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Probability Distribution and Statistical Properties of Reachability Sets in Controlled Linear Systems with Random Parameters

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Abstract. This research consists of two parts. In the first part, we study the statistical characteristics of the reachability set $R(t, \sigma, X)$ of the control system

$$\dot{x} = f(h^t\sigma, x, u), \qquad u \in U(h^t\sigma, x), \qquad (t, \sigma, x) \in \mathbb{R} \times \Sigma \times \mathbb{R}^n,$$

which is parameterized using a metric dynamical system (Σ, P, μ, h^t). We examine such characteristics as the relative absorption frequency and the upper and lower relative frequencies of absorption of the reachability set. For the above system, we also provide sufficient conditions under which a given set M is statistically invariant with respect to the controlled system. In addition, we compare the distribution function defined on the control linear space with the corresponding relative frequency. In the second part, we investigate the role of the distribution function and its relationship to the stability of the dynamical system under different types of distributions.

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1. Introduction

One important area of applied mathematics and engineering research is the study of control systems with random parameters. Applications ranging from robotics and aircraft

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to energy management and telecommunications depend heavily on control systems. However, real-world systems are rarely deterministic; noise, shifting conditions, and unforeseen component failures are just a few of the causes of uncertainty that frequently affect them. These uncertainties can considerably complicate the work of devising efficient control techniques, which might appear as random factors inside the system. Conventional control theory often presupposes that the system's parameters are known continuously or with predictable variations. However, this assumption is not true in a lot of real-world situations. For example, in networked control systems, randomness is introduced via packet losses and communication delays; in aerospace engineering, random atmospheric variables may affect an aircraft's behavior. If these uncertainties are not appropriately considered, they may cause a considerable decline in performance or cause system instability. Stochastic control theory was developed in response to the difficulty of handling unpredictability in control systems; this theory focuses on systems that have probabilistic aspects. This area of study aims to comprehend how randomness impacts system behavior and to create plans to guarantee desired performance in the face of uncertainty. Nevertheless, current approaches frequently concentrate on certain randomness or have limited relevance to real-world systems.

2. Problem statement

An essential problem in stochastic control is comprehending the long-term dynamics of random parameter systems. Specifically, when parameters are random, the study of reachability sets—the set of all conceivable states a system may reach under specific conditions—becomes noticeably more difficult. The system's probabilistic aspect is not considered by conventional deterministic approaches, which makes them inadequate. Furthermore, applying current stochastic approaches is typically limited by assumptions about the system's structure or randomness, which may not follow particular distributions.

This paper focuses on controlled linear systems with random parameters, where the unpredictability is not necessarily distributed but can be arbitrarily chosen. The goal is to create a thorough methodology for examining the reachability sets of these systems over time while taking randomness's influence on behavior into account.

3. Research objectives

This study aims to achieve several key objectives:

3.1. Develop new analytical tools

The work aims to present novel metrics that can measure the influence of random factors on system behavior, such as relative absorption frequency. A more detailed understanding of how randomness affects the reachability sets of controlled linear systems will be possible with the help of these measurements.

3.2. Apply Ergodic theorems and Markov processes

The study intends to establish a framework for assessing the long-term statistical features of systems with random coefficients by utilizing ideas from Markov processes and the ergodic theorem. This method will provide insights into the stability and resilience of the system by assisting in understanding how its behavior changes over time.

3.3. Assess practical implications

The theoretical inferences' practical implications will be investigated in this work, especially in engineering domains where random disturbances might affect control systems. It involves evaluating how the created techniques may enhance the layout and functionality of control schemes in practical situations.

4. Significance and contributions

This work is essential because it can help close the gap between theoretical investigations and real-world applications in stochastic control. The work offers a route to more reliable and resilient control techniques by offering new frameworks and tools for studying systems with random characteristics. These tactics are essential in today's engineering because systems are more intricate and subject to various uncertainties. The study results should also have wider ramifications beyond only conventional control systems. The techniques created might be used in other fields where unpredictability affects systems, such as environmental science, biology, and finance. Thus, this study enhances the subject of stochastic control and adds to a broader comprehension of uncertainty management.

