EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2026, Vol. 19, Issue 2, Article Number 7195
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

New Proof of the Wang-Sun Identity with Applications to Super Catalan Numbers

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Abstract. In this work, we establish an identity originally due to Wang and Sun by leveraging a classical hypergeometric series identity of Bailey. As an application, we derive new identities for the Catalan numbers and introduce novel identities for the Super Catalan numbers involving the Wallis ratio. Furthermore, we present new formulas related to Touchard's identity and Koshy's identity. Finally, we provide integral representations that stem from the Wang-Sun identity.

2020 Mathematics Subject Classifications: 05A05, 05A10, 33C05, 05A19
Key Words and Phrases: Combinatorial identities, generalized hypergeometric function, terminating series, Catalan numbers, Super Catalan numbers

1. Introduction

Let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers respectively and z be a complex variable. For complex parameters a and b, define the generalized binomial coefficient as

$$\binom{a}{b} = \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)} = \binom{a}{a-b} \quad (a, b \in \mathbb{C}), \tag{1}$$

in which

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx, \quad (\mathbb{R}(z) > 0),$$

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DOI: https://doi.org/10.29020/nybg.ejpam.v19i2.7195

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denotes the well-known gamma function.
Diaz and Pariguan [[1], p. 180] introduced the Pochhammer k-symbol as

$$(x)_{n,k} = x(x+k)(x+2k)...(x+(n-1)k), \quad k \in \mathbb{R}, \quad n \geq 0. \tag{2}$$

In particular, we have [2, 3],

$$(x)_{n,1} = (x)_n = \begin{cases} x(x+1)(x+2)...(x+n-1) & , n \geq 1, \\ 1 & , n = 0. \end{cases}$$

The relation (1) can be reduced to the particular case

$$\binom{a}{n} = \frac{(-1)^n(-a)_n}{n!} \tag{3}$$

Further, in terms of Pochhammer symbol, the generalized hypergeometric series with p numerator and q denominator parameters is defined [4] by

$${}_pF_q\left[\begin{array}{c} a_1, ..., a_p \\ b_1, ..., b_q \end{array} ; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n...(a_p)_n}{(b_1)_n...(b_q)_n} \frac{z^n}{n!}. \tag{4}$$

Here p and q are positive integers or zero (interpreting any empty product as 1), and we assume that the variable z of the series, the numerator parameters $a_1, ..., a_p$ and the denominator parameters $b_1, ..., b_q$ take on complex values, provided that b_j for $j = 1, 2, .., q$ should not be zero or a negative integer. Further, assuming that none of the numerator parameters is zero or a negative integer (otherwise the question of convergence will not arise), and with the restriction that b_j $(j = 0, 1, ..., q)$ should not be zero or a negative integer, by well-known ratio tests, it can easily be seen that the generalized hypergeometric series defined in (4):
(i) convergent for all $|z| < \infty$ provided $p \leq q$,
(ii) converges for all $|z| < 1$ provided $p = q + 1$,
and **(iii)** diverges for all z, $z \neq 0$ provided $p > q + 1$.
Furthermore, if we set

$$w = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j,$$

then with the relation $p = q + 1$, the series (4) is seen to be
(i) absolutely convergent for $|z| = 1$ provided $\mathbb{R}(w) > 0$,
(ii) conditionally convergent for $|z| = 1$, $z \neq 1$, provided $-1 < \mathbb{R}(w) \leq 0$,
and **(iii)** divergent for $|z| = 1$ provided $\mathbb{R}(w) \leq -1$.
Next, we mention a curious identity due to Simons [5] written here in an equivalent form viz.

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} x^k = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} (1+x)^k \tag{5}$$

He established the identity (5) using repeated differentiation. For several different and interesting proofs of (5), we refer research papers by Hirschhorn [6] and Munarini [7].

In 2007, Wang and Sun [8] derived Simons identity (5) using operator method in a very simple manner.

In the same paper, they have also established the following known identity viz.

$$\sum_{k=0}^{n}\binom{n}{k}^2 x^k = \sum_{k=0}^{n}\binom{n}{k}\binom{2n-k}{n}(x-1)^k \tag{6}$$

which is recorded, for example, in the Stanley's book [9].

