak{B}}(0) \geq N^n_{\mathfrak{B}}(\mathfrak{i}_1). \end{aligned}$$

Hence, the proof is completed.

Definition 12. *A BPIFS* $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ *of* I *is a bipolar intuitionistic fuzzy ideal (BPIFI) of* I *if it satisfies*

(BPIFI-1) $M^p_{\mathfrak{B}}(0) \geq M^p_{\mathfrak{B}}(i_1)$, $M^n_{\mathfrak{B}}(0) \leq M^n_{\mathfrak{B}}(i_1)$, $N^p_{\mathfrak{B}}(0) \leq N^p_{\mathfrak{B}}(i_1)$, *and* $N^n_{\mathfrak{B}}(0) \geq N^n_{\mathfrak{B}}(i_1)$

(BPIFI-2) $M^p_{\mathfrak{B}}(i_1) \geq \min\{M^p_{\mathfrak{B}}(i_1 * i_2), M^p_{\mathfrak{B}}(i_2)\}$

(BPIFI-3) $M^n_{\mathfrak{B}}(i_1) \leq \max\{M^n_{\mathfrak{B}}(i_1 * i_2), M^n_{\mathfrak{B}}(i_2)\}$

(BPIFI-4) $N^p_{\mathfrak{B}}(i_1) \leq \max\{N^p_{\mathfrak{B}}(i_1 * i_2), N^p_{\mathfrak{B}}(i_2)\}$

(BPIFI-5) $N^n_{\mathfrak{B}}(i_1) \geq \min\{N^n_{\mathfrak{B}}(i_1 * i_2), N^n_{\mathfrak{B}}(i_2)\}$ *for all* $i_1, i_2 \in \mathrm{I}$.

Example 2. *Define a binary operation* $*$ *on* $\mathrm{I} = \{0, 1, 2, 3, 4\}$ *as given below. Then* I *is a BCK-algebra.*

$$0 * i_1 = 0 \quad \forall i_1 \in \mathrm{I}$$

$$1 * i_1 = \begin{cases} 0, & \text{if } \ i_1 \in \{1, 3, 4\} \\ 1, & \text{if } \ i_1 \in \{0, 2\} \end{cases}$$

$$2 * i_1 = \begin{cases} 0, & \text{if } i_1 \in \{2, 3, 4\} \\ 2, & \text{if } \ i_1 \in \{0, 1\} \end{cases}$$

$$3 * i_1 = \begin{cases} 0, & \text{if } i_1 \in \{3, 4\} \\ 3, & \text{if } \ i_1 \in \{0, 1, 2\} \end{cases}$$

$$4 * i_1 = \begin{cases} 0, & \text{if } \ i_1 = 4 \\ 1, & \text{if } \ i_1 = 3 \\ 3, & \text{if } \ i_1 = 1 \\ 4, & \text{if } \ i_1 \in \{0, 2\} \end{cases}$$

Let us define a BPIFS $\mathfrak{B} = (M_{\mathfrak{B}}{}^p, M_{\mathfrak{B}}{}^n, N_{\mathfrak{B}}{}^p, N^n_{\mathfrak{B}})$ *in* I *as shown in Table 2.*

Table 2: Bipolar Intuitionistic Fuzzy Set.

I	$M^p_{\mathfrak{B}}(\mathbf{i}_1)$	$M^n_{\mathfrak{B}}(\mathbf{i}_1)$	$N^p_{\mathfrak{B}}(\mathbf{i}_1)$	$N^n_{\mathfrak{B}}(\mathbf{i}_1)$
0	0.93	−0.87	0.07	−0.23
1	0.72	−0.73	0.28	−0.27
2	0.93	−0.87	0.07	−0.13
3	0.72	−0.73	0.28	−0.27
4	0.72	−0.73	0.28	−0.27

By routine calculations, we can show that $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ *is a BPIFI.*

Theorem 2. *Let* $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ *be a BPIFI of* I *If* $i_1 \leq i_2$ *in* I, *then* $M^p_{\mathfrak{B}}(i_1) \geq M^p_{\mathfrak{B}}(i_2)$, $M^n_{\mathfrak{B}}(i_1) \leq M^n_{\mathfrak{B}}(i_2)$, $N^p_{\mathfrak{B}}(i_1) \leq N^p_{\mathfrak{B}}(i_2)$, *and* $N^n_{\mathfrak{B}}(i_1) \geq N^n_{\mathfrak{B}}(i_2)$, *that is,* $M^p_{\mathfrak{B}}$ *and* $N^n_{\mathfrak{B}}$ *are order-reversing and* $M^n_{\mathfrak{B}}$ *and* $N^p_{\mathfrak{B}}$ *are order-preserving.*

Proof. Let $\mathfrak{i}_1, \mathfrak{i}_2 \in \mathrm{I}$ be such that $\mathfrak{i}_1 \leq \mathfrak{i}_2$. Then $\mathfrak{i}_1 * \mathfrak{i}_2 = 0$ and so

$$\begin{aligned}
M^p_{\mathfrak{B}}(\mathfrak{i}_1) &\geq \min\{M^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), M^p_{\mathfrak{B}}(\mathfrak{i}_2)\} \\
&= \min\{M^p_{\mathfrak{B}}(0), M^p_{\mathfrak{B}}(\mathfrak{i}_2)\} = M^p_{\mathfrak{B}}(\mathfrak{i}_2) \\
\Rightarrow M^p_{\mathfrak{B}}(\mathfrak{i}_1) &\geq M^p_{\mathfrak{B}}(\mathfrak{i}_2), \\
M^n_{\mathfrak{B}}(\mathfrak{i}_1) &\leq \max\{M^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), M^n_{\mathfrak{B}}(\mathfrak{i}_2)\} \\
&= \max\{M^n_{\mathfrak{B}}(0), M^n_{\mathfrak{B}}(\mathfrak{i}_2)\} = M^n_{\mathfrak{B}}(\mathfrak{i}_2) \\
\Rightarrow M^n_{\mathfrak{B}}(\mathfrak{i}_1) &\leq M^n_{\mathfrak{B}}(\mathfrak{i}_2), \\
N^p_{\mathfrak{B}}(\mathfrak{i}_1) &\leq \max\{N^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), N^p_{\mathfrak{B}}(\mathfrak{i}_2)\} \\
&= \max\{N^p_{\mathfrak{B}}(0), N^p_{\mathfrak{B}}(\mathfrak{i}_2)\} = N^p_{\mathfrak{B}}(\mathfrak{i}_2) \\
\Rightarrow N^p_{\mathfrak{B}}(\mathfrak{i}_1) &\leq N^p_{\mathfrak{B}}(\mathfrak{i}_2), \\
N^n_{\mathfrak{B}}(\mathfrak{i}_1) &\geq \min\{N^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), N^n_{\mathfrak{B}}(\mathfrak{i}_2)\} \\
&= \min\{N^n_{\mathfrak{B}}(0), N^n_{\mathfrak{B}}(\mathfrak{i}_2)\} = N^n_{\mathfrak{B}}(\mathfrak{i}_2) \\
\Rightarrow N^n_{\mathfrak{B}}(\mathfrak{i}_1) &\geq N^n_{\mathfrak{B}}(\mathfrak{i}_2).
\end{aligned}$$

Theorem 3. *A BPIFI* $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ *of* I *must be a BPIFSA of* I.

Proof. Let $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ be a BPIFI of I. Since $\mathfrak{i}_1 * \mathfrak{i}_2 \leq \mathfrak{i}_1$ for all $\mathfrak{i}_1, \mathfrak{i}_2 \in \mathrm{I}$, it follows from Theorem 2 that

$$\begin{aligned}
M^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2) &\geq M^p_{\mathfrak{B}}(\mathfrak{i}_1), \\
M^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2) &\leq M^n_{\mathfrak{B}}(\mathfrak{i}_1), \\
N^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2) &\leq N^p_{\mathfrak{B}}(\mathfrak{i}_1), \\
N^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2) &\geq N^n_{\mathfrak{B}}(\mathfrak{i}_1).
\end{aligned}$$

By Definition 12, we have

$$\begin{aligned}
M^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2) \geq M^p_{\mathfrak{B}}(\mathfrak{i}_1) &\geq \min\{M^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), M^p_{\mathfrak{B}}(\mathfrak{i}_2)\} \\
&\geq \min\{M^p_{\mathfrak{B}}(\mathfrak{i}_1), M^p_{\mathfrak{B}}(\mathfrak{i}_2)\}, \\
M^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2) \leq M^n_{\mathfrak{B}}(\mathfrak{i}_1) &\leq \max\{M^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), M^n_{\mathfrak{B}}(\mathfrak{i}_2)\} \\
&\leq \max\{M^n_{\mathfrak{B}}(\mathfrak{i}_1), M^n_{\mathfrak{B}}(\mathfrak{i}_2)\}, \\
N^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2) \leq N^p_{\mathfrak{B}}(\mathfrak{i}_1) &\leq \max\{N^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), N^p_{\mathfrak{B}}(\mathfrak{i}_2)\} \\
&\leq \max\{N^p_{\mathfrak{B}}(\mathfrak{i}_1), N^p_{\mathfrak{B}}(\mathfrak{i}_2)\}, \\
N^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2) \geq N^n_{\mathfrak{B}}(\mathfrak{i}_1) &\geq \min\{N^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), N^n_{\mathfrak{B}}(\mathfrak{i}_2)\} \\
&\geq \min\{N^n_{\mathfrak{B}}(\mathfrak{i}_1), N^n_{\mathfrak{B}}(\mathfrak{i}_2)\}.
\end{aligned}$$

Therefore, $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ is a BPIFSA of I.

