EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2026, Vol. 19, Issue 2, Article Number 7198
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

The Theory and Implementation of the Sehgal Fixed Point Theorem in an Expanded G-Metric Space Using Banach's Contraction Principle

Elhadi Dalam

Department of Mathematics, Faculty of Science, Al-Baha University, SaudiArabia

Abstract. The present paper establishes fixed-point results in extended G-metric spaces byleveraging the Banach contraction principle and Sehgal-type iterate conditions. Motivated by recent developments, we prove a genuine Sehgal–Gusman type theoremwhere the contractive iterate depends on the pointwise structure. Furthermore, weintroduce a generalized contraction condition incorporating the extension function $\Theta(x; y; z)$,ensuring the results are nontrivial extensions of standard G-metric theory. To validate our theoretical findings, we provide a concrete numerical exampleand an application to nonlinear integral equations. Our results enrich the theoreticalstructure of functional analysis and topology, as well as providing more applicationsin dynamical systems and optimization problems.

2020 Mathematics Subject Classifications: 47B38, 47H10

Key Words and Phrases: G-metric space, Banach space, fixed point theorem, operators.

1. Introduction

Proving the existence and uniqueness of solutions for several issues in applied sciences,physics, and mathematics necessitates rigorous formulations that encompass multiple theories. By proving the contraction principle, which holds that every contraction mappingon a complete metric space has a unique fixed point, Banach [1] founded the disciplineof fixed-point theory in 1922. Sehgal [2] presented a novel kind of mapping that employscontractive iterates at each location, acting as a generalization of the Banach contractionprinciple. A multitude of academics have generalized this discovery to more contexts andmore extensive metric spaces (see [3–6]). Many important works have investigatedthe characteristics of fixed points in diverse environments, including 2-metric spaces [7, 8]. Current advancements in fixed-point theory focus on enhancing the metric structure of spaces to overcome the limits of classical metrics (see, e.g.,[9] for a recent survey). In 2017, Kamran et al. presented 1a notably intriguing extension of the b-metric

DOI: https://doi.org/10.29020/nybg.ejpam.v19i2.7198

Email address: adalam@bu.edu.sa (E. Dalam)

space concept, referred to as the extendedb-metric space [10]. Subsequently, many scholars have meticulously analyzed fixed pointtheory under these novel situations. Readers seeking further information may consult references [11–18]. Z. Mustafa and B. Sims were the earliest pioneers ofthe G-metric space [19]. For further foundational studies, see [19].

A cohort of academics subsequently employed, refined, and expanded this innovative metric space [20–25], with recent developments in extended metric structures summarized in [20]. Section 1 reviews previous studies on b, G, and 2-metric space; Section 2 discusses important preliminaries and definitions; Section 3 presents the main results of the study;Section 4 examines applications and core motivations.

2. Preliminaries

Definition 1 (Extended G-metric space)**.** *Let X be a non-empty set and $G : X \times X \times X \to [0, \infty)$ be a function satisfying:*

(i) $G(x, y, z) = 0$ *if and only if* $x = y = z$*;*

(ii) $G(x, x, y) > 0$ *for all* $x, y \in X$ *with* $x \neq y$*;*

(iii) $G(x, y, z) = G(x, z, y) = G(y, z, x)$ *(symmetry);*

(iv) $G(x, y, z) \leq \Theta(x, y, z)\,[G(x, a, a) + G(a, y, z)]$ *for all* $x, y, z, a \in X$*.*

Then the pair (X, G) is called an extended G-metric space.

Remark 1. *If the extended condition in Definition 1 is replaced by the standard one (i.e., $\Theta(x, y, z) \equiv 1$), we recover the usual definition of a G-metric space. In our subsequent theorems, we explicitly utilize Θ in the contractive conditions to ensure the results are nontrivial generalizations. Specifically, we require conditions where*

$$\alpha \cdot \sup_{x,y,z} \Theta(x, y, z) < 1$$

to guarantee convergence.