It is interesting to mention here that after replacing x by $-x$ in (6) and then applying the operator method to each side of (6), they have obtained the following result:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3 = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n-k}{n}\binom{n+k}{n}. \tag{7}$$

In addition to this, the well-known Dixon's identity [10] states that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3 = \begin{cases} (-1)^m\dfrac{(3m)!}{(m!)^3} & , for\ n=2m \quad \text{(even)} \\ 0 & , for\ n=2m+1 \quad \text{(odd)}. \end{cases} \tag{8}$$

Finally, upon equating (7) and (8), they have obtained the following beautiful identity viz.

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n-k}{n}\binom{n+k}{k} = \begin{cases} (-1)^m\dfrac{(3m)!}{(m!)^3} & , for\ n=2m \quad \text{(even)} \\ 0 & , for\ n=2m+1 \quad \text{(odd)}. \end{cases} \tag{9}$$

On the other hand, in combinatorial mathematics, the Catalan numbers are a sequence of natural numbers that occur in various counting problems, often involving recursively defined objects. The nth Catalan number can be expressed in terms of the central binomial coefficients defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!}, \quad for \quad n \geq 0. \tag{10}$$

In [[11], relation (1.2)], the Catalan numbers can be represented in terms of the hypergeometric function by

$${}_2F_1\left[\begin{matrix} -n, -n+1 \\ 2 \end{matrix} ; 1\right], \quad for \quad n \geq 0. \tag{11}$$

It is not out of place to mention here that if we use the classical Gauss's summation theorem [3] viz.

$${}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix} ; 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad (\mathbb{R}(c-a-b)>0) \tag{12}$$

in (11), we immediately recover (10).
Recently, an alternatively and analytical generalization of the Catalan numbers C_n was introduced in the literature and now popularly known as the Catalan–Qi numbers $C(a, b; n)$ which is defined by

$$C(a, b; n) = \left(\frac{b}{a}\right)^n \frac{(a)_n}{(b)_n}, \quad n \geq 0 \tag{13}$$

for $(a) > 0$ and $\mathbb{R}(b) > 0$.
It is not difficult to verify that

$$C\left(\frac{1}{2}, 2; n\right) = C_n. \tag{14}$$

For a comprehensive study, we refer to Koshy's standard text [12] and related research papers [2, 13].
It is interesting to mention here that the generalization of the Catalan numbers known as the Super Catalan numbers is defined in [14] as follows:

$$S_{p,n} = \frac{(2p)!(2n)!}{p!n!(p+n)!}, \quad p, n \geq 0. \tag{15}$$

In particular, we have

$$S_{0,n} = \frac{(2n)!}{(n!)^2}, \quad n \geq 0, \tag{16}$$

$$S_{1,n} = 2\frac{(2n)!}{n!(n+1)!}, \quad n \geq 0, \tag{17}$$

$$S_{n,n} = \frac{(2n)!}{(n!)^2}, \quad n \geq 0, \tag{18}$$

$$S_{p,n} = S_{n,p}, \quad p, n \geq 0. \tag{19}$$

It is clear that

$$S_{1,n} = 2C_n, \quad n \geq 0. \tag{20}$$

It is well known that the Wallis ratio is defined by

$$W_n = \frac{(2n-1)!!}{(2n)!!} = \frac{(2n)!}{2^{2n}(n!)^2} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(n+1/2)}{\Gamma(n+1)}, \; n \geq 0. \tag{21}$$

Hence, it is easy to see that

$$C_n = \frac{4^n}{n+1} W_n, \; n \geq 0. \tag{22}$$

The Wallis ratio has been studied and applied by many mathematicians and plenty of literature therein.
The aim of this paper is to establish the identity (9) via a hypergeometric series approach.
For this, we need the following known identity due to Bailey [4] viz.

$${}_3F_2\left[\begin{matrix} a, b, -n \\ \frac{1}{2}(a+b+1), -2n \end{matrix}; 1\right] = \frac{(\frac{1}{2}a+\frac{1}{2})_n(\frac{1}{2}b+\frac{1}{2})_n}{(\frac{1}{2})_n(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})_n}. \tag{23}$$

Also, we present new identities for the Catalan numbers by employing the relation (10). As an application, we introduce a new identities for the Super Catalan numbers. Similarly, we obtain new result for the Wallis ratio Moreover, we provide new formulas involving the Touchard's identity and Koshy's identity. Finally, we give new integral representations for the identity (9).