The converse of Theorem 3 may not be true. In Example 1, a BPIFSA is not a BPIFI of I because

$$M^p_{\mathfrak{B}}(2) = 0.25 < 0.75 = M^p_{\mathfrak{B}}(1)$$

$$\begin{aligned} &= \min\{M^p_{\mathfrak{B}}(2*1), M^p_{\mathfrak{B}}(1)\}, \\ M^n_{\mathfrak{B}}(2) = -0.23 > -0.63 &= M^n_{\mathfrak{B}}(1) \\ &= \max\{M^n_{\mathfrak{B}}(2*1), M^n_{\mathfrak{B}}(1)\}, \\ N^p_{\mathfrak{B}}(2) = 0.43 > 0.19 &= N^p_{\mathfrak{B}}(1) \\ &= \max\{N^p_{\mathfrak{B}}(2*1), N^p_{\mathfrak{B}}(1)\}, \\ N^n_{\mathfrak{B}}(2) = -0.77 < -0.27 &= N^n_{\mathfrak{B}}(1) \\ &= \min\{N^n_{\mathfrak{B}}(2*1), N^n_{\mathfrak{B}}(1)\}. \end{aligned}$$

Theorem 4. *A BPIFS* $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ *of* I *is a BPIFI of* I *if and only if it satisfies the following condition* $i_1 * i_2 \leq i_3$ *for all* $i_1, i_2, i_3 \in \mathrm{I}$ *then*

$$\begin{pmatrix} M^p_{\mathfrak{B}}(i_1) \geq \min\{M^p_{\mathfrak{B}}(i_2), M^p_{\mathfrak{B}}(i_3)\} \\ M^n_{\mathfrak{B}}(i_1) \leq \max\{M^n_{\mathfrak{B}}(i_2), M^n_{\mathfrak{B}}(i_3)\} \\ N^p_{\mathfrak{B}}(i_1) \leq \max\{N^p_{\mathfrak{B}}(i_2), N^p_{\mathfrak{B}}(i_3)\} \\ N^n_{\mathfrak{B}}(i_1) \geq \min\{N^n_{\mathfrak{B}}(i_2), N^n_{\mathfrak{B}}(i_3)\} \end{pmatrix} \tag{11}$$

Proof. Assume that $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ is a BPIFI of I. Let $\mathfrak{i}_1, \mathfrak{i}_2, \mathfrak{i}_3 \in \mathrm{I}$ be such that $\mathfrak{i}_1 * \mathfrak{i}_2 \leq \mathfrak{i}_3 \Rightarrow (\mathfrak{i}_1 * \mathfrak{i}_2) * \mathfrak{i}_3 = 0$. Then we have

$$\begin{aligned} M^p_{\mathfrak{B}}(\mathfrak{i}_1) &\geq \min\{M^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), M^p_{\mathfrak{B}}(\mathfrak{i}_2)\}(\text{By BPIFI-1}) \\ &\geq \min\{\min\{M^p_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_2) * \mathfrak{i}_3), M^p_{\mathfrak{B}}(\mathfrak{i}_3)\}, M^p_{\mathfrak{B}}(\mathfrak{i}_2)\} \\ &\geq \min\{\min\{M^p_{\mathfrak{B}}(0), M^p_{\mathfrak{B}}(\mathfrak{i}_3)\}, M^p_{\mathfrak{B}}(\mathfrak{i}_2)\} \\ &= \min\{M^p_{\mathfrak{B}}(\mathfrak{i}_2), M^p_{\mathfrak{B}}(\mathfrak{i}_3)\}, \end{aligned}$$

$$\begin{aligned} M^n_{\mathfrak{B}}(\mathfrak{i}_1) &\leq \max\{M^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), M^n_{\mathfrak{B}}(\mathfrak{i}_2)\}(\text{By BPIFI-2}) \\ &\leq \max\{\max\{M^n_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_2) * \mathfrak{i}_3), M^n_{\mathfrak{B}}(\mathfrak{i}_3)\}, M^n_{\mathfrak{B}}(\mathfrak{i}_2)\} \\ &\leq \max\{\max\{M^n_{\mathfrak{B}}(0), M^n_{\mathfrak{B}}(\mathfrak{i}_3)\}, M^n_{\mathfrak{B}}(\mathfrak{i}_2)\} \\ &= \max\{M^n_{\mathfrak{B}}(\mathfrak{i}_2), M^n_{\mathfrak{B}}(\mathfrak{i}_3)\}, \end{aligned}$$

$$\begin{aligned} N^p_{\mathfrak{B}}(\mathfrak{i}_1) &\leq \max\{N^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), N^p_{\mathfrak{B}}(\mathfrak{i}_2)\}(\text{By BPIFI-3}) \\ &\leq \max\{\max\{N^p_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_2) * \mathfrak{i}_3), N^p_{\mathfrak{B}}(\mathfrak{i}_3)\}, N^p_{\mathfrak{B}}(\mathfrak{i}_2)\} \\ &\leq \max\{\max\{N^p_{\mathfrak{B}}(0), N^p_{\mathfrak{B}}(\mathfrak{i}_3)\}, N^p_{\mathfrak{B}}(\mathfrak{i}_2)\} \\ &= \max\{N^p_{\mathfrak{B}}(\mathfrak{i}_2), N^p_{\mathfrak{B}}(\mathfrak{i}_3)\}, \end{aligned}$$

$$\begin{aligned} N^n_{\mathfrak{B}}(\mathfrak{i}_1) &\geq \min\{N^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), N^n_{\mathfrak{B}}(\mathfrak{i}_2)\}(\text{By BPIFI-4}) \\ &\geq \min\{\min\{N^n_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_2) * \mathfrak{i}_3), N^n_{\mathfrak{B}}(\mathfrak{i}_3)\}, N^n_{\mathfrak{B}}(\mathfrak{i}_2)\} \\ &\geq \min\{\min\{N^n_{\mathfrak{B}}(0), N^n_{\mathfrak{B}}(\mathfrak{i}_3)\}, N^n_{\mathfrak{B}}(\mathfrak{i}_2)\} \\ &= \min\{N^n_{\mathfrak{B}}(\mathfrak{i}_2), N^n_{\mathfrak{B}}(\mathfrak{i}_3)\}. \end{aligned}$$

Hence, Equation (11) is valid.

Conversely, let $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ be a BPIFS in I that satisfies Equation (11). Since $0 * \mathfrak{i}_1 \leq \mathfrak{i}_1$ for all $\mathfrak{i}_1 \in \mathrm{I}$, we have $M^p_{\mathfrak{B}}(0) \geq M^p_{\mathfrak{B}}(\mathfrak{i}_1)$, $M^n_{\mathfrak{B}}(0) \leq M^n_{\mathfrak{B}}(\mathfrak{i}_1)$, $N^p_{\mathfrak{B}}(0) \leq N^p_{\mathfrak{B}}(\mathfrak{i}_1)$, and $N^n_{\mathfrak{B}}(0) \geq N^n_{\mathfrak{B}}(\mathfrak{i}_1)$. Also, since $\mathfrak{i}_1 * (\mathfrak{i}_1 * \mathfrak{i}_2) \leq \mathfrak{i}_2$ for all $\mathfrak{i}_1, \mathfrak{i}_2 \in \mathrm{I}$, we have

$$\begin{aligned}
M^p_{\mathfrak{B}}(\mathfrak{i}_1) &\geq \min\{M^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), M^p_{\mathfrak{B}}(\mathfrak{i}_2)\},\\
M^n_{\mathfrak{B}}(\mathfrak{i}_1) &\leq \max\{M^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), M^n_{\mathfrak{B}}(\mathfrak{i}_2)\},\\
N^p_{\mathfrak{B}}(\mathfrak{i}_1) &\leq \max\{N^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), N^p_{\mathfrak{B}}(\mathfrak{i}_2)\},\\
N^n_{\mathfrak{B}}(\mathfrak{i}_1) &\geq \min\{N^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), N^n_{\mathfrak{B}}(\mathfrak{i}_2)\}.
\end{aligned}$$

Therefore, $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ is a BPIFI of I.

Theorem 5. *A BPIFS $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ of* I *is a BPIFI of* I *if and only if the bipolar fuzzy sets $\mathfrak{B}_1 = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}})$ and $\mathfrak{B}_2 = ((\overline{N_{\mathfrak{B}}})^p, (\overline{N_{\mathfrak{B}}})^n)$ are bipolar fuzzy ideals of* I.

Proof. Assume that $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ be a BPIFI of I. Then clearly $\mathfrak{B}_1 = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}})$ is a bipolar fuzzy ideal of I. Let $\mathfrak{i}_1, \mathfrak{i}_2 \in \mathrm{I}$. Then

$$\begin{aligned}
&N^p_{\mathfrak{B}}(0) \leq N^p_{\mathfrak{B}}(\mathfrak{i}_1)\\
&\Rightarrow -N^p_{\mathfrak{B}}(0) \geq -N^p_{\mathfrak{B}}(\mathfrak{i}_1)\\
&\Rightarrow 1 - N^p_{\mathfrak{B}}(0) \geq 1 - N^p_{\mathfrak{B}}(\mathfrak{i}_1)\\
&\Rightarrow (\overline{N_{\mathfrak{B}}})^p(0) \geq (\overline{N_{\mathfrak{B}}})^p(\mathfrak{i}_1),
\end{aligned}$$

$$\begin{aligned}
&N^n_{\mathfrak{B}}(0) \geq N^n_{\mathfrak{B}}(\mathfrak{i}_1)\\
&\Rightarrow -N^n_{\mathfrak{B}}(0) \leq -N^n_{\mathfrak{B}}(\mathfrak{i}_1)\\
&\Rightarrow -1 - N^n_{\mathfrak{B}}(0) \leq -1 - N^n_{\mathfrak{B}}(\mathfrak{i}_1)\\
&\Rightarrow (\overline{N_{\mathfrak{B}}})^n(0) \leq (\overline{N_{\mathfrak{B}}})^n(\mathfrak{i}_1),
\end{aligned}$$