Theorem 1. *Let (X, G) be a complete G-metric space and $T : X \to X$ a continuous mapping [26]. Suppose there exists a function $\varphi : [0, 1) \to [0, 1)$ such that*

$$G(Tx, Ty, Tz) \leq \varphi(G(x, y, z)), \quad \forall x, y, z \in X,$$

and $\varphi(t) < t$ for all $t > 0$. Then T possesses a unique fixed point $x^* \in X$ with $Tx^* = x^*$.

Theorem 2. *Let (X, G) be a complete G-metric space and $T : X \to X$ satisfy*

$$G(Tx, Ty, Tz) \leq \alpha G(x, y, z), \quad \forall x, y, z \in X,$$

for some constant $\alpha \in (0, 1)$. Then T has a unique fixed point in X [12].

Theorem 3. *Let (X, G) be a complete G-metric space and $T : X \to X$ be a mapping satisfying*

$$G(Tx, Ty, Tz) \leq \alpha\big[G(x, y, z) + G(Tx, x, x) + G(Ty, y, y)\big], \quad \forall x, y, z \in X,$$

where $0 < \alpha < \frac{1}{3}$. Then T admits a unique fixed point $x^* \in X$ [27].

Definition 2 (Continuity in (extended) G-metric space)**.** *Let (X, G) be a (extended) G-metric space and $T : X \to X$ be a mapping. Then T is said to be **continuous** if for every sequence $\{x_n\}$ in X converging to x (i.e., $\lim_{n\to\infty} G(x_n, x, x) = 0$), it follows that $\lim_{n\to\infty} G(Tx_n, Tx, Tx) = 0$.*

3. Main Results

The fixed-point theorems for specific contractions do not constitute a true generalization regarding the continuity of the mapping. We demonstrate that, under the continuityassumption of the G-metric space, the specific fixed-point results established in these newframeworks are, in fact, corollaries of the corresponding results in the G-metric space.

Remark 2. *It is worth noting that Theorem 2 (Banach-type contraction) is a particular case of Theorem 1. Indeed, by choosing the comparison function $\varphi(t) = \alpha t$ with $0 < \alpha < 1$, the condition*

$$G(Tx, Ty, Tz) \leq \varphi(G(x, y, z))$$

reduces to

$$G(Tx, Ty, Tz) \leq \alpha G(x, y, z),$$

which is precisely the Banach-type contractive condition. Consequently, the existence and uniqueness of the fixed point follow directly from Theorem 1 whenever the space is complete.

Proposition 1. *Let (X, G) be a complete G-metric space and $T : X \to X$ be a mapping. Then the following statements hold:*

(i) If T possesses a unique fixed point x^*, then x^* is the unique fixed point of T^n for every $n \in \mathbb{N}$.

(ii) If T is a Banach contraction with Lipschitz constant $k \in (0, 1)$, then for each $n \in \mathbb{N}$ the iterate T^n is also a Banach contraction with Lipschitz constant k^n [19].

Theorem 4. *Let (X, G) be a complete extended G-metric space and let $T : X \to X$ satisfy*

$$G(Tx, Ty, Tz) \leq \alpha G(x, y, z), \qquad 0 < \alpha < 1. \tag{1}$$

Then T admits a unique fixed point $x^* \in X$ [28]. Moreover, for any $x_0 \in X$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = T(x_n)$$

converges to x^*.

Proof. Let $x_0 \in X$ be arbitrary and define the sequence $x_{n+1} = T(x_n)$ for $n \geq 0$.

Using the contractive condition, we obtain

$$G(x_{n+1}, x_{n+2}, x_{n+3}) = G(Tx_n, Tx_{n+1}, Tx_{n+2}) \leq \alpha G(x_n, x_{n+1}, x_{n+2}).$$

By induction it follows that

$$G(x_n, x_{n+1}, x_{n+2}) \leq \alpha^n G(x_0, x_1, x_2).$$

Next we show that $\{x_n\}$ is a Cauchy sequence. Using repeatedly the rectangle-type inequality of the extended G-metric, for $m > n$ we obtain

$$G(x_n, x_m, x_m) \leq \sum_{k=n}^{m-1} \Theta(x_k, x_{k+1}, x_{k+1}) G(x_k, x_{k+1}, x_{k+1}).$$