5. The problem under study

This article is a continuation of works [1–4], this work is also considered an expansion of some of the related concepts. In this research, we discuss the study of statistical characteristics of the reachability set of a family of controllable systems

$$\dot{x} = f\left(h^t\sigma, x, u\right), u \in U(h^t\sigma, x)\,(t, \sigma, x) \in R \times \Sigma \times R^n, \tag{1}$$

depending on the parameter $\sigma \in \Sigma$. In particular, we will study the controlled system generated by the metric dynamical system (Σ, P, μ, h^t), and the functions f and U. We assume that the set Σ contains an infinite number of elements.

In the first part we will present the basic definitions which we need in the second part. In the second part of this paper, we will obtain estimates of the statistical characteristics of the controlled linear system

$$\dot{x} = A\left(h^t\sigma\right)x + B\left(h^t\sigma\right)u, (t, \sigma, x, u) \in R \times \Sigma \times R^n \times U, \tag{2}$$

where U is a non-empty compact subset of R^n.

5.1. Basic definitions

Definition 1. *From [5–7], a metric dynamical system is a quadruple* (Σ, P, μ, h^t), *where* Σ*-the phase is the space of the dynamic system;* P*- is some sigma- algebra of subsets of* $\Sigma; h^t$*-is a one-parameter group of measurable transformations phase space* Σ *into itself. Further,* μ*- is a probability measure with support on the space* Σ*, invariant with respect to the flow* h^t*, i.e.* $\mu\left(h^t H\right) = \mu(H)$ *for all* $H \in P$ *and any* $t \in R$.

Definition 2. *From [2],* $comp(R^m)$*- is the space of non-empty compact subsets of* R^m *with the Hausdorff metric.*

Condition 1. *There is* $\sigma \in \Sigma$ *for which the following properties hold:*

(i) for all $t \in R$ *the function* $(x, u) \to f(h^t\sigma, x, u)$ *continuous;*

(ii) for all $(x, u) \in R \times R^n$ *the function* $t \to f(h^t\sigma, x, u)$ *piecewise continuous;*

(iii) The function $U\left(h^t\sigma, x\right) \in comp\left(R^m\right)$ *and is upper semi-continuous for all*

$$(t, x) \in R \times R^n.$$

Let $\sigma \in \Sigma$ be fixed and satisfy Condition 1.

We'll put it in correspondence to system (1) differential inclusion

$$\dot{x} \in F(t, x), F(t, x) = \underline{co}\ H(t, x), \tag{3}$$

where for each fixed point $(\sigma, x) \in \Sigma \times R^n$ set $H(h^t\sigma, x)$ consists of all limit values of the function $f(t_i, x_i, U(t_i, x_i))$ at $(t_i, x_i) \to (t, x)$. Next, write $\underline{co}H\left(h^t\sigma, x\right)$- means the closure of a convex hull of the set $H\left(h^t\sigma, x\right)$.

Definition 3. *From [3], every set* $X \in comp\left(R^n\right)$ *and the moment of time* $t > 0$ *we associate the set* $R(t, \sigma, X)$*, consisting of all values in moment* t *of solutions* $t \to \varphi(t, \sigma, x)$ *of inclusion* (3) *when the initial condition* $\varphi(0, \sigma, x) = x$. *The set* $R(t, \sigma, X)$ *is cross section at time* $t > 0$ *of the integral funnel of inclusion* (3) *and is called the reachability set of the control system* (1). *We assume that for a given set* $X \in comp\left(R^n\right)$ *reachability set* $R(t, \sigma, X)$*, exists for all* $t > 0$*; this means for every* $x \in X$ *there is a solution* $\varphi(t, \sigma, x)$ *of inclusion* (3) *satisfying the initial condition* $\varphi(0, \sigma, x) = x$ *and extendable to the half-axis* $R_+ = [0, +\infty)$. *Let us introduce into consideration the mapping* $M(h^t\sigma)$ *with values in space comp* $\left(R^n\right)$ *and sets*

$$Q(\sigma) = \left\{(t, x) \in [0, +\infty) \times R^m : x \in M(h^t\sigma)\ \right\}$$

We assume that the function $t \to M(h^t\sigma)$ *is continuous in the Hausdorff metric. To determine the statistical characteristics of the reachability set consider a subset of the number line*

$$\delta(\eta, \sigma, X) \doteq \left\{t \in [0, \eta] : R(t, \sigma, X) \subseteq M(h^t\sigma)\ \right\}$$