2. Derivation of (9)

In this section, we shall derive the identity due to Wang and Sun (9) via a hypergeometric series approach.
In order to derive the identity (9), we proceed as follows. Denote the left-hand side of (9) by S, we have

$$S = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n-k}{n} \binom{n+k}{n}$$

using the definition of Binomial Coefficient (1), it is not difficult to see that

(i) $\binom{n}{k} = \dfrac{(-1)^k(-n)_k}{k!}$,

(ii) $\binom{n+k}{n} = \dfrac{(n+1)_k}{(1)_k}$

(iii) $\binom{2n-k}{n} = \dfrac{\Gamma(2n+1)}{\Gamma^2(n+1)} \dfrac{(-n)_k}{(-2n)_k}$,

and

(iv) $\Gamma(\alpha - n) = \dfrac{(-1)^n \Gamma(\alpha)}{(1-\alpha)_n}$

we have, after some simplification

$$S = \frac{\Gamma(2n+1)}{\Gamma^2(n+1)} \sum_{k=0}^{n} \frac{(-n)_k(-n)_k(n+1)_k}{(1)_k(-2n)_k k!}.$$

Finally, summing up the series, we have

$$S = \frac{\Gamma(2n+1)}{\Gamma^2(n+1)} \, {}_3F_2 \left[\begin{array}{c} -n, n+1, -n \\ 1, -2n \end{array} ; 1 \right]$$

We now observe that the ${}_3F_2$ appearing can be evaluated with the help of Bailey's identity (23) by letting $a = -n$ and $b = n+1$, we have

$$S = \frac{\Gamma(2n+1)}{\Gamma^2(n+1)} \frac{(-\frac{1}{2}n + \frac{1}{2})_n (\frac{1}{2}n + 1)_n}{(1)_n (\frac{1}{2})_n}.$$

after some algebra, we get

$$S = \frac{(2n)!}{(n!)^3} \frac{(1-n)_{n,2}(n+2)_{n,2}}{2^{2n}} \frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}+n)}$$

Then, using the formula

$$\frac{(2n)!}{n!} = \frac{4^n \Gamma(n+\frac{1}{2})}{\sqrt{\pi}},$$

we obtain set

$$S = \frac{(1-n)_{n,2}(n+2)_{n,2}}{(n!)^2}.$$

The right-hand side of (9) now follows by considering the even and odd n. This completes the proof of the identity (9) due to Wang and Sun.

3. New identity for the Catalan numbers

In this section, we establish new identity for the Catalan numbers asserted in the following theorem.

Theorem 1. *The following result for the Catalan numbers holds true.*

$$\frac{(n+1)(1-n)_{n,2}(n+2)_{n,2}}{(2n)!} C_n = \begin{cases} (-1)^m \dfrac{(3m)!}{(m!)^3} & , for \ n = 2m \quad (even) \\ 0 & , for \ n = 2m+1 \quad (odd). \end{cases} \tag{24}$$

Proof. Using the formula for the Catalan numbers

$$C_n = \frac{(2n)!}{n!(n+1)!},$$

we have

$$\begin{aligned} \frac{(1-n)_{n,2}(n+2)_{n,2}}{(n!)^2} &= \frac{(n+1)}{(n+1)} \frac{(1-n)_{n,2}(n+2)_{n,2}}{n!n!} \\ &= \frac{(n+1)(1-n)_{n,2}(n+2)_{n,2}}{n!(n+1)!} \\ &= \frac{(n+1)(1-n)_{n,2}(n+2)_{n,2}}{n!(n+1)!} \frac{(2n)!}{(2n)!} \\ &= \frac{(n+1)(1-n)_{n,2}(n+2)_{n,2}}{(2n)!} \frac{(2n)!}{(n+1)!n!} \\ &= \frac{(n+1)(1-n)_{n,2}(n+2)_{n,2}}{(2n)!} C_n. \end{aligned}$$