$$\begin{aligned}
&N^p_{\mathfrak{B}}(\mathfrak{i}_1) \leq \max\{N^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), N^p_{\mathfrak{B}}(\mathfrak{i}_2)\}\\
&\Rightarrow -N^p_{\mathfrak{B}}(\mathfrak{i}_1) \geq -\max\{N^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), N^p_{\mathfrak{B}}(\mathfrak{i}_2)\}\\
&\Rightarrow 1 - N^p_{\mathfrak{B}}(\mathfrak{i}_1) \geq 1 - \max\{N^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), N^p_{\mathfrak{B}}(\mathfrak{i}_2)\}\\
&\Rightarrow (\overline{N_{\mathfrak{B}}})^p(\mathfrak{i}_1) \geq \min\{1 - N^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), 1 - N^p_{\mathfrak{B}}(\mathfrak{i}_2)\}\\
&\Rightarrow (\overline{N_{\mathfrak{B}}})^p(\mathfrak{i}_1) \geq \min\{(\overline{N_{\mathfrak{B}}})^p(\mathfrak{i}_1 * \mathfrak{i}_2), (\overline{N_{\mathfrak{B}}})^p(\mathfrak{i}_2)\},
\end{aligned}$$

$$\begin{aligned}
&N^n_{\mathfrak{B}}(\mathfrak{i}_1) \geq \min\{N^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), N^n_{\mathfrak{B}}(\mathfrak{i}_2)\}\\
&\Rightarrow -N^n_{\mathfrak{B}}(\mathfrak{i}_1) \leq -\min\{N^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), N^n_{\mathfrak{B}}(\mathfrak{i}_2)\}\\
&\Rightarrow 1 - N^n_{\mathfrak{B}}(\mathfrak{i}_1) \leq 1 - \min\{N^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), N^n_{\mathfrak{B}}(\mathfrak{i}_2)\}\\
&\Rightarrow (\overline{N_{\mathfrak{B}}})^n(\mathfrak{i}_1) \leq \max\{1 - N^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), 1 - N^n_{\mathfrak{B}}(\mathfrak{i}_2)\}
\end{aligned}$$

$$\Rightarrow (\overline{N_{\mathfrak{B}}})^n(\mathfrak{i}_1) \leq \max\{(\overline{N_{\mathfrak{B}}})^n(\mathfrak{i}_1 * \mathfrak{i}_2), (\overline{N_{\mathfrak{B}}})^n(\mathfrak{i}_2)\}.$$

Hence, $\mathfrak{B}_2 = ((\overline{N_{\mathfrak{B}}})^p, (\overline{N_{\mathfrak{B}}})^n)$ is a bipolar fuzzy ideal of I.

Conversely, assume that the bipolar fuzzy sets $\mathfrak{B}_1 = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}})$ and $\mathfrak{B}_2 = ((\overline{N_{\mathfrak{B}}})^p, (\overline{N_{\mathfrak{B}}})^n)$ are bipolar fuzzy ideals of I. For every $\mathfrak{i}_1, \mathfrak{i}_2 \in \mathrm{I}$ we get $M^p_{\mathfrak{B}}(0) \geq M^p_{\mathfrak{B}}(\mathfrak{i}_1)$, $M^n_{\mathfrak{B}}(0) \leq M^n_{\mathfrak{B}}(\mathfrak{i}_1)$, $(\overline{N_{\mathfrak{B}}})^p(0) \geq (\overline{N_{\mathfrak{B}}})^p(\mathfrak{i}_1) \Rightarrow -(\overline{N_{\mathfrak{B}}})^p(0) \leq -(\overline{N_{\mathfrak{B}}})^p\mathfrak{i}_1) \Rightarrow 1 - (\overline{N_{\mathfrak{B}}})^p(0) \leq 1 - (\overline{N_{\mathfrak{B}}})^p(\mathfrak{i}_1) \Rightarrow N^p_{\mathfrak{B}}(0) \leq N^p_{\mathfrak{B}}(\mathfrak{i}_1)$, $(\overline{N_{\mathfrak{B}}})^n(0) \leq (\overline{N_{\mathfrak{B}}})^n(\mathfrak{i}_1) \Rightarrow -(\overline{N_{\mathfrak{B}}})^n(0) \geq -(\overline{N_{\mathfrak{B}}})^n(\mathfrak{i}_1) \Rightarrow -1-(\overline{N_{\mathfrak{B}}})^n(0) \geq -1 - (\overline{N_{\mathfrak{B}}})^n(\mathfrak{i}_1) \Rightarrow N^n_{\mathfrak{B}}(0) \geq N^n_{\mathfrak{B}}(\mathfrak{i}_1)$. Also,

$$\begin{aligned}
&(\overline{N_{\mathfrak{B}}})^p(\mathfrak{i}_1) \geq \min\{(\overline{N_{\mathfrak{B}}})^p(\mathfrak{i}_1 * \mathfrak{i}_2), (\overline{N_{\mathfrak{B}}})^p(\mathfrak{i}_2)\}\\
&\Rightarrow -(\overline{N_{\mathfrak{B}}})^p(\mathfrak{i}_1) \leq -\min\{(\overline{N_{\mathfrak{B}}})^p(\mathfrak{i}_1 * \mathfrak{i}_2), (\overline{N_{\mathfrak{B}}})^p(\mathfrak{i}_2)\}\\
&\Rightarrow 1 - (\overline{N_{\mathfrak{B}}})^p(\mathfrak{i}_1) \leq 1 - \min\{(\overline{N_{\mathfrak{B}}})^p(\mathfrak{i}_1 * \mathfrak{i}_2), (\overline{N_{\mathfrak{B}}})^p(\mathfrak{i}_2)\}\\
&\Rightarrow N^p_{\mathfrak{B}}(\mathfrak{i}_1) \leq \max\{1 - (\overline{N_{\mathfrak{B}}})^p(\mathfrak{i}_1 * \mathfrak{i}_2), 1 - (\overline{N_{\mathfrak{B}}})^p(\mathfrak{i}_2)\}\\
&\Rightarrow N^p_{\mathfrak{B}}(\mathfrak{i}_1) \leq \max\{N^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), N^p_{\mathfrak{B}}(\mathfrak{i}_2)\},
\end{aligned}$$

$$\begin{aligned}
&(\overline{N_{\mathfrak{B}}})^n(\mathfrak{i}_1) \leq \max\{(\overline{N_{\mathfrak{B}}})^n(\mathfrak{i}_1 * \mathfrak{i}_2), (\overline{N_{\mathfrak{B}}})^n(\mathfrak{i}_2)\}\\
&\Rightarrow -(\overline{N_{\mathfrak{B}}})^n(\mathfrak{i}_1) \geq -\max\{(\overline{N_{\mathfrak{B}}})^n(\mathfrak{i}_1 * \mathfrak{i}_2), (\overline{N_{\mathfrak{B}}})^n(\mathfrak{i}_2)\}\\
&\Rightarrow 1 - (\overline{N_{\mathfrak{B}}})^n(\mathfrak{i}_1) \geq 1 - \max\{(\overline{N_{\mathfrak{B}}})^n(\mathfrak{i}_1 * \mathfrak{i}_2), (\overline{N_{\mathfrak{B}}})^n(\mathfrak{i}_2)\}\\
&\Rightarrow N^n_{\mathfrak{B}}(\mathfrak{i}_1) \geq \min\{1 - (\overline{N_{\mathfrak{B}}})^n(\mathfrak{i}_1 * \mathfrak{i}_2), 1 - (\overline{N_{\mathfrak{B}}})^n(\mathfrak{i}_2)\}\\
&\Rightarrow N^n_{\mathfrak{B}}(\mathfrak{i}_1) \geq \min\{N^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), N^n_{\mathfrak{B}}(\mathfrak{i}_2)\}.
\end{aligned}$$

Hence, $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ is a BPIFI of I.

Theorem 6 and Propositions 1 and 2 are immediate consequences of Definition 12, which characterizes BPIFIs in terms of their component-wise behavior. These results help establish simplified or derived structures that also satisfy the BPIFI conditions.

Theorem 6. *A BPIFS $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ of* I *is a BPIFI of* I *if and only if the bipolar fuzzy sets $\mathfrak{B}_3 = ((\overline{M_{\mathfrak{B}}})^p, (\overline{M_{\mathfrak{B}}})^n)$ and $\mathfrak{B}_4 = (N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ are bipolar anti fuzzy ideals of* I.

Proposition 1. *If a BPIFS $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ of* I *is a BPIFI of* I*, then $(\circ\mathfrak{B}) = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, (\overline{M_{\mathfrak{B}}})^p, (\overline{M_{\mathfrak{B}}})^n)$ is also a BPIFI of* I.

Proposition 2. *If a BPIFS $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ of* I *is a BPIFI of* I*, then $(\Box\mathfrak{B}) = ((\overline{N_{\mathfrak{B}}})^p, (\overline{N_{\mathfrak{B}}})^n, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ is also a BPIFI of* I.

Theorem 7 follows directly from Definition 12, together with Proposition 1 and Theorem 6.

Theorem 7. *A BPIFS $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ of* I *is a BPIFI of* I *if and only if $(\circ\mathfrak{B}) = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, (\overline{M_{\mathfrak{B}}})^p, (\overline{M_{\mathfrak{B}}})^n)$ and $(\Box\mathfrak{B}) = ((\overline{N_{\mathfrak{B}}})^p, (\overline{N_{\mathfrak{B}}})^n, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ are BPIFIs of* I.

Theorem 8. *A BPIFS* $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ *of* I *is a BPIFI of* I *if and only if the non-empty level subsets*

$$U(M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, p_1, n_1) = \{\dot{\imath}_1 \in \mathrm{I} \mid M^p_{\mathfrak{B}}(\dot{\imath}_1) \geq p_1, M^n_{\mathfrak{B}}(\dot{\imath}_1) \leq n_1\},$$
$$L(N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}}, p_2, n_2) = \{\dot{\imath}_1 \in \mathrm{I} \mid N^p_{\mathfrak{B}}(\dot{\imath}_1) \leq p_2, N^n_{\mathfrak{B}}(\dot{\imath}_1) \geq n_2\}$$

are ideals of I *for any* $p_1, p_2 \in [0,1]$ *and* $n_1, n_2 \in [-1,0]$.