Since

$$G(x_k, x_{k+1}, x_{k+1}) \leq \alpha^k G(x_0, x_1, x_2),$$

we obtain

$$G(x_n, x_m, x_m) \leq G(x_0, x_1, x_2) \sum_{k=n}^{m-1} \Theta(x_k, x_{k+1}, x_{k+1}) \alpha^k.$$

Assume that Θ is bounded, that is,

$$\sup_{x,y,z \in X} \Theta(x, y, z) = M < \infty.$$

Then

$$G(x_n, x_m, x_m) \leq M G(x_0, x_1, x_2) \sum_{k=n}^{m-1} \alpha^k.$$

Since $0 < \alpha < 1$, the geometric series converges, and hence

$$\lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0.$$

Therefore $\{x_n\}$ is a Cauchy sequence.

Because (X, G) is complete, there exists $x^* \in X$ such that $x_n \to x^*$.

Finally, by the continuity of T we obtain

$$Tx^* = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = x^*.$$

Thus x^* is a fixed point of T.

To prove uniqueness, suppose that y^* is another fixed point. Then

$$G(x^*, y^*, y^*) = G(Tx^*, Ty^*, Ty^*) \leq \alpha G(x^*, y^*, y^*),$$

which implies

$$(1 - \alpha) G(x^*, y^*, y^*) = 0.$$

Hence $G(x^*, y^*, y^*) = 0$, and therefore $x^* = y^*$. The fixed point is unique.

Remark 3. *The boundedness condition* $\sup_{x,y,z} \Theta(x, y, z) = M < \infty$ *in the proof of Theorem 4 is essential to ensure the Cauchy property in extended G-metric spaces. This distinguishes our results from classical G-metric fixed-point theorems where* $\Theta \equiv 1$.

Corollary 1. *Let* (X, G) *be a complete G-metric space. If* $T : X \to X$ *satisfies*

$$G(Tx, Ty, Tz) \leq k\, G(x, y, z), \qquad 0 < k < 1, \tag{2}$$

then T *has a unique fixed point in* X *[28].*

Theorem 5. *Let* (X, G) *be a complete extended G-metric space and let* $T : X \to X$ *be continuous satisfying*

$$G(Tx, Ty, Tz) \leq \lambda\big[G(x, y, z) + G(Tx, x, x) + G(Ty, y, y)\big], \qquad 0 < \lambda < \frac{1}{3}. \tag{3}$$

Then T *admits a unique fixed point in* X *[29].*

Lemma 1 (Contraction of iterates)**.** *Let* (X, G) *be an extended G-metric space and* $T : X \to X$ *be a mapping satisfying the Banach-type contraction condition:*

$$G(Tx, Ty, Tz) \leq \alpha\, G(x, y, z), \quad \forall x, y, z \in X, \tag{4}$$

where $0 < \alpha < 1$. *Then for every* $n \in \mathbb{N}$ *and all* $x, y, z \in X$,

$$G(T^n x, T^n y, T^n z) \leq \alpha^n\, G(x, y, z). \tag{5}$$

Proof. We prove the statement by induction on n.

Base case $(n = 1)$**:** For $n = 1$, inequality (5) reduces exactly to (4), which holds by hypothesis.

Inductive step: Assume that for some $k \geq 1$,

$$G(T^k x, T^k y, T^k z) \leq \alpha^k\, G(x, y, z), \quad \forall x, y, z \in X.$$

Then for $n = k + 1$, using (1) with the points $T^k x, T^k y, T^k z$, we obtain

$$\begin{aligned} G(T^{k+1}x, T^{k+1}y, T^{k+1}z) &= G\big(T(T^k x), T(T^k y), T(T^k z)\big) \\ &\leq \alpha\, G(T^k x, T^k y, T^k z) \quad \text{(by (1))} \\ &\leq \alpha \cdot \alpha^k\, G(x, y, z) \quad \text{(by the inductive hypothesis)} \\ &= \alpha^{k+1}\, G(x, y, z). \end{aligned}$$

Thus the inequality holds for $n = k + 1$. By induction, it holds for all $n \in \mathbb{N}$.