Then

$$\frac{(n+1)(1-n)_{n,2}(n+2)_{n,2}}{(2n)!} C_n = \begin{cases} (-1)^m \dfrac{(3m)!}{(m!)^3} & , for \ n = 2m \quad \text{(even)} \\ 0 & , for \ n = 2m+1 \quad \text{(odd)}. \end{cases}$$

The proof of Theorem 1 is complete.

Corollary 1. *The following results for the Super Catalan numbers holds true.*

$$\frac{(1-n)_{n,2}(n+2)_{n,2}}{2(2n)!}S_{n,n} = \begin{cases} (-1)^m \dfrac{(3m)!}{(m!)^3} & , for\ n = 2m \quad (even) \\ 0 & , for\ n = 2m+1 \quad (odd) \end{cases} \tag{25}$$

and

$$\frac{(n+1)(1-n)_{n,2}(n+2)_{n,2}}{2(2n)!}S_{1,n} = \begin{cases} (-1)^m \dfrac{(3m)!}{(m!)^3} & , for\ n = 2m \quad (even) \\ 0 & , for\ n = 2m+1 \quad (odd). \end{cases} \tag{26}$$

Proof. Using the formula for the Super Catalan numbers with $p = n \geq 0$

$$S_{n,n} = \frac{(2n)!}{(n!)^2},$$

we have

$$\begin{aligned} \frac{(n+1)(1-n)_{n,2}(n+2)_{n,2}}{(2n)!}C_n &= \frac{(n+1)(1-n)_{n,2}(n+2)_{n,2}}{(2n)!}\frac{(2n)!}{(n+1)!n!} \\ &= \frac{(1-n)_{n,2}(n+2)_{n,2}}{(2n)!}\frac{(2n)!}{n!n!} \\ &= \frac{(1-n)_{n,2}(n+2)_{n,2}}{(2n)!}S_{n,n}. \end{aligned}$$

Then, we obtain the result (25)

$$\frac{(1-n)_{n,2}(n+2)_{n,2}}{(2n)!}S_{n,n} = \begin{cases} (-1)^m \dfrac{(3m)!}{(m!)^3} & , for\ n = 2m \quad \text{(even)} \\ 0 & , for\ n = 2m+1 \quad \text{(odd)}. \end{cases}$$

Now, we applying equation (20) to the relation (24), we obtain the result (26). The required proof is complete.

Corollary 2. *The following result for the Wallis ratio holds true.*

$$\frac{4^n(1-n)_{n,2}(n+2)_{n,2}}{(2n)!}W_n = \begin{cases} (-1)^m \dfrac{(3m)!}{(m!)^3} & , for\ n = 2m \quad (even) \\ 0 & , for\ n = 2m+1 \quad (odd). \end{cases} \tag{27}$$

Proof. Using equation (22) in the relation (24), we obtain the result (27). The proof of the Corollary 2 is complete.

Corollary 3. *The following result for the hypergeometric function holds true.*

$$\frac{(n+1)(1-n)_{n,2}(n+2)_{n,2}}{(2n)!}\ {}_2F_1\left[\begin{matrix}-n,-n+1\\2\end{matrix};1\right] = \begin{cases}(-1)^m\dfrac{(3m)!}{(m!)^3} & ,for\ n=2m\quad (even)\\ 0 & ,for\ n=2m+1\quad (odd).\end{cases} \tag{28}$$

Proof. Applying relation (11) to the equation (24), we have the relation (28).

In [15], Qi et al. discovered that ${}_2F_1\left[\begin{matrix}a,1\\b\end{matrix};\frac{bt}{a}\right]$ is a generating function for the Catalan–Qi numbers $C(a,b;n)$.