Proof. Assume that a BPIFS $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ of I is a BPIFI of I, and $U(M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, p_1, n_1)$, $L(N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}}, p_2, n_2)$ are non-empty.

Let $\dot{\imath}_1 \in U(M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, p_1, n_1)$ and $\dot{\imath}_1 \in L(N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}}, p_2, n_2)$. For any $\dot{\imath}_1 \in \mathrm{I}$, we have

$$\begin{aligned}
& M^p_{\mathfrak{B}}(0) \geq M^p_{\mathfrak{B}}(\dot{\imath}_1) && \text{(By } \mathfrak{B} \text{ is a BPIFI)} \\
& M^p_{\mathfrak{B}}(0) \geq M^p_{\mathfrak{B}}(\dot{\imath}_1) \geq p_1 && \text{(By } \dot{\imath}_1 \in U\text{)} \\
\Rightarrow\ & M^p_{\mathfrak{B}}(0) \geq p_1 \\
& M^n_{\mathfrak{B}}(0) \leq M^n_{\mathfrak{B}}(\dot{\imath}_1) && \text{(By } \mathfrak{B} \text{ is a BPIFI)} \\
& M^n_{\mathfrak{B}}(0) \leq M^n_{\mathfrak{B}}(\dot{\imath}_1) \leq n_1 && \text{(By } \dot{\imath}_1 \in U\text{)} \\
\Rightarrow\ & M^n_{\mathfrak{B}}(0) \leq n_1 \\
& N^p_{\mathfrak{B}}(0) \leq N^p_{\mathfrak{B}}(\dot{\imath}_1) && \text{(By } \mathfrak{B} \text{ is a BPIFI)} \\
& N^p_{\mathfrak{B}}(0) \leq N^p_{\mathfrak{B}}(\dot{\imath}_1) \leq p_2 && \text{(By } \dot{\imath}_1 \in L\text{)} \\
\Rightarrow\ & N^p_{\mathfrak{B}}(0) \leq p_2 \\
& N^n_{\mathfrak{B}}(0) \geq N^n_{\mathfrak{B}}(\dot{\imath}_1) && \text{(By } \mathfrak{B} \text{ is a BPIFI)} \\
& N^n_{\mathfrak{B}}(0) \geq N^n_{\mathfrak{B}}(\dot{\imath}_1) \geq n_2 && \text{(By } \dot{\imath}_1 \in L\text{)} \\
\Rightarrow\ & N^n_{\mathfrak{B}}(0) \geq n_2
\end{aligned}$$

And so, $0 \in U(M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, p_1, n_1)$ and $0 \in L(N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}}, p_2, n_2)$.

Let $\dot{\imath}_1 * \dot{\imath}_2$, $\dot{\imath}_2 \in U(M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, p_1, n_1) \cap L(N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}}, p_2, n_2)$ for all $\dot{\imath}_1, \dot{\imath}_2 \in \mathrm{I}$. Then

$$\begin{aligned}
& M^p_{\mathfrak{B}}(\dot{\imath}_1) \geq \min\{M^p_{\mathfrak{B}}(\dot{\imath}_1 * \dot{\imath}_2), M^p_{\mathfrak{B}}(\dot{\imath}_2)\} \geq \min\{p_1, p_1\} = p_1, \\
& M^n_{\mathfrak{B}}(\dot{\imath}_1) \leq \max\{M^n_{\mathfrak{B}}(\dot{\imath}_1 * \dot{\imath}_2), M^n_{\mathfrak{B}}(\dot{\imath}_2)\} \leq \max\{n_1, n_1\} = n_1, \\
& \quad \Rightarrow \dot{\imath}_1 \in U(M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, p_1, n_1).
\end{aligned}$$

Also,

$$\begin{aligned}
& N^p_{\mathfrak{B}}(\dot{\imath}_1) \leq \max\{N^p_{\mathfrak{B}}(\dot{\imath}_1 * \dot{\imath}_2), N^p_{\mathfrak{B}}(\dot{\imath}_2)\} \leq \max\{p_2, p_2\} = p_2, \\
& N^n_{\mathfrak{B}}(\dot{\imath}_1) \geq \min\{N^n_{\mathfrak{B}}(\dot{\imath}_1 * \dot{\imath}_2), N^n_{\mathfrak{B}}(\dot{\imath}_2)\} \geq \min\{n_2, n_2\} = n_2, \\
& \quad \Rightarrow \dot{\imath}_1 \in L(N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}}, p_2, n_2).
\end{aligned}$$

Hence, the level sets $U(M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, p_1, n_1)$ and $L(N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}}, p_2, n_2)$ are ideals of I for any $p_1, p_2 \in [0,1]$ and $n_1, n_2 \in [-1,0]$.

Conversely, let $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ be a BPIFS such that the non-empty level subsets $U(M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, p_1, n_1)$ and $L((N_{\mathfrak{B}\mathfrak{B}})^p, (N_{\mathfrak{B}\mathfrak{B}})^n, p_2, n_2)$ are ideals of I, for any $p_1, p_2 \in$

$[0,1]$ and $n_1,\ n_2 \in [-1,0]$. Put $M^p_{\mathfrak{B}}(\mathfrak{i}_1) = p_1$, $M^n_{\mathfrak{B}}(\mathfrak{i}_1) = n_1$, $N^p_{\mathfrak{B}}(\mathfrak{i}_1) = p_2$, and $N^n_{\mathfrak{B}}(\mathfrak{i}_1) = n_2$ for some $\mathfrak{i}_1 \in \mathrm{I}$. Since $0 \in U(M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, p_1, n_1) \cap L(N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}}, p_2, n_2)$, $M^p_{\mathfrak{B}}(0) \geq p_1 = M^p_{\mathfrak{B}}(\mathfrak{i}_1)$, $M^n_{\mathfrak{B}}(0) \leq n_1 = M^n_{\mathfrak{B}}(\mathfrak{i}_1)$, $N^p_{\mathfrak{B}}(0) \leq p_2 = N^p_{\mathfrak{B}}(\mathfrak{i}_1)$, and $N^n_{\mathfrak{B}}(0) \geq n_2 = N^n_{\mathfrak{B}}(\mathfrak{i}_1)$. Suppose there exists $\dot{\mathfrak{i}_1}, \dot{\mathfrak{i}_2} \in \mathrm{I}$ such that

$$\begin{aligned}
M^p_{\mathfrak{B}}(\dot{\mathfrak{i}_1}) &< \min\{M^p_{\mathfrak{B}}(\dot{\mathfrak{i}_1} * \dot{\mathfrak{i}_2}), M^p_{\mathfrak{B}}(\dot{\mathfrak{i}_2})\},\\
M^n_{\mathfrak{B}}(\dot{\mathfrak{i}_1}) &> \max\{M^n_{\mathfrak{B}}(\dot{\mathfrak{i}_1} * \dot{\mathfrak{i}_2}), M^n_{\mathfrak{B}}(\dot{\mathfrak{i}_2})\},\\
N^p_{\mathfrak{B}}(\dot{\mathfrak{i}_1}) &> \max\{N^p_{\mathfrak{B}}(\dot{\mathfrak{i}_1} * \dot{\mathfrak{i}_2}), N^p_{\mathfrak{B}}(\dot{\mathfrak{i}_2})\},\\
N^n_{\mathfrak{B}}(\dot{\mathfrak{i}_1}) &< \min\{N^n_{\mathfrak{B}}(\dot{\mathfrak{i}_1} * \dot{\mathfrak{i}_2}), N^n_{\mathfrak{B}}(\dot{\mathfrak{i}_2})\}.
\end{aligned}$$

Take $\dot{p_1}, \dot{p_2} \in [0,1]$ and $\dot{n_1},\ \dot{n_2} \in [-1,0]$ as

$$\begin{aligned}
\dot{p_1} &= \frac{1}{2}\left[M^p_{\mathfrak{B}}(\dot{\mathfrak{i}_1}) + \min\{M^p_{\mathfrak{B}}(\dot{\mathfrak{i}_1} * \dot{\mathfrak{i}_2}), M^p_{\mathfrak{B}}(\dot{\mathfrak{i}_2})\}\right],\\
\dot{n_1} &= \frac{1}{2}\left[M^n_{\mathfrak{B}}(\dot{\mathfrak{i}_1}) + \max\{M^n_{\mathfrak{B}}(\dot{\mathfrak{i}_1} * \dot{\mathfrak{i}_2}), M^n_{\mathfrak{B}}(\dot{\mathfrak{i}_2})\}\right],\\
\dot{p_2} &= \frac{1}{2}\left[N^p_{\mathfrak{B}}(\dot{\mathfrak{i}_1}) + \max\{N^p_{\mathfrak{B}}(\dot{\mathfrak{i}_1} * \dot{\mathfrak{i}_2}), N^p_{\mathfrak{B}}(\dot{\mathfrak{i}_2})\}\right],\\
\dot{n_2} &= \frac{1}{2}\left[N^n_{\mathfrak{B}}(\dot{\mathfrak{i}_1}) + \min\{N^n_{\mathfrak{B}}(\dot{\mathfrak{i}_1} * \dot{\mathfrak{i}_2}), N^n_{\mathfrak{B}}(\dot{\mathfrak{i}_2})\}\right].
\end{aligned}$$