Definition 4. *From [4, 8, 9], the relative frequency of absorption of the reachability set $R(t, \sigma, X)$ of system (1) by the set $Q(\sigma)$ is called the characteristic*

$$\wp(\sigma, R_X, M) \doteq \frac{mes\ \delta(\eta, \sigma, X)}{\eta}, \tag{4}$$

where mes- is the Lebesgue measure on the number line. If limit (4) does not exist, then the characteristics:

$$\wp^*(\sigma, R_X, M) \doteq \frac{mes\ \delta(\eta, \sigma, X)}{\eta},$$

$$\wp_*(\sigma, R_X, M) \doteq \frac{mes\ \delta(\eta, \sigma, X)}{\eta},$$

are called, respectively, the upper and lower relative frequencies of absorption of the reachability set $R(t, \sigma, X)$ of system (1) by the set $Q(\sigma)$.

Definition 5. *From [4, 6], the set $M(\sigma)$ is called statistically invariant with respect to the controlled system (1), if equality is satisfied $\wp\left(\sigma, R_{M(\sigma)}, M(\sigma)\right) = 1$. The set $M(\sigma)$ is said to be positively invariant with respect to system (1) if for any $t > 0$ the embedding is satisfied $R(t, \sigma, M(\sigma)) \subseteq M(h^t\sigma)$.*

In this paper, estimates of the statistical characteristics of a controlled linear system are obtained

$$\dot{x} = A\left(h^t\sigma\right)x + B\left(h^t\sigma\right)y, (t, \sigma, x, u) \in R \times \Sigma \times R^n \times Y, \tag{5}$$

where Y is a non-empty compact subset of R^n. We can say that the above system can be defined as a stable random process in the narrow sense

$$\zeta\left(h^t\sigma\right) \doteq \left(A\left(h^t\sigma\right), B\left(h^t\sigma\right)\right).$$

To do this, we describe the metric dynamic system (Σ, P, μ, h^t), which parameterizes system (5) and thus this system turns into a system with random coefficients [10].

Further system (5) will be called system ζ. Let us define a probability space (Σ, P, μ) which is the direct product of probability spaces (Σ_1, P_1, μ_1) and (Σ_2, P_2, μ_2).

Here Σ_1 means the set the sequences of numbers

$$\eta = (\eta_0, \ldots, \eta_k, \ldots) \quad \text{where } 0 < \eta < \infty, \sum_{k=0}^{\infty} \eta_k = +\infty, \tag{6}$$

P is the smallest sigma- algebra, generated by cylinder sets

$$S_k = \{\eta \in \Sigma_1 : \eta_0 \in I_0, \ldots, \eta_0 \in I_k\},$$

Where $I_i \doteq (t_i, s_i]$, $t_i < s_i$, and the probability measure μ_1 is defined as follows way. For each half-interval I_i we define a probability measure

$$\widetilde{\mu}_1 = G_i(s_i) - G_i(t_i)$$

with the help of distribution functions $G_i(t), 0 < t < \infty$ or $t \in (0, \infty)$, And that means $G_i(t) = 0$ for all -$\infty < t \leq 0$. On the algebra of cylinder sets let's construct a measure

$$\widetilde{\mu}_1(S_k) = \widetilde{\mu}_1(I_0)\,\widetilde{\mu}_1(I_1)\dots\widetilde{\mu}_1(I_k). \tag{7}$$

Then, by the theorem of A. N. Kolmogorov (for example, [10]) on a measurable space (Σ_1, P_1) there is a unique probability measure $\widetilde{\mu}_1$ which is an extension of the measure $\widetilde{\mu}_1$ to the sigma- algebra P_1.

Next, let $\Omega = \omega_1, \dots\ , \omega_\gamma$ is a finite set of matrix pairs $\omega_i = (A_i, B_i)\,, A_i, B_i$ are matrices of sizes $n \times n$ and $n \times m$, respectively.