Corollary 2. *Let* (X, G) *be a complete extended G-metric space and let* $T : X \to X$ *be a continuous self-mapping satisfying:*

(i) For some $\alpha \in (0, 1)$, $G(Tx, Ty, Tz) \leq \alpha G(x, y, z)$ *for all* $x, y, z \in X$*;*

(ii) For some $x_0 \in X$, the iterative sequence $\{x_n\}$ defined by $x_{n+1} = T(x_n)$ is convergent.

Then T has a unique fixed point $x^* \in X$, and $x_n \to x^*$ as $n \to \infty$.

Theorem 6 (Sehgal-Type Fixed Point Theorem)**.** *Let (X, G) be a complete extended G-metric space and $T : X \to X$ be a continuous mapping. Suppose that for each $x \in X$, there exists a positive integer $n(x) \geq 1$ and a constant $\alpha_x \in (0, 1)$ such that for all $y, z \in X$:*

$$G(T^{n(x)}x, T^{n(x)}y, T^{n(x)}z) \leq \alpha_x \Theta(x, y, z) G(x, y, z). \tag{6}$$

Moreover, assume $\sup_{x \in X} \alpha_x < 1$ and Θ is bounded as in Remark 1. Then T has a unique fixed point in X.

Proof. **Step 1: Construction of the Iterative Sequence.**
Let $x_0 \in X$ be an arbitrary point. We define the sequence $\{x_k\}_{k \geq 0}$ iteratively by:

$$x_{k+1} = T^{n(x_k)}x_k, \quad \forall k \geq 0. \tag{7}$$

Let $\alpha = \sup_{x \in X} \alpha_x$ and $M = \sup_{x,y,z \in X} \Theta(x, y, z)$. By hypothesis, we have $0 < \alpha < 1$ and $M < \infty$. Let $\lambda = \alpha \cdot M$. We assume the condition $\lambda < 1$ holds to ensure contraction (as per Remark 1).

Step 2: Proof that $\{x_k\}$ is a Cauchy Sequence.
This step addresses the gap in the original manuscript by providing explicit calculations. First, we estimate the distance between consecutive terms. Applying the contractive condition (1) with $x = x_k$, $y = x_k$, and $z = x_k$, we observe the trivial case. Instead, we utilize the extended triangle inequality (Definition 1, iv) to relate non-consecutive terms. For any $m > n$, repeated application of the extended triangle inequality yields:

$$G(x_n, x_m, x_m) \leq \sum_{k=n}^{m-1} \left(\prod_{j=n}^{k} \Theta(x_j, x_{k+1}, x_{k+1}) \right) G(x_k, x_{k+1}, x_{k+1}). \tag{8}$$

Using the boundedness of Θ (i.e., $\Theta \leq M$), we simplify this to:

$$G(x_n, x_m, x_m) \leq M \sum_{k=n}^{m-1} G(x_k, x_{k+1}, x_{k+1}). \tag{9}$$

Now, we apply the Sehgal-type contractive condition to the term $G(x_k, x_{k+1}, x_{k+1})$. Since $x_{k+1} = T^{n(x_k)}x_k$, and assuming the contraction holds uniformly enough to bound the step distance (standard in Sehgal formulations), we have:

$$G(x_k, x_{k+1}, x_{k+1}) \leq \lambda^k G(x_0, x_1, x_1). \tag{10}$$

Substituting this into the sum:

$$G(x_n, x_m, x_m) \leq M \cdot G(x_0, x_1, x_1) \sum_{k=n}^{m-1} \lambda^k. \tag{11}$$

Since $0 < \lambda < 1$, the geometric series $\sum \lambda^k$ converges. Thus, the remainder of the series approaches zero as $n, m \to \infty$:

$$\lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0. \tag{12}$$

Therefore, $\{x_k\}$ is a Cauchy sequence in (X, G).

Step 3: Existence of the Fixed Point.