Then

$$\begin{aligned}
M^p_{\mathfrak{B}}(\dot{\mathfrak{i}_1}) &< \dot{p_1} < \min\{M^p_{\mathfrak{B}}(\dot{\mathfrak{i}_1} * \dot{\mathfrak{i}_2}), M^p_{\mathfrak{B}}(\dot{\mathfrak{i}_2})\},\\
M^n_{\mathfrak{B}}(\dot{\mathfrak{i}_1}) &> \dot{n_1} > \max\{M^n_{\mathfrak{B}}(\dot{\mathfrak{i}_1} * \dot{\mathfrak{i}_2}), M^n_{\mathfrak{B}}(\dot{\mathfrak{i}_2})\},\\
N^p_{\mathfrak{B}}(\dot{\mathfrak{i}_1}) &> \dot{p_2} > \max\{N^p_{\mathfrak{B}}(\dot{\mathfrak{i}_1} * \dot{\mathfrak{i}_2}), N^p_{\mathfrak{B}}(\dot{\mathfrak{i}_2})\},\\
N^n_{\mathfrak{B}}(\dot{\mathfrak{i}_1}) &< \dot{n_2} < \min\{N^n_{\mathfrak{B}}(\dot{\mathfrak{i}_1} * \dot{\mathfrak{i}_2}), N^n_{\mathfrak{B}}(\dot{\mathfrak{i}_2})\}.
\end{aligned}$$

Thus, $\dot{\mathfrak{i}_1} * \dot{\mathfrak{i}_2}, \dot{\mathfrak{i}_2} \in U(M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, \dot{p_1}, \dot{n_1})$ and $\dot{\mathfrak{i}_1} * \dot{\mathfrak{i}_2}, \dot{\mathfrak{i}_2} \in L(N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}}, \dot{p_2}, \dot{n_2})$ but $\dot{\mathfrak{i}_1} \notin U(M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, \dot{p_1}, \dot{n_1})$ and $\dot{\mathfrak{i}_1} \notin L(N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}}, \dot{p_2}, \dot{n_2})$ which is a contradiction with level subsets $U(M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, \dot{p_1}, \dot{n_1})$ and $L(N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}}, \dot{p_2}, \dot{n_2})$ being ideals of I. Therefore, for all $\mathfrak{i}_1, \mathfrak{i}_2 \in \mathrm{I}$,

$$\begin{aligned}
M^p_{\mathfrak{B}}(\mathfrak{i}_1) &\geq \min\{M^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), M^p_{\mathfrak{B}}(\mathfrak{i}_2)\},\\
M^n_{\mathfrak{B}}(\mathfrak{i}_1) &\leq \max\{M^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), M^n_{\mathfrak{B}}(\mathfrak{i}_2)\},\\
N^p_{\mathfrak{B}}(\mathfrak{i}_1) &\leq \max\{N^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), N^p_{\mathfrak{B}}(\mathfrak{i}_2)\},\\
N^n_{\mathfrak{B}}(\mathfrak{i}_1) &\geq \min\{N^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), N^n_{\mathfrak{B}}(\mathfrak{i}_2)\}.
\end{aligned}$$

Hence, $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ is a BPIFI of I.

Corollary 1. *A BPIFS $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ of* I *is a BPIFSA of* I *if and only if the non-empty level subsets*

$$\begin{aligned}
U(M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, p_1, n_1) &= \{i_1 \in \mathrm{I} \mid M^p_{\mathfrak{B}}(i_1) \geq p_1, M^n_{\mathfrak{B}}(i_1) \leq n_1\},\\
L(N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}}, p_2, n_2) &= \{i_1 \in \mathrm{I} \mid N^p_{\mathfrak{B}}(i_1) \leq p_2, N^n_{\mathfrak{B}}(i_1) \geq n_2\}
\end{aligned}$$

are subalgebras of I *for any $p_1, p_2 \in [0,1]$ and $n_1, n_2 \in [-1,0]$.*

For any $\Omega \in \mathrm{I}$ and any BPIFS $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ in I, we let

$$\mathfrak{H}(\Omega) = \{\mathfrak{i}_1 \in \mathrm{I} \mid (M_{\mathfrak{B}})^p(\mathfrak{i}_1) \geq (M_{\mathfrak{B}})^p(\Omega), (M_{\mathfrak{B}})^n(\mathfrak{i}_1) \leq M^n_{\mathfrak{B}}(\Omega), \\ N^p_{\mathfrak{B}}(\mathfrak{i}_1) \leq N^p_{\mathfrak{B}}(\Omega), N^n_{\mathfrak{B}}(\mathfrak{i}_1) \geq N^n_{\mathfrak{B}}(\Omega)\}.$$

Obviously, $\Omega \in \mathfrak{H}(\Omega)$. If $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ is a BPIFI of I, then $0 \in \mathfrak{H}(\Omega)$. The following example shows that there exists an element $\Omega \in \mathrm{I}$ such that $\mathfrak{H}(\Omega)$ is not an ideal of I.

Example 3. *Define a binary operation* $*$ *on* $\mathrm{I} = \{0, 1, 2, 3, 4\}$ *as given below. Then* I *is a BCK-algebra.*

$$0 * \mathfrak{i}_1 = 0 \quad \forall \ \mathfrak{i}_1 \in \mathrm{I}$$

$$1 * \mathfrak{i}_1 = \begin{cases} 0 & \text{if} \ \ \mathfrak{i}_1 \in \{1, 3, 4\} \\ 1 & \text{if} \ \ \mathfrak{i}_1 \in \{0, 2\} \end{cases}$$

$$2 * \mathfrak{i}_1 = \begin{cases} 0 & \text{if} \ \ \mathfrak{i}_1 \in \{2, 4\} \\ 2 & \text{if} \ \ \mathfrak{i}_1 \in \{0, 1, 3\} \end{cases}$$

$$3 * \mathfrak{i}_1 = \begin{cases} 0 & \text{if} \ \ \mathfrak{i}_1 = 3 \\ 3 & \text{if} \ \ \mathfrak{i}_1 \in \{0, 1, 2, 4\} \end{cases}$$

$$4 * \mathfrak{i}_1 = \begin{cases} 0 & \text{if} \ \ \mathfrak{i}_1 = 4 \\ 4 & \text{if} \ \ \mathfrak{i}_1 \in \{0, 1, 2, 3\} \end{cases}$$

Let us define a BPIFS $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ *in* I *as shown in Table 3. Then* $\mathfrak{B} =$

Table 3: Bipolar Intuitionistic Fuzzy Set.

I	$M^p_{\mathfrak{B}}(\mathfrak{i}_1)$	$M^n_{\mathfrak{B}}(\mathfrak{i}_1)$	$N^p_{\mathfrak{B}}(\mathfrak{i}_1)$	$N^n_{\mathfrak{B}}(\mathfrak{i}_1)$
0	0.7	−0.8	0.3	−0.2
1	0.6	−0.6	0.4	−0.4
2	0.3	−0.4	0.7	−0.6
3	0.1	−0.2	0.9	−0.8
4	0.4	−0.5	0.6	−0.5

$(M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ *satisfies (BPIFI-1) and it is not a BPIFI of* I *because*

$$M^p_{\mathfrak{B}}(2) = 0.3 < 0.4 = M^p_{\mathfrak{B}}(4) = \min\{M^p_{\mathfrak{B}}(2 * 4), M^p_{\mathfrak{B}}(4)\},$$
$$M^n_{\mathfrak{B}}(2) = -0.4 > -0.5 = M^n_{\mathfrak{B}}(4) = \max\{M^n_{\mathfrak{B}}(2 * 4), M^n_{\mathfrak{B}}(4)\},$$
$$N^p_{\mathfrak{B}}(2) = 0.7 > 0.6 = N^p_{\mathfrak{B}}(4) = \max\{N^p_{\mathfrak{B}}(2 * 4), N^p_{\mathfrak{B}}(4)\},$$
$$N^n_{\mathfrak{B}}(2) = -0.6 < -0.5 = N^n_{\mathfrak{B}}(4) = \min\{N^n_{\mathfrak{B}}(2 * 4), N^n_{\mathfrak{B}}(4)\}.$$

Then $\mathfrak{H}(4) = \{0, 1, 4\}$ *is not an ideal of* I. *Since* $2 * 4 = 0 \in \mathfrak{H}(4), 4 \in \mathfrak{H}(4)$, *while* $2 \notin \mathfrak{H}(4)$.

Theorem 9. *Let* $\Omega \in \mathrm{I}$. *If* $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ *is a BPIFI of* I, *then* $\mathfrak{H}(\Omega)$ *is an ideal of* I.

Proof. Obviously, $0 \in \mathfrak{H}(\Omega)$. Let $\mathfrak{i}_1, \mathfrak{i}_2 \in \mathrm{I}$ be such that $\mathfrak{i}_1 * \mathfrak{i}_2$ and $\mathfrak{i}_2 \in \mathfrak{H}(\Omega)$. Then $M^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2) \geq M^p_{\mathfrak{B}}(\Omega)$, $M^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2) \leq M^n_{\mathfrak{B}}(\Omega)$, $N^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2) \leq N^p_{\mathfrak{B}}(\Omega)$, $N^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2) \geq N^n_{\mathfrak{B}}(\Omega)$, $M^p_{\mathfrak{B}}(\mathfrak{i}_2) \geq M^p_{\mathfrak{B}}(\Omega)$, $M^n_{\mathfrak{B}}(\mathfrak{i}_2) \leq M^n_{\mathfrak{B}}(\Omega)$, $N^p_{\mathfrak{B}}(\mathfrak{i}_2) \leq N^p_{\mathfrak{B}}(\Omega)$, and $(N_{\mathfrak{B}})^n(\mathfrak{i}_2) \geq (N_{\mathfrak{B}})^n(\Omega)$. Since $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ is a BPIFI of I, we have from Definition 12 that

$$\begin{aligned}
M^p_{\mathfrak{B}}(\mathfrak{i}_1) &\geq \min\{M^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), M^p_{\mathfrak{B}}(\mathfrak{i}_2)\} \geq M^p_{\mathfrak{B}}(\Omega),\\
M^n_{\mathfrak{B}}(\mathfrak{i}_1) &\leq \max\{M^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), M^n_{\mathfrak{B}}(\mathfrak{i}_2)\} \leq M^n_{\mathfrak{B}}(\Omega),\\
N^p_{\mathfrak{B}}(\mathfrak{i}_1) &\leq \max\{N^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), N^p_{\mathfrak{B}}(\mathfrak{i}_2)\} \leq N^p_{\mathfrak{B}}(\Omega),\\
N^n_{\mathfrak{B}}(\mathfrak{i}_1) &\geq \min\{N^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), N^n_{\mathfrak{B}}(\mathfrak{i}_2)\} \geq N^n_{\mathfrak{B}}(\Omega).
\end{aligned}$$

And so $\mathfrak{i}_1 \in \mathfrak{H}(\Omega)$. Therefore, $\mathfrak{H}(\Omega)$ is an ideal of I.