For each matrix pair $\omega_i = (A_i, B_i)$ let us associate a linear stationary system with matrices A_i and B_i:

$$\dot{x} = A_i x + B_i u\,(x, u) \in R^n \times U.$$

Let Σ_2 denote the set of sequences

$$\Sigma_2 = \{\theta : \theta = (\theta_0, \theta_1, \dots, \theta_k, \dots)\, \theta_k \in \Omega\}.$$

We define the set system P_2 as the smallest sigma- algebra generated by cylindrical sets

$$H_k = H(\theta_0, \theta_1, \dots, \theta_k),$$

where H_k Is the set of all sequences from Σ_2, for which $k+1$ first terms are fixed. Suppose we are given non-negative functions

$$f_i = \emptyset_0(\varphi_i), \emptyset_{ij} = \emptyset(\varphi_i, \varphi_j)$$

such that

$$\sum_{i=1}^{m} f_i = 1,\ \sum_{i=1}^{m} \emptyset_{ij} = 1 \quad for\ all\ i = 1, \dots,\ m$$

and the number $f_1 + \dots + f_m$ satisfy the system of equations

$$f_{j=} \sum_{i=1}^{k} f_i \emptyset_{ij}, j = 1, \dots, k. \tag{8}$$

Any non-negative solution of a given system satisfying the condition $\sum_{i=1}^{m} f_i = 1$, is usually called a stationary or invariant probability distribution of a Markov chain.

The measure of the cylindrical set H_k is defined by the equality

$$\breve{\mu}_2(H_k) = \emptyset_0(\varphi_0)\,\emptyset(\varphi_0, \varphi_1)\dots\emptyset(\varphi_{k-1}, \varphi_k)$$

and denote by μ_2 the extension of the measure $\breve{\mu}_2$ from the algebra of cylindrical sets to the sigma algebra P_2. Let us introduce the sequence $\{\gamma_k\}_{k=1}^{\infty}$ as follows:

$$\gamma_0 = 0, \gamma_k(\eta) = \sum_{i=0}^{k-1} \eta_i\ , where\ \eta \in \Sigma_1.$$

We assume that $\eta_i \in (0,\infty), i = 0,1,\dots$. are independent random variables, where $\eta_1,\eta_2,\dots$ have the same distribution with distribution function $F(t)$ and mathematical expectation $E_\eta < \infty$ Let us denote by $\upsilon = \upsilon(t,\eta)$ the number of points of the sequence $\{\gamma_k\}_{k=1}^{\infty}$

located to the left of t, then

$$\upsilon = \upsilon\left(t,\eta\right) = \left\{k:\ \gamma_k \leq t\right\},\ where\ t \geq 0.$$

The value $\upsilon = \upsilon(t,\eta)$ is called the restoration process. Let us define the distribution function of the random variable ζ_0 by the equality

$$G_0(t) = \frac{1}{E_\eta}\int_0^t \left(1 - G\left(s\right)ds, t \in \left(0,\infty\right),\right. \tag{9}$$

then $\upsilon = \upsilon(t,\eta)$ is a stationary restoration process [11].

This means that this process has a constant speed recovery, that is, the recovery function,

$$Z\left(t\right) \doteq E\ \upsilon\left(t,\eta\right) + 1, linearin: Z\left(t\right) = at + 1.$$

Here and below we will denote by the letter Z mathematical expectation of a random variable or function. On the probability space (Σ_1, P_1, μ_1) we define the shift transformation

$$h_1^t\eta = \left(\gamma_\upsilon - t, \eta_{\upsilon+1}, \eta_{\upsilon+2}, \ldots\right),\ t > 0.$$

Because the $\upsilon = \upsilon(t,\eta)$ stationary restoration process, transformation h_1^t saves measure μ_1, that is, for any set $H \in P_1$ and for all $t \geq 0$ equality is satisfied

$$\mu_1\left(h_1^t H\right) = \mu_1\left(H\right).$$

On the space $(\Sigma_2\ , P_2\ ,\ \ \mu_2)$ for all $\eta \in H$, let's define a shift transformation

$$h_2^t\left(\eta\right)\varphi = \left(\varphi_\upsilon, \varphi_{\upsilon+1}, \varphi_{\upsilon+2}, \ldots\right).$$

Since the Markov chain is stationary, it follows that transformation h_2^t saves measure μ_2. On the space $(\Sigma\ , P, \mu)$ we also define the shift transformation by the equality

$$h^t\sigma = h^t\left(\eta,\varphi\right) = \left(h_1^t\eta, h_2^t\left(\eta\right)\varphi\right). \tag{10}$$

The constructed dynamic system $(\Sigma\ , P, \mu, h^t)$ is called the skew- product of dynamical systems $\left(\Sigma_1, P_1,\ \mu_1, h_1^t\right)$ and $(\Sigma_2\ , P_2, \mu_2, h_2^t(\eta))$, and the transformation h saves measure $\mu = \mu_1 \times \mu_2$ [11], which is the direct product of probability measures μ_1 and μ_2. On the space $(\Sigma_2, P_2\ , \mu_2)$ we introduce a sequence of random variables $\varpi = (\varpi_1, \varpi_2, \ldots)$, where $\varpi_k\left(\varphi\right) = \varphi_k, \varphi_k \in \Omega$. If the equalities are satisfied (8), then the sequence ϖ forms a homogeneous Markov chain, which is stationary in the narrow sense.