Theorem 2. *([15, Theorem 10]) For $a,b>0$ and $n\geq 0$, the Catalan–Qi numbers $C(a,b;n)$ can be generated by*

$${}_2F_1\left[\begin{matrix}a,1\\b\end{matrix};\frac{bt}{a}\right] = \sum_{n=0}^{\infty} C(a,b;n)t^n \tag{29}$$

and satisfy

$$C(a,b;n) = (-1)^n\sum_{k=0}^{n}(-1)^k\binom{n}{k}\ {}_2F_1\left[\begin{matrix}a,-k\\b\end{matrix};-\frac{b}{a}\right]. \tag{30}$$

Corollary 4. *The following result for the Catalan numbers holds true.*

$$\frac{(-1)^n(n+1)(1-n)_{n,2}(n+2)_{n,2}}{(2n)!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\ {}_2F_1\left[\begin{matrix}\frac{1}{2},-k\\2\end{matrix};-4\right] = \begin{cases}(-1)^m\dfrac{(3m)!}{(m!)^3} & ,for\ n=2m\quad (even)\\ 0 & ,for\ n=2m+1\quad (odd).\end{cases} \tag{31}$$

Proof. If $a=\frac{1}{2}$ and $b=2$ in (30) and using (14) we obtain

$$C\left(\frac{1}{2},2;n\right) = C_n = (-1)^n\sum_{k=0}^{n}(-1)^k\binom{n}{k}\ {}_2F_1\left[\begin{matrix}\frac{1}{2},-k\\2\end{matrix};-4\right]$$

Then, we substitute this result into the left-hand side of the equation (24), we obtain the result (31)

Now, we will present two important mathematical identities: Touchard's identity and Koshy's identity [12]. Touchard's identity is a powerful tool in analyzing algebraic structures, enabling precise expression of coefficients. On the other hand, Koshy's identity is used in coefficient calculations and analysis of sets. Touchard's identity can be expressed as:

$$C_{n+1} = \sum_{r=0}^{n} 2^{n-2r} \binom{n}{2r} C_r, \tag{32}$$

Koshy's identity expresses the following relationship:

$$C_n = \sum_{r=1}^{n} (-1)^{r-1} \binom{n-r+1}{r} C_{n-r}. \tag{33}$$

These two identities are considered cornerstones in mathematical literature due to their broad applications in many areas of mathematics.

Theorem 3. *The following result for the Touchard's identity holds true.*

$$\begin{aligned} &\frac{(n+1)(n+2)(1-n)_{n,2}(n+2)_{n,2}}{(2n+1)!} \sum_{r=0}^{n} 2^{n-2r-1} \binom{n}{2r} C_r \\ &\qquad = \begin{cases} (-1)^m \dfrac{(3m)!}{(m!)^3} & , for\ n = 2m \quad (even) \\ 0 & , for\ n = 2m+1 \quad (odd). \end{cases} \end{aligned} \tag{34}$$

Proof. Using the following recurrence relation for the Catalan numbers:

$$C_0 = 1 \quad and \quad C_{n+1} = \frac{2(2n+1)}{n+2} C_n, \quad n \geq 0,$$

we have

$$\begin{aligned} \frac{(n+1)(1-n)_{n,2}(n+2)_{n,2}}{(2n)!} C_n &= \frac{(n+1)(1-n)_{n,2}(n+2)_{n,2}}{(2n)!} \frac{(n+2)}{2(2n+1)} C_{n+1} \\ &= \frac{(n+1)(n+2)(1-n)_{n,2}(n+2)_{n,2}}{2(2n+1)!} C_{n+1} \\ &= \frac{(n+1)(n+2)(1-n)_{n,2}(n+2)_{n,2}}{2(2n+1)!} \sum_{r=0}^{n} 2^{n-2r} \binom{n}{2r} C_r \\ &= \frac{(n+1)(n+2)(1-n)_{n,2}(n+2)_{n,2}}{(2n+1)!} \sum_{r=0}^{n} 2^{n-2r-1} \binom{n}{2r} C_r. \end{aligned}$$

Now, we substitute this result into the left-hand side of the equation (24), we obtain the result (34)

Theorem 4. *The following result for the Koshy's identity holds true.*

$$\begin{aligned} &\frac{(n+1)(1-n)_{n,2}(n+2)_{n,2}}{(2n)!} \sum_{r=1}^{n} (-1)^{r-1} \binom{n-r+1}{r} C_{n-r} \\ &\qquad = \begin{cases} (-1)^m \dfrac{(3m)!}{(m!)^3} & , for\ n = 2m \quad (even) \\ 0 & , for\ n = 2m+1 \quad (odd). \end{cases} \end{aligned} \tag{35}$$

Proof. By applying Koshy's identity (33), to the right-hand side of (24), we obtain the result.