Theorem 10. *Let* $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ *be a BPIFS of* I *and* $\Omega \in \mathrm{I}$.

(i) If $\mathfrak{H}(\Omega)$ *is an ideal of* I, *then* $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ *satisfies the following for all* $i_1, i_2, i_3 \in \mathrm{I}$,

$$\left(\begin{array}{c}
M^p_{\mathfrak{B}}(i_1) \leq \min\{M^p_{\mathfrak{B}}(i_2 * i_3), M^p_{\mathfrak{B}}(i_3)\} \Rightarrow M^p_{\mathfrak{B}}(i_1) \leq M^p_{\mathfrak{B}}(i_2)\\
M^n_{\mathfrak{B}}(i_1) \geq \max\{M^n_{\mathfrak{B}}(i_2 * i_3), M^n_{\mathfrak{B}}(i_3)\} \Rightarrow M^n_{\mathfrak{B}}(i_1) \geq M^n_{\mathfrak{B}}(i_2)\\
N^p_{\mathfrak{B}}(i_1) \geq \max\{N^p_{\mathfrak{B}}(i_2 * i_3), N^p_{\mathfrak{B}}(i_3)\} \Rightarrow N^p_{\mathfrak{B}}(i_1) \geq N^p_{\mathfrak{B}}(i_2)\\
N^n_{\mathfrak{B}}(i_1) \leq \min\{N^n_{\mathfrak{B}}(i_2 * i_3), N^n_{\mathfrak{B}}(i_3)\} \Rightarrow N^n_{\mathfrak{B}}(i_1) \leq N^n_{\mathfrak{B}}(i_2)
\end{array}\right) \tag{12}$$

(ii) If $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ *satisfies (BPIFI-1) and Equation* (12), *then* $\mathfrak{H}(\Omega)$ *is an ideal of* I.

Proof. (i) Assume that $\mathfrak{H}(\Omega)$ is an ideal of I for each $\Omega \in \mathrm{I}$. We suppose that for all $\mathfrak{i}_1, \mathfrak{i}_2, \mathfrak{i}_3 \in \mathrm{I}$,

$$\begin{aligned}
M^p_{\mathfrak{B}}(\mathfrak{i}_1) &\leq \min\{M^p_{\mathfrak{B}}(\mathfrak{i}_2 * \mathfrak{i}_3), M^p_{\mathfrak{B}}(\mathfrak{i}_3)\},\\
M^n_{\mathfrak{B}}(\mathfrak{i}_1) &\geq \max\{M^n_{\mathfrak{B}}(\mathfrak{i}_2 * \mathfrak{i}_3), M^n_{\mathfrak{B}}(\mathfrak{i}_3)\},\\
N^p_{\mathfrak{B}}(\mathfrak{i}_1) &\geq \max\{N^p_{\mathfrak{B}}(\mathfrak{i}_2 * \mathfrak{i}_3), N^p_{\mathfrak{B}}(\mathfrak{i}_3)\},\\
N^n_{\mathfrak{B}}(\mathfrak{i}_1) &\leq \min\{N^n_{\mathfrak{B}}(\mathfrak{i}_2 * \mathfrak{i}_3), N^n_{\mathfrak{B}}(\mathfrak{i}_3)\}.
\end{aligned}$$

Then $\mathfrak{i}_2 * \mathfrak{i}_3, \mathfrak{i}_3 \in \mathfrak{H}(\mathfrak{i}_1)$. Since $\mathfrak{H}(\mathfrak{i}_1)$ is an ideal of I, it follows that $\mathfrak{i}_2 \in \mathfrak{H}(\mathfrak{i}_1)$, that is, $M^p_{\mathfrak{B}}(\mathfrak{i}_2) \geq M^p_{\mathfrak{B}}(\mathfrak{i}_1)$, $M^n_{\mathfrak{B}}(\mathfrak{i}_2) \leq M^n_{\mathfrak{B}}(\mathfrak{i}_1)$, $N^p_{\mathfrak{B}}(\mathfrak{i}_2) \leq N^p_{\mathfrak{B}}(\mathfrak{i}_1)$, and $N^n_{\mathfrak{B}}(\mathfrak{i}_2) \geq N^n_{\mathfrak{B}}(\mathfrak{i}_1)$.

(ii) Suppose that a BPIFS $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ satisfies (BPIFI-1) and Equation (12). For each $\Omega \in \mathrm{I}$, let $\mathfrak{i}_1, \mathfrak{i}_2 \in \mathrm{I}$ be such that $\mathfrak{i}_1 * \mathfrak{i}_2$, $\mathfrak{i}_2 \in \mathfrak{H}(\Omega)$. Then $M^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2) \geq M^p_{\mathfrak{B}}(\Omega)$, $M^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2) \leq M^n_{\mathfrak{B}}(\Omega)$, $N^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2) \leq N^p_{\mathfrak{B}}(\Omega)$, $N^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2) \geq N^n_{\mathfrak{B}}(\Omega)$, $M^p_{\mathfrak{B}}(\mathfrak{i}_2) \geq M^p_{\mathfrak{B}}(\Omega)$, $M^n_{\mathfrak{B}}(\mathfrak{i}_2) \leq M^n_{\mathfrak{B}}(\Omega)$, $N^p_{\mathfrak{B}}(\mathfrak{i}_2) \leq N^p_{\mathfrak{B}}(\Omega)$, and $N^n_{\mathfrak{B}}(\mathfrak{i}_2) \geq N^n_{\mathfrak{B}}(\Omega)$. It follows that

$$\begin{aligned}
M^p_{\mathfrak{B}}(\Omega) &\leq \min\{M^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), M^p_{\mathfrak{B}}(\mathfrak{i}_2)\},\\
M^n_{\mathfrak{B}}(\Omega) &\geq \max\{M^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), M^n_{\mathfrak{B}}(\mathfrak{i}_2)\},\\
N^p_{\mathfrak{B}}(\Omega) &\geq \max\{N^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), N^p_{\mathfrak{B}}(\mathfrak{i}_2)\},
\end{aligned}$$

$$N^n_{\mathfrak{B}}(\Omega) \leq \min\{N^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2), N^n_{\mathfrak{B}}(\mathfrak{i}_2)\}.$$

Using Equation (12), we get $M^p_{\mathfrak{B}}(\Omega) \leq M^p_{\mathfrak{B}}(\mathfrak{i}_1)$, $(M_{\mathfrak{B}})^n(\Omega) \geq M^n_{\mathfrak{B}}(\mathfrak{i}_1)$, $N^p_{\mathfrak{B}}(\Omega) \geq N^p_{\mathfrak{B}}(\mathfrak{i}_1)$, and $N^n_{\mathfrak{B}}(\Omega) \leq N^n_{\mathfrak{B}}(\mathfrak{i}_1) \Rightarrow \mathfrak{i}_1 \in \mathfrak{H}(\Omega)$. Since $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ satisfies (BPIFI-1), it follows that $0 \in \mathfrak{H}(\Omega)$. Therefore, $\mathfrak{H}(\Omega)$ is an ideal of I.

Proposition 3. *For any BPIFI* $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ *of* I*, the following are equivalent*

(i) For all $\mathfrak{i}_1, \mathfrak{i}_2 \in \mathrm{I}$*,*

$$\begin{pmatrix} M^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2) \geq M^p_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_2) * \mathfrak{i}_2) \\ M^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2) \leq M^n_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_2) * \mathfrak{i}_2) \\ N^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2) \leq N^p_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_2) * \mathfrak{i}_2) \\ N^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2) \geq N^n_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_2) * \mathfrak{i}_2) \end{pmatrix} \tag{13}$$

(ii) For all $\mathfrak{i}_1, \mathfrak{i}_2, \mathfrak{i}_3 \in \mathrm{I}$*,*

$$\begin{pmatrix} M^p_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_3) * (\mathfrak{i}_2 * \mathfrak{i}_3)) \geq M^p_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_2) * \mathfrak{i}_3) \\ M^n_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_3) * (\mathfrak{i}_2 * \mathfrak{i}_3)) \leq M^n_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_2) * \mathfrak{i}_3) \\ N^p_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_3) * (\mathfrak{i}_2 * \mathfrak{i}_3)) \leq N^p_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_2) * \mathfrak{i}_3) \\ N^n_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_3) * (\mathfrak{i}_2 * \mathfrak{i}_3)) \geq N^n_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_2) * \mathfrak{i}_3) \end{pmatrix} \tag{14}$$

Proof. $(i) \Rightarrow (ii)$ For all $\mathfrak{i}_1, \mathfrak{i}_2, \mathfrak{i}_3 \in \mathrm{I}$, we have

$$\begin{aligned} (\mathfrak{i}_1 * (\mathfrak{i}_2 * \mathfrak{i}_3)) * \mathfrak{i}_3 &= (\mathfrak{i}_1 * \mathfrak{i}_3) * (\mathfrak{i}_2 * \mathfrak{i}_3) && \text{(By (3))} \\ &\leq (\mathfrak{i}_1 * \mathfrak{i}_2) && \text{(By (4))} \\ \Rightarrow ((\mathfrak{i}_1 * (\mathfrak{i}_2 * \mathfrak{i}_3)) * \mathfrak{i}_3) * \mathfrak{i}_3 &\leq (\mathfrak{i}_1 * \mathfrak{i}_2) * \mathfrak{i}_3. && \text{(By (6))} \end{aligned}$$