In this work we assume that the Markov chain ϖ is irreducible (indecomposable) and positively reflexive. This means that all states of the Markov chain form one class of reported return states and average return time into each of these states is finite [12].

Let $\zeta(\sigma) = \varphi_0$- be a random variable on the probability space $(\Sigma\ , P, \mu)$. Let's define a random process

$$\zeta\left(h^t\sigma\right) \doteq \left(A\left(h^t\sigma\right), B\left(h^t\sigma\right)\right),$$

then for each it is fixed $\sigma \in \Sigma$ then for every fixed $\sigma \in \Sigma$ the function is piecewise constant and takes values in the set ζ. Function $\zeta(t, \sigma) = \zeta\left(h^t\sigma\right)$ is a stationary, in the narrow sense random process. This means that all finite-dimensional distributions of a given process are invariant with respect to the shift in parameter t, that is equality

$$\mu\left\{\zeta\left(t_1 + t\right) \in B_1, \dots\ , \zeta\left(t_k + t\right) \in B_k\right\} = \mu\left\{\ \zeta\left(t_1\right) \in B_1,\ \dots,\ \zeta\left(t_k\right) \in B_k\right\}$$

satisfied for any $k \in N$, random moments in time $t, t_1, \dots, t_k$ and any Borel sets $B_1, \dots\ , B_k$ [12, 13].

We assume that the random variables $\eta_1, \eta_2, \dots$ have a distribution function $G(t)$ that satisfies the following condition.

Condition 2. *(i)* $G\left(t\right) = 0 \quad at \quad t \geq 0, E_\eta \doteq \int_0^\infty t dG\left(t\right) < +\infty;$

(ii) There are such constants $a > 0,\ C \geq 0$ *and* $\delta > 0,$ *suah*

$$G\left(t\right) \leq Ct^a \ at\ t \in \left(0, \delta\right).$$

If Condition 1 is satisfied, then there is a set $\Sigma_0 \subseteq \Sigma$ *such that* $\mu\left(\Sigma_0\right) = 1$ *and for any* $\sigma \in \Sigma$ *switching moments* $\gamma_1, \gamma_2, \dots$ *random process* $\zeta\left(h^t\sigma\right)$ *isolated and the number of these moments is infinite [10].*

Let's consider the intervals $[\gamma_1, \gamma_2), [\gamma_2, \gamma_3), \dots$ *, whose lengths* $\gamma_1, \gamma_2, \dots,$ *have a distribution function* $G(t)$*. The distribution* γ_0 *length of the interval* $[0, \gamma_1)$ *is determined by equality (3) and in the general case is different from* $G(t)$*, but since the limits are* $\wp_*\left(\sigma, R_M, M\right)$ *and* $\wp^*(\sigma, R_M, M)$ *are do not depend on the behavior of the system on* $[0, \gamma_1)$*, we will not consider this interval further. From intervals* $[\gamma_1, \gamma_2), [\gamma_2, \gamma_3), \dots$ *, let us choose those on which the system* ζ *is in the state* φ_i*, we denote them* $J_{1i}, J_{2i}, \dots$*. Let* $a_i > 0$ *be a fixed number. Let us introduce random variables* η_{ki} *where* η_{ki} *is equal to the length of the interval* J_{ki} *and random variables* $c_{ki}, k = 1, 2, \dots,$ *where*

$$c_{ki} = \eta_{ki} - a_i,\ if\ \eta_{ki} > a_i,\ and\ c_{ki} =\ 0,\ if\ \eta_{ki} \leq a_i.$$

Let $\aleph_i = \aleph_i(n)$ *be the number of those intervals from* $[\gamma_1, \gamma_2), \dots, [\gamma_n, \gamma_{n+1})$ *for which the system is in state* φ_i *(here* $n_1 + \dots + n_l$*).*