4. Integral representations involving the identity due to Wang and Sun

In this section, we establish some integral representations involving the identity due to Wang and Sun.

Theorem 5. *[[11], Theorem 1.4] The Catalan numbers C_n for $n \geq 0$, can be represented by:*

$$C_n = \frac{1}{\pi}\int_0^\infty \frac{\sqrt{x}}{(x+\frac{1}{4})^{n+2}}dx = \frac{2}{\pi}\int_0^\infty \frac{x^2}{(x^2+\frac{1}{4})^{n+2}}dx. \tag{36}$$

Corollary 5. *The following results holds true.*

$$\frac{(n+1)(1-n)_{n,2}(n+2)_{n,2}}{(2n)!\pi}\int_0^\infty \frac{\sqrt{x}}{(x+\frac{1}{4})^{n+2}}dx = \begin{cases} (-1)^m\dfrac{(3m)!}{(m!)^3} & , for\ n=2m \quad (even) \\ 0 & , for\ n=2m+1 \quad (odd) \end{cases} \tag{37}$$

and

$$\frac{2(n+1)(1-n)_{n,2}(n+2)_{n,2}}{(2n)!\pi}\int_0^\infty \frac{x^2}{(x^2+\frac{1}{4})^{n+2}}dx = \begin{cases} (-1)^m\dfrac{(3m)!}{(m!)^3} & , for\ n=2m \quad (even) \\ 0 & , for\ n=2m+1 \quad (odd). \end{cases} \tag{38}$$

Proof. By applying the integral representations for the Catalan numbers from Theorem 5, to the left-hand side of (24), we obtain the results (37) and (38).

Theorem 6. *[[16], Proposition 4.1] For $n \geq 0$, the Catalan numbers C_n can be represented by:*

$$C_n = \frac{2^{2n+5}}{\pi}\int_0^1 \frac{x^2(1-x^2)^{2n}}{(1+x^2)^{2n+3}}dx. \tag{39}$$

Corollary 6. *The following result holds true.*

$$\frac{2^{2n+5}(n+1)(1-n)_{n,2}(n+2)_{n,2}}{(2n)!\pi}\int_0^1 \frac{x^2(1-x^2)^{2n}}{(1+x^2)^{2n+3}}dx = \begin{cases} (-1)^m\dfrac{(3m)!}{(m!)^3} & , for\ n=2m \quad (even) \\ 0 & , for\ n=2m+1 \quad (odd). \end{cases} \tag{40}$$

Proof. Using the integral representations for the Catalan numbers in Theorem 6, on the left-hand side of (24), we obtain the result (40).

Concluding Remark: In this paper, we establish a fundamental identity originally derived by Wang and Sun through the application of a hypergeometric series transformation attributed to Bailey. This result not only highlights the deep interplay between hypergeometric series and combinatorial structures but also provides a novel perspective on summation techniques and special function identities. Additionally, we derive new combinatorial and analytical identities for the Catalan numbers, which play a crucial role in various branches of discrete mathematics. As an application, we extend our findings to obtain new expressions for the Super Catalan numbers, which generalize the classical Catalan numbers and frequently arise in enumerative combinatorics and representation theory. Furthermore, we explore their connection to the Wallis ratio. Moreover, we establish novel algebraic and combinatorial formulations involving Touchard's identity and Koshy's identity. Finally, we derive integral representations associated with the identity of Wang and Sun.

Acknowledgements

The authors extends the appreciation to the Deanship of Postgraduate Studies and Scientific Research at Majmaah University for funding this research work through the project number(R-2026-219).

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