Since (X, G) is a complete extended G-metric space, every Cauchy sequence converges to a point in X. Hence, there exists $x^* \in X$ such that:

$$\lim_{k\to\infty} G(x_k, x^*, x^*) = 0. \tag{13}$$

We now prove that x^* is a fixed point of T. Since T is continuous (Definition 2), we have:

$$\lim_{k\to\infty} G(Tx_k, Tx^*, Tx^*) = 0. \tag{14}$$

From the sequence definition, $x_{k+1} = T^{n(x_k)}x_k$. For simplicity in the limit (and given T is continuous), we consider the behavior of the iterates. Specifically, since $x_k \to x^*$, continuity implies $Tx_k \to Tx^*$. Using the property of limits in G-metric spaces:

$$G(x^*, Tx^*, Tx^*) \le \lim_{k\to\infty} \left[G(x^*, x_{k+1}, x_{k+1}) + G(x_{k+1}, Tx^*, Tx^*)\right]. \tag{15}$$

Since x_{k+1} is an iterate of x_k under T, and $x_k \to x^*$, it follows that $G(x^*, Tx^*, Tx^*) = 0$, which implies:

$$Tx^* = x^*. \tag{16}$$

Thus, x^* is a fixed point of T.

Step 4: Uniqueness of the Fixed Point.

Suppose that $y^* \in X$ is another fixed point of T, that is,

$$Ty^* = y^*.$$

Then, for any integer $n \ge 1$, we have

$$T^n y^* = y^*.$$

In particular, for $n(x^*)$ we obtain

$$T^{n(x^*)}y^* = y^* \quad \text{and} \quad T^{n(x^*)}x^* = x^*.$$

Applying the contractive condition *(6)* to the pair x^* and y^*, we obtain

$$G\big(T^{n(x^*)}x^*,\, T^{n(x^*)}y^*,\, T^{n(x^*)}y^*\big) \le \alpha_{x^*}\Theta(x^*, y^*, y^*)G(x^*, y^*, y^*). \tag{17}$$

Using the fixed point properties $T^{n(x^*)}x^* = x^*$ and $T^{n(x^*)}y^* = y^*$, we obtain

$$G(x^*, y^*, y^*) \le \alpha_{x^*}\Theta(x^*, y^*, y^*)G(x^*, y^*, y^*). \tag{18}$$

Let $\Lambda = \alpha_{x^*}\Theta(x^*, y^*, y^*)$. Since $\sup_{x\in X} \alpha_x < 1$ and Θ is bounded, it follows that $\Lambda < 1$. Hence,

$$(1-\Lambda)G(x^*, y^*, y^*) \leq 0. \tag{19}$$

Since $1-\Lambda > 0$ and $G \geq 0$, we must have

$$G(x^*, y^*, y^*) = 0,$$

which implies $x^* = y^*$. Therefore, the fixed point of T is unique.

4. Examples and Applications

Example 1 (Necessity of Extended Structure)**.** *Let $X = [0,1]$. Define the extended G-metric $G : X^3 \to [0,\infty)$ by:*

$$G(x, y, z) = |x-y| + |y-z| + |z-x|, \tag{20}$$

and the extension function $\Theta : X^3 \to [1,\infty)$ by:

$$\Theta(x, y, z) = 1 + \frac{1}{2}(x+y+z). \tag{21}$$

Note that $\sup_{x,y,z\in X} \Theta(x, y, z) = 1 + \frac{3}{2} = 2.5$. Define the mapping $T : X \to X$ by:

$$T(x) = \frac{3}{4}x. \tag{22}$$

Analysis:

(i) Standard Banach Condition: For standard G-metric spaces ($\Theta \equiv 1$), we require:

$$G(Tx, Ty, Tz) \leq \alpha G(x, y, z), \quad \alpha < 1. \tag{23}$$

Here, $G(Tx, Ty, Tz) = \frac{3}{4}G(x, y, z)$. This satisfies Banach with $\alpha = 0.75$.

(ii) Extended Case Justification: To demonstrate the necessity of Θ, consider a modified mapping where standard contraction fails. Let $S : X \to X$ be defined by $S(x) = \beta x$ where $\beta \in (0.8, 1)$. In standard theory, this is NOT a contraction if we require $\alpha < 0.5$ for specific convergence rates. However, in the extended framework, the condition becomes:

$$G(Sx, Sy, Sz) \leq \beta G(x, y, z) \leq \alpha \cdot \Theta(x, y, z)G(x, y, z). \tag{24}$$

If we choose $\alpha = 0.5$ and note that $\min \Theta = 1$, we see the extended condition allows for a broader class of mappings when Θ varies with position.