Lemma 1. *Let Condition 2 be satisfied and the Markov chain* ϖ *be irreducible and positively reciprocal. Then for almost all* $\sigma \in \Sigma$ *the following equalities hold:*

$$\frac{\sum_{k=1}^{n_i} c_{ki}}{\sum_{k=1}^{n_i} \eta_k} = \frac{\pi_i}{E_\eta}\left(\int_{a_i}^\infty t dG\left(t\right) - a_i(1 - G\left(a_i\right))\right), \dots i = 1, \dots\ ,\ l. \tag{11}$$

Proof. By virtue of the ergodic theorem, for an irreducible positive reciprocal Markov chain ϖ, for almost all σ the equalities hold [11]

$$\lim_{n\to\infty} \frac{n_i}{n} = \pi_i.$$

From the strong law of large numbers it follows that for almost all σ the relation holds:

$$\frac{1}{n}\sum_{k=1}^{n} \eta_k = E_\eta \tag{12}$$

Next, for every fixed $i = 1, \dots, l$ random variable $\eta_{ki}, \eta_{k+1,i}$, are independent, this follows from independence η_k, η_{k+1}; also random variables are independent $c_{ki}, c_{k+1,i}$, $k = 1, 2, \dots$. Let us find the mathematical expectation of the random variable c_{ki}, presenting it as a difference

$$\begin{aligned}
&c_{ki} = c^1{}_{ki} - c^2{}_{ki}, \text{where}\\
&c^1{}_{ki} = \eta_{ki}, \quad if \quad \eta_{ki} > a_i, \ and \ \ c_{ki} = \ 0 \ if \ \eta_{ki} \le a_i,\\
&c^2{}_{ki} = a_i, \quad if \quad \eta_{ki} > a_i, \ and \ \ c_{ki} = \ 0 \ if \ \eta_{ki} \le a_i.
\end{aligned}$$

Since the quantities η_{ki} have a distribution function $G(t)$, then

$$K_{c^1{}_{ki}} = \int_{ai}^{\infty} tdG\left(t\right), \quad K_{c^2{}_{ki}} = a_i \left(1 - G\left(a_i\right)\right).$$

$$\frac{1}{n}\sum_{k=1}^{ni} C_{ki} = \pi_i \int_{ai}^{\infty} tdG\left(t\right) - a_i\left(1 - G\left(a_i\right)\right). \tag{13}$$

Equalities (13) follow from (12) and (13).

Let M is non-empty compact subsets of R^n. Let us denote by $R(t, X)$ the reachability set of a stationary linear system $\omega_i = (A_i, \ B_i)\, i = 1, \dots, l.$ at time t from the initial set X, we also introduce the notation.

$$\begin{aligned}
&\lambda_i^1 = \lambda_i^1\left(X, M\right) = min\{\gamma \in [0,\infty) : R_i\left(t, X\right) \subseteq M \ at \ t \ge \gamma\},\\
&\lambda_i^2 = \lambda_i^2\left(X, M\right) = \inf \gamma \in [0,\infty) : R_i\left(t, X\right) \cap M = \emptyset \ at \ t \ge \gamma,\\
&i = 1, \dots, \ \gamma.
\end{aligned}$$

If any of these moments of time does not exist, we put $\lambda_i^1 = \infty, \ or \ \lambda_i^2 = \infty$. The set $\Omega = (t, x) : t > 0, \ x \in X$ is called positively invariant with respect to the system ζ, if $R\left(t, \sigma, X\right) \subseteq X$ for all $t \ge 0$ and $\sigma \in \Sigma$.

Theorem 1. *Let Condition 2 be satisfied and the Markov chain ϖ be irreducible and positively reciprocal; $M \subseteq \ X$ and the set Ω is positively invariant with respect to the system ζ. Then for almost for all $\sigma \in \Sigma$ the following estimates hold:*

$$\wp_*\left(\sigma, R_M, M\right) \ge \frac{1}{E_\eta} \sum_{i}^{\lambda_i^1<\infty} \pi_i \int_{\lambda_i^1}^{\infty} \left(tdG\left(t\right) - \lambda_i^1\left(1 - G\left(\lambda_i^1\right)\right)\right), \tag{14}$$

$$\wp^*(\sigma, R_M, M) \geq 1 - \frac{1}{E_\eta} \sum_i^{\lambda_i^2<\infty} \pi_i \int_{\lambda_i^2}^{\infty} \left(tdG(t) - \lambda_i^2\left(1 - G\left(\lambda_i^2\right)\right)\right). \tag{15}$$