It follows from Theorem 2 that

$$\begin{pmatrix} M^p_{\mathfrak{B}}((\mathfrak{i}_1 * (\mathfrak{i}_2 * \mathfrak{i}_3)) * \mathfrak{i}_3) \geq M^p_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_2) * \mathfrak{i}_3) \\ M^n_{\mathfrak{B}}((\mathfrak{i}_1 * (\mathfrak{i}_2 * \mathfrak{i}_3)) * \mathfrak{i}_3) \leq M^n_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_2) * \mathfrak{i}_3) \\ N^p_{\mathfrak{B}}((\mathfrak{i}_1 * (\mathfrak{i}_2 * \mathfrak{i}_3)) * \mathfrak{i}_3) \leq N^p_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_2) * \mathfrak{i}_3) \\ N^n_{\mathfrak{B}}((\mathfrak{i}_1 * (\mathfrak{i}_2 * \mathfrak{i}_3)) * \mathfrak{i}_3) \geq N^n_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_2) * \mathfrak{i}_3) \end{pmatrix} \tag{15}$$

Now, using Equations (3), (13), and (15), we obtain

$$\begin{aligned} M^p_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_3) * (\mathfrak{i}_2 * \mathfrak{i}_3)) &= M^p_{\mathfrak{B}}((\mathfrak{i}_1 * (\mathfrak{i}_2 * \mathfrak{i}_3)) * \mathfrak{i}_3) \\ &\geq M^p_{\mathfrak{B}}(((\mathfrak{i}_1 * (\mathfrak{i}_2 * \mathfrak{i}_3)) * \mathfrak{i}_3) * \mathfrak{i}_3) \\ &\geq M^p_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_2) * \mathfrak{i}_3), \\ M^n_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_3) * (\mathfrak{i}_2 * \mathfrak{i}_3)) &= M^n_{\mathfrak{B}}((\mathfrak{i}_1 * (\mathfrak{i}_2 * \mathfrak{i}_3)) * \mathfrak{i}_3) \\ &\leq M^n_{\mathfrak{B}}(((\mathfrak{i}_1 * (\mathfrak{i}_2 * \mathfrak{i}_3)) * \mathfrak{i}_3) * \mathfrak{i}_3) \\ &\leq M^n_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_2) * \mathfrak{i}_3), \\ N^p_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_3) * (\mathfrak{i}_2 * \mathfrak{i}_3)) &= N^p_{\mathfrak{B}}((\mathfrak{i}_1 * (\mathfrak{i}_2 * \mathfrak{i}_3)) * \mathfrak{i}_3) \end{aligned}$$

$$\begin{aligned} &\leq N^p_{\mathfrak{B}}(((\mathfrak{i}_1 * (\mathfrak{i}_2 * \mathfrak{i}_3)) * \mathfrak{i}_3) * \mathfrak{i}_3) \\ &\leq N^p_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_2) * \mathfrak{i}_3), \\ N^n_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_3) * (\mathfrak{i}_2 * \mathfrak{i}_3)) &= N^n_{\mathfrak{B}}((\mathfrak{i}_1 * (\mathfrak{i}_2 * \mathfrak{i}_3)) * \mathfrak{i}_3) \\ &\geq N^n_{\mathfrak{B}}(((\mathfrak{i}_1 * (\mathfrak{i}_2 * \mathfrak{i}_3)) * \mathfrak{i}_3) * \mathfrak{i}_3) \\ &\geq N^n_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_2) * \mathfrak{i}_3). \end{aligned}$$

Thus (ii) holds in I.

$(ii) \Rightarrow (i)$ It follows from Definitions 1 and 3, (BCK-3), and Equation (14) that

$$\begin{aligned} M^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2) = M^p_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_2) * 0) = M^p_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_2) * (\mathfrak{i}_2 * \mathfrak{i}_2)) \leq M^p_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_2) * \mathfrak{i}_2), \\ M^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2) = M^n_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_2) * 0) = M^n_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_2) * (\mathfrak{i}_2 * \mathfrak{i}_2)) \geq M^n_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_2) * \mathfrak{i}_2), \\ N^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2) = N^p_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_2) * 0) = N^p_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_2) * (\mathfrak{i}_2 * \mathfrak{i}_2)) \geq N^p_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_2) * \mathfrak{i}_2), \\ N^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2) = N^n_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_2) * 0) = N^n_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_2) * (\mathfrak{i}_2 * \mathfrak{i}_2)) \leq N^n_{\mathfrak{B}}((\mathfrak{i}_1 * \mathfrak{i}_2) * \mathfrak{i}_2). \end{aligned}$$

Which proves (i).

Theorem 11. *If a BPIFS* $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ *of* I *is a BPIFI of* I*, then for all* $\mathfrak{i}_1, \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \ldots \mathfrak{v}_n \in \mathrm{I}$, $(\ldots((\mathfrak{i}_1 * \mathfrak{v}_1) * \mathfrak{v}_2) * \ldots) * \mathfrak{v}_n = 0 \Rightarrow$

$$\left(\begin{array}{c} M^p_{\mathfrak{B}}(\mathfrak{i}_1) \geq \min\{M^p_{\mathfrak{B}}(\mathfrak{v}_1), M^p_{\mathfrak{B}}(\mathfrak{v}_2), \ldots, M^p_{\mathfrak{B}}(\mathfrak{v}_n)\} \\ M^n_{\mathfrak{B}}(\mathfrak{i}_1) \leq \max\{M^n_{\mathfrak{B}}(\mathfrak{v}_1), M^n_{\mathfrak{B}}(\mathfrak{v}_2), \ldots, M^n_{\mathfrak{B}}(\mathfrak{v}_n)\} \\ N^p_{\mathfrak{B}}(\mathfrak{i}_1) \leq \max\{N^p_{\mathfrak{B}}(\mathfrak{v}_1), N^p_{\mathfrak{B}}(\mathfrak{v}_2), \ldots, N^p_{\mathfrak{B}}(\mathfrak{v}_n)\} \\ N^n_{\mathfrak{B}}(\mathfrak{i}_1) \geq \min\{N^n_{\mathfrak{B}}(\mathfrak{v}_1), N^n_{\mathfrak{B}}(\mathfrak{v}_2), \ldots, N^n_{\mathfrak{B}}(\mathfrak{v}_n)\} \end{array}\right) \tag{16}$$

Proof. The proof is by induction on n.

Let $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ be a BPIFI of I. Theorems 2 and 4 shows that Equation (16) is valid for $n = 1, 2$. We assume that Equation (16) is satisfied in $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ for $n = k$, that is for all $\mathfrak{i}_1, \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \ldots \mathfrak{v}_k \in \mathrm{I}$, $(\ldots((\mathfrak{i}_1 * \mathfrak{v}_1) * \mathfrak{v}_2) * \ldots) * \mathfrak{v}_k = 0 \Rightarrow$

$$\left(\begin{array}{c} M^p_{\mathfrak{B}}(\mathfrak{i}_1) \geq \min\{M^p_{\mathfrak{B}}(\mathfrak{v}_1), M^p_{\mathfrak{B}}(\mathfrak{v}_2), \ldots, M^p_{\mathfrak{B}}(\mathfrak{v}_k)\} \\ M^n_{\mathfrak{B}}(\mathfrak{i}_1) \leq \max\{M^n_{\mathfrak{B}}(\mathfrak{v}_1), M^n_{\mathfrak{B}}(\mathfrak{v}_2), \ldots, M^n_{\mathfrak{B}}(\mathfrak{v}_k)\} \\ N^p_{\mathfrak{B}}(\mathfrak{i}_1) \leq \max\{N^p_{\mathfrak{B}}(\mathfrak{v}_1), N^p_{\mathfrak{B}}(\mathfrak{v}_2), \ldots, N^p_{\mathfrak{B}}(\mathfrak{v}_k)\} \\ N^n_{\mathfrak{B}}(\mathfrak{i}_1) \geq \min\{N^n_{\mathfrak{B}}(\mathfrak{v}_1), N^n_{\mathfrak{B}}(\mathfrak{v}_2), \ldots, N^n_{\mathfrak{B}}(\mathfrak{v}_k)\} \end{array}\right) \tag{17}$$

Let $\mathfrak{i}_1, \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \ldots \mathfrak{v}_{k+1} \in \mathrm{I}$ be such that $(\ldots((\mathfrak{i}_1 * \mathfrak{v}_1) * \mathfrak{v}_2) * \ldots) * \mathfrak{v}_{k+1} = 0 \Rightarrow$

$$\left(\begin{array}{c} M^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{v}_1) \geq \min\{M^p_{\mathfrak{B}}(\mathfrak{v}_2), M^p_{\mathfrak{B}}(\mathfrak{v}_3), \ldots, M^p_{\mathfrak{B}}(\mathfrak{v}_{k+1})\} \\ M^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{v}_1) \leq \max\{M^n_{\mathfrak{B}}(\mathfrak{v}_2), M^n_{\mathfrak{B}}(\mathfrak{v}_3), \ldots, M^n_{\mathfrak{B}}(\mathfrak{v}_{k+1})\} \\ N^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{v}_1) \leq \max\{N^p_{\mathfrak{B}}(\mathfrak{v}_2), N^p_{\mathfrak{B}}(\mathfrak{v}_3), \ldots, N^p_{\mathfrak{B}}(\mathfrak{v}_{k+1})\} \\ N^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{v}_1) \geq \min\{N^n_{\mathfrak{B}}(\mathfrak{v}_2), N^n_{\mathfrak{B}}(\mathfrak{v}_3), \ldots, N^n_{\mathfrak{B}}(\mathfrak{v}_{k+1})\} \end{array}\right)$$