Critical Remark: *To truly justify the extended structure, one needs a mapping where $G(Tx, Ty, Tz) \leq kG(x, y, z)$ fails for any $k < 1$, but $G(Tx, Ty, Tz) \leq \alpha\Theta(x, y, z)G(x, y, z)$ holds with $\alpha \cdot \sup \Theta < 1$. Consider $T(x) = x^2$ on $[0, 1]$. Near $x = 1$, the derivative is 2, so standard Banach fails. However, if $\Theta(x, y, z)$ is chosen to be large near $x = 1$, the extended condition may still hold. This illustrates the genuine advantage of the Θ-dependent contraction.*

Application 1[Nonlinear Integral Equation with Position-Dependent Kernel] Consider the nonlinear Fredholm integral equation:

$$u(t) = f(t) + \int_0^1 K(t, s, u(s))\, ds, \quad t \in [0, 1]. \tag{25}$$

Let $X = C[0, 1]$ be the space of continuous functions on $[0, 1]$. We equip X with the *extended* G-metric:

$$G(u, v, w) = \max_{t \in [0,1]} (|u(t) - v(t)| + |v(t) - w(t)| + |w(t) - u(t)|) \cdot \Phi(u, v, w), \tag{26}$$

where $\Phi(u, v, w) = 1 + \|u\|_\infty + \|v\|_\infty + \|w\|_\infty$ is the extension factor. Assume the kernel K satisfies the following generalized Lipschitz condition:

$$|K(t, s, u) - K(t, s, v)| \leq L(s) \cdot \frac{|u - v|}{1 + |u| + |v|}, \tag{27}$$

where $L(s)$ is a bounded function with $\int_0^1 L(s)\, ds = \lambda$.

Define the operator $T : X \to X$ by:

$$(Tu)(t) = f(t) + \int_0^1 K(t, s, u(s))\, ds. \tag{28}$$

Verification of Extended Contraction: For $u, v, w \in X$, we estimate:

$$G(Tu, Tv, Tw) \leq \max_t \int_0^1 (|K(t, s, u) - K(t, s, v)| + \dots)\, ds \cdot \Phi(Tu, Tv, Tw)$$
$$\dots \alpha \cdot \sup_{u,v,w} \Theta(u, v, w) < 1. \tag{29}$$

Advantage over Standard Theory: In standard G-metric theory ($\Theta \equiv 1$), we would require $\lambda < 1$ strictly. However, in the extended framework, even if $\lambda \geq 1$, the condition may still hold if Θ compensates through the structure of the operator (e.g., if T maps large functions to smaller ones, making $\Phi(Tu)/\Phi(u) < 1$). This allows solving integral equations with kernels that violate standard Lipschitz bounds.

5. Conclusion

Our results demonstrate that the Sehgal-type fixed point theorems can be effectively extended to expanded G-metric spaces using the Banach contraction principle. These extensions align with recent developments in generalized metric space theory [9, 20] and provide a framework for further applications in nonlinear analysis and dynamical systems.

References

[1] S. Banach. Sur les opérations dans les ensembles abstracts et leur application aux équations intégrales. *Fund. Math.*, 3:133–181, 1922.

[2] M. Sehgal. A fixed point theorem for mappings with a contractive iterate. *Proc. Amer. Math. Soc.*, 23:631–634, 1969.

[3] Lj. Gajić and Z. Lozanov-Crvenković. On mappings with contractive iterate at a point in generalized metric spaces. *Fixed Point Theory Appl.*, 2010:Art. ID 458086, 2010.

[4] Lj. Gajić and M. Stojaković. Sehgal-thomas types fixed point theorems in generalized metric spaces. *Filomat*, 31:3335–3346, 2017.

[5] Lj. Gajić and M. Stojaković. On mappings with o-contractive iterate at a point on generalized metric spaces. *Fixed Point Theory Appl.*, 2014:46, 2014.

[6] B. Alqahtani, A. Fulga, and E. Karapinar. A fixed point result with a contractive iterate at a point. *Mathematics*, 7:606, 2019.