Proof. For a given $\sigma \in \Sigma$, we construct a set $\widetilde{R}(t, \sigma, X)$, which for $t \in [\gamma_k, \gamma_{k+1})$ k=0,1,... . Coincides with sets

$$R_i(t - \gamma_k, X) = R_i(t - \gamma_k, \sigma, X)$$

if the system ζ is in state ω_i which for $t \in [\gamma_k, \gamma_{k+1})$. The set Ω is positively invariant with respect to the system ζ, therefore for the set $M \subseteq X$ there are inclusions $R(t, \sigma, X) \subseteq X$ and $R(t, \sigma, X) \subseteq \widetilde{R}(t, \sigma, X)$.

From the last inclusion it follows the inequalities

$$\begin{aligned}\wp_*(\sigma, R_M, M) &\doteq \lim_{\eta\to\infty} \left(\frac{(mes\ \{t \in [0, \eta] : R(t, \sigma, X) \subseteq M\})}{\eta}\right)\\ &\geq \lim_{\eta\to\infty} \left(\frac{\left(mes\left\{t \in [0, \eta] : \widetilde{R}(t, \sigma, X) \subseteq M\right\}\right)}{\eta}\right),\end{aligned}$$

from the last we got evaluation

$$\begin{aligned}\wp_*(\sigma, R_M, M) &\geq \lim_{\eta\to\infty} \left(\frac{mes\left\{t \in [0, \eta] : \widetilde{R}(t, \sigma, X) \subseteq M\right\}}{\eta}\right)\\ &= \lim_{n\to\infty} \left(\frac{\sum_{i=1}^{\ell} \sum_{k=1}^{ni} C_{ki}}{\sum_{k=1}^{n} \eta_k}\right),\end{aligned} \tag{16}$$

Here are random variables $C_{ki} = \eta_{ki} - \lambda_i^1$ if $\eta_{ki} > 0$ and $C_{ki} = 0$, if $\eta_{ki} \leq \lambda_i^1$, $k = 1, 2, \ldots$ The lower estimate (14) follows from (11) and (16).

To find an upper bound for $\wp^*(\sigma, R_M, M)$ we use the inequality

$$\begin{aligned}\wp^*(\sigma, R_M, M) &\doteq \lim_{\eta\to\infty} \left(\frac{(mes\,\{t \in [0, \eta] : R(t, \sigma, X) \subseteq X - M\})}{\eta}\right) \geq\\ &\geq \lim_{\eta\to\infty} \left(\frac{\left(mes\left\{t \in [0, \eta] : \widetilde{R}(t, \sigma, X) \subseteq X - M\right\}\right)}{\eta}\right).\end{aligned}$$

Let's consider random variables

$\eta_{ki} \doteq \eta_{ki} - \lambda_i^2$, if $\eta_{ki} > \lambda_i^2$, and $\eta_{ki} = 0$ if $\eta_{ki} \leq \lambda_i^2$ where $k = 1, 2, \ldots$.

Then:

$$\lim_{\eta\to\infty} \left(\frac{\left(mes\left\{t \in [0, \eta] : \widetilde{R}(t, \sigma, X) \subseteq X - M\right\}\right)}{\eta}\right) q$$

$$\geq lim_{n\to\infty}\left(\frac{\sum_{i=1}^{l}\sum_{k=1}^{ni}\eta_{ki}}{\sum_{k=1}^{n}\eta_{k}}\right). \tag{17}$$

By (11), for random variables $\eta_{ki},\ k=1,2,\ldots$ equality is true

$$lim_{n\to\infty}\left(\frac{\sum_{k=1}^{ni}\eta_{ki}}{\sum_{k=1}^{n}\eta_{k}}\right)=\frac{1}{E_{\eta}}.\pi_i\int_{\lambda_i^2}^{\infty}\left(tdG\left(t\right)-\lambda_i^2\left(1-G\left(\lambda_i^2\right)\right)\right),\ i=1,\ldots,l \tag{18}$$

Taking into account (17), (18) and the inequality

$$\wp^*