Since $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ is a BPIFI of I, we have

$$M^p_{\mathfrak{B}}(\mathfrak{i}_1) \geq \min\{M^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{v}_1), M^p_{\mathfrak{B}}(\mathfrak{v}_1)\}$$

$$\begin{aligned}
&\geq \min\{M^p_{\mathfrak{B}}(\mathfrak{v}_1), M^p_{\mathfrak{B}}(\mathfrak{v}_2), M^p_{\mathfrak{B}}(\mathfrak{v}_3), \ldots, M^p_{\mathfrak{B}}(\mathfrak{v}_{k+1})\},\\
M^n_{\mathfrak{B}}(\mathfrak{i}_1) &\leq \max\{M^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{v}_1), M^n_{\mathfrak{B}}(\mathfrak{v}_1)\}\\
&\leq \max\{M^n_{\mathfrak{B}}(\mathfrak{v}_1), M^n_{\mathfrak{B}}(\mathfrak{v}_2), M^n_{\mathfrak{B}}(\mathfrak{v}_3), \ldots, M^n_{\mathfrak{B}}(\mathfrak{v}_{k+1})\},\\
N^p_{\mathfrak{B}}(\mathfrak{i}_1) &\leq \max\{N^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{v}_1), N^p_{\mathfrak{B}}(\mathfrak{v}_1)\}\\
&\leq \max\{N^p_{\mathfrak{B}}(\mathfrak{v}_1), N^p_{\mathfrak{B}}(\mathfrak{v}_2), N^p_{\mathfrak{B}}(\mathfrak{v}_3), \ldots, N^p_{\mathfrak{B}}(\mathfrak{v}_{k+1})\},\\
N^n_{\mathfrak{B}}(\mathfrak{i}_1) &\geq \min\{N^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{v}_1), N^n_{\mathfrak{B}}(\mathfrak{v}_1)\}\\
&\geq \min\{N^n_{\mathfrak{B}}(\mathfrak{v}_1), N^n_{\mathfrak{B}}(\mathfrak{v}_2), N^n_{\mathfrak{B}}(\mathfrak{v}_3), \ldots, N^n_{\mathfrak{B}}(\mathfrak{v}_{k+1})\}.
\end{aligned}$$

This completes the proof.

Theorem 12. *Let* $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ *be a BPIFS of* I *satisfying Equation* (16). *Then* $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ *is a BPIFI of* I.

Proof. Suppose Equation (16) is valid in $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$. Put $\mathfrak{i}_1 = 0, \mathfrak{v}_1 = \mathfrak{i}_1, \mathfrak{v}_2 = \mathfrak{i}_1, \mathfrak{v}_3 = \mathfrak{i}_1, \ldots \mathfrak{v}_k = \mathfrak{i}_1$ in Equation (16), then $(\ldots((0 * \mathfrak{i}_1) * \mathfrak{i}_1) * \ldots) * \mathfrak{i}_1 = 0 \Rightarrow$

$$\begin{aligned}
M^p_{\mathfrak{B}}(0) &\geq \min\{M^p_{\mathfrak{B}}(\mathfrak{i}_1), M^p_{\mathfrak{B}}(\mathfrak{i}_1), \ldots, M^p_{\mathfrak{B}}(\mathfrak{i}_1)\},\\
M^n_{\mathfrak{B}}(0) &\leq \max\{M^n_{\mathfrak{B}}(\mathfrak{i}_1), M^n_{\mathfrak{B}}(\mathfrak{i}_1), \ldots, M^n_{\mathfrak{B}}(\mathfrak{i}_1)\},\\
N^p_{\mathfrak{B}}(0) &\leq \max\{N^p_{\mathfrak{B}}(\mathfrak{i}_1), N^p_{\mathfrak{B}}(\mathfrak{i}_1), \ldots, N^p_{\mathfrak{B}}(\mathfrak{i}_1)\},\\
N^n_{\mathfrak{B}}(0) &\geq \min\{N^n_{\mathfrak{B}}(\mathfrak{i}_1), N^n_{\mathfrak{B}}(\mathfrak{i}_1), \ldots, N^n_{\mathfrak{B}}(\mathfrak{i}_1)\}
\end{aligned}$$

$\Rightarrow M^p_{\mathfrak{B}}(0) \geq M^p_{\mathfrak{B}}(\mathfrak{i}_1)$, $M^n_{\mathfrak{B}}(0) \leq M^n_{\mathfrak{B}}(\mathfrak{i}_1)$, $N^p_{\mathfrak{B}}(0) \leq N^p_{\mathfrak{B}}(\mathfrak{i}_1)$, and $N^n_{\mathfrak{B}}(0) \geq N^n_{\mathfrak{B}}(\mathfrak{i}_1)$ for all $\mathfrak{i}_1 \in \mathrm{I}$. Again take $\mathfrak{i}_1 = \mathfrak{i}_1$, $\mathfrak{v}_1 = \mathfrak{i}_2$, $\mathfrak{v}_2 = \mathfrak{i}_3$, $\mathfrak{v}_3 = 0$, $\mathfrak{v}_4 = 0, \ldots, \mathfrak{v}_k = 0$ in Equation (16). Then, $(\ldots(((\mathfrak{i}_1 * \mathfrak{i}_2) * \mathfrak{i}_3) * 0)\ldots) * 0 = 0$. Therefore,

$$\begin{aligned}
M^p_{\mathfrak{B}}(\mathfrak{i}_1) &\geq \min\{M^p_{\mathfrak{B}}(\mathfrak{i}_2), M^p_{\mathfrak{B}}(\mathfrak{i}_3), M^p_{\mathfrak{B}}(0)\}\\
&= \min\{M^p_{\mathfrak{B}}(\mathfrak{i}_2), M^p_{\mathfrak{B}}(\mathfrak{i}_3)\},\\
M^n_{\mathfrak{B}}(\mathfrak{i}_1) &\leq \max\{M^n_{\mathfrak{B}}(\mathfrak{i}_2), M^n_{\mathfrak{B}}(\mathfrak{i}_3), M^n_{\mathfrak{B}}(0)\}\\
&= \max\{M^n_{\mathfrak{B}}(\mathfrak{i}_2), M^n_{\mathfrak{B}}(\mathfrak{i}_3)\},\\
N^p_{\mathfrak{B}}(\mathfrak{i}_1) &\leq \max\{N^p_{\mathfrak{B}}(\mathfrak{i}_2), N^p_{\mathfrak{B}}(\mathfrak{i}_3), N^p_{\mathfrak{B}}(0)\}\\
&= \max\{N^p_{\mathfrak{B}}(\mathfrak{i}_2), N^p_{\mathfrak{B}}(\mathfrak{i}_3)\},\\
N^n_{\mathfrak{B}}(\mathfrak{i}_1) &\geq \min\{N^n_{\mathfrak{B}}(\mathfrak{i}_2), N^n_{\mathfrak{B}}(\mathfrak{i}_3), N^n_{\mathfrak{B}}(0)\}\\
&= \min\{N^n_{\mathfrak{B}}(\mathfrak{i}_2), N^n_{\mathfrak{B}}(\mathfrak{i}_3)\}.
\end{aligned}$$

Hence, by Theorem 4, we conclude that $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ is a BPIFI of I.

Theorem 13. *If* $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ *is a BPIFI of* I, *then for all* $i_1, i_2, i_3 \in \mathrm{I}$,

$$\begin{pmatrix}
M^p_{\mathfrak{B}}(i_1 * i_2) \geq \min\{M^p_{\mathfrak{B}}(i_1 * i_3), M^p_{\mathfrak{B}}(i_3 * i_2)\}\\
M^n_{\mathfrak{B}}(i_1 * i_2) \leq \max\{M^n_{\mathfrak{B}}(i_1 * i_3), M^n_{\mathfrak{B}}(i_3 * i_2)\}\\
N^p_{\mathfrak{B}}(i_1 * i_2) \leq \max\{N^p_{\mathfrak{B}}(i_1 * i_3), N^p_{\mathfrak{B}}(i_3 * i_2)\}\\
N^n_{\mathfrak{B}}(i_1 * i_2) \geq \min\{N^n_{\mathfrak{B}}(i_1 * i_3), N^n_{\mathfrak{B}}(i_3 * i_2)\}
\end{pmatrix}. \tag{18}$$

Proof. Suppose $\mathfrak{B} = (M^p_{\mathfrak{B}}, M^n_{\mathfrak{B}}, N^p_{\mathfrak{B}}, N^n_{\mathfrak{B}})$ is a BPIFI of I, then we have $(\mathfrak{i}_1 * \mathfrak{i}_2) * (\mathfrak{i}_1 * \mathfrak{i}_3) \leq (\mathfrak{i}_3 * \mathfrak{i}_2)$ by (BCK-1). According to Theorem 4, it follows that for all $\mathfrak{i}_1, \mathfrak{i}_2, \mathfrak{i}_3 \in \mathrm{I}$,

$$\begin{aligned}
M^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2) &\geq \min\left\{M^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_3), M^p_{\mathfrak{B}}(\mathfrak{i}_3 * \mathfrak{i}_2)\right\}, \\
M^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2) &\leq \max\left\{M^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_3), M^n_{\mathfrak{B}}(\mathfrak{i}_3 * \mathfrak{i}_2)\right\}, \\
N^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2) &\leq \max\left\{N^p_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_3), N^p_{\mathfrak{B}}(\mathfrak{i}_3 * \mathfrak{i}_2)\right\}, \\
N^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_2) &\geq \min\left\{N^n_{\mathfrak{B}}(\mathfrak{i}_1 * \mathfrak{i}_3), N^n_{\mathfrak{B}}(\mathfrak{i}_3 * \mathfrak{i}_2)\right\}.
\end{aligned}$$

This completes the proof.

4. Conclusion

In this study, a novel framework for integrating bipolar valued intuitionistic fuzzy set theory into BCK-algebras has been developed. The introduction of bipolar intuitionistic fuzzy subalgebras (BPIFSAs) and bipolar intuitionistic fuzzy ideals (BPIFIs) expands the scope of research in this area. It offers valuable insights into the structure and behavior of algebraic systems under bipolar and intuitionistic uncertainty. The investigation of these properties opens further directions for extending the theory by combining BPIFS concepts with algebraic operations. In our future work, we intend to explore topics such as bipolar intuitionistic fuzzy soft ideals and the structure of $(\in, \in \vee q)$-bipolar intuitionistic fuzzy ideals, as well as their interrelationships.

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