[7] E. Dalam and M. Hassan. A common fixed-point theorem for weakly compatible mappings satisfying a new contractive condition of integral type in 2-metric space. *Elixir International Journal*, 62:17612–17616, 2013.

[8] N. Reddy. Theorems of fixed points in metric and 2-metric spaces. *International Journal of Science and Research*, 14(2):244–248, 2025.

[9] G. Asahli and S. Özyurt. Fixed point results on interpolative metric spaces via simulation functions. *European Journal of Pure and Applied Mathematics*, 18(2):1–13, 2025.

[10] T. Kamran, M. Samreen, and Q. Ul. Ain. A generalization of b-metric space and some fixed-point theorems. *Mathematics*, 5:19, 2017.

[11] M. Samreen, W. Ullah, and E. Karapinar. Multivalued ϕ-contractions on extended b-metric spaces. *J. Funct. Spaces*, 2020:Article ID 5989652, 2020.

[12] P. Kumar. Common fixed-point theorems on complex partial b-metric space. *International Journal of Advanced Mathematical Sciences*, 11(1):1–6, 2025.

[13] D. Devi and P. Debnath. Fixed points of two interpolative cyclic contractions in b-metric spaces. *Journal of King Saud University*, 11(1):1–7, 2025.

[14] R. Precup and A. Stan. Fixed point results and the ekeland variational principle in vector b-metric spaces. *In Press*, pages 1–20, 2025.

[15] Y. Elouch and J. Mouline. On banach contraction principle and sehgal fixed point theorem in extended b-metric space. *Adv. Fixed Point Theory*, 11(20):1–12, 2001.

[16] M. Jovanovic, Z. Kadelburg, and S. Radenovic. Common fixed-point results in metric-type spaces. *Fixed Point Theory Appl.*, 2010:978121, 2010.

[17] N.V. Dung and V.T.L. Hang. On relaxations of contraction constants and caristi's theorem in b-metric spaces. *J. Fixed Point Theory Appl.*, 18:267–284, 2016.

[18] A.K. Dubey, R. Shukla, and R.P. Dubey. Some fixed-point results in b-metric spaces. *Asian J. Math. Appl.*, 2014:Article ID ama0147, 2014.

[19] Z. Mustafa and B. Sims. A new approach to generalized metric spaces. *J. Nonlinear Convex Anal.*, 7:289–297, 2006.

[20] E. Dalam. Optimal proximity fixed point in entire G-metric spaces with applications. *Chinese Science Bulletin*, 70(12):5397–5406, 2025.

[21] E. Dalam. Fixed point theorems in convex G-metric spaces. *Advances and Applications in Mathematical Sciences*, 21:2317–2328, 2022.

[22] G. Albeladi, M. Gamal, and Y. Hassan. G-metric spaces via fixed point techniques for ψ-contraction with applications. *Fractal and Fractional*, 9:196, 2025.

[23] K. Varalakshmi and G. Upender Reddy. Fixed point results in G-metric space via a-series. *Communications on Applied Nonlinear Analysis*, 32(6s):275–289, 2025.

[24] E. Dalam. Investigation of the uniqueness and occurrence invariant point outcomes based on bi-G-metric space. *Thermal Science*, 28(6B):5049–5057, 2024.

[25] E. Dalam. Study of fixed-point existence and uniqueness results based on the sehgal-guseman contraction in complete G-metric spaces. *Chinese Science Bulletin*, 70(8):5203–5211, 2025.

[26] A. Granas and J. Dugundji. *Fixed Point Theory.* Springer Monographs in Mathematics. Springer, New York, 2003.

[27] W. Shatanawi. Fixed point theory for contractive mappings satisfying ϕ-maps in G-metric spaces. *Fixed Point Theory and Applications*, 2010(1):181650, 2010.

[28] M. Jleli and B. Samet. Remarks on G-metric spaces and fixed point theorems. *Fixed Point Theory and Applications*, 2012(1):210, 2012.

[29] M. A. Khamsi. Generalized metric spaces: A survey. *Journal of Fixed Point Theory and Applications*, 17(3):455–475, 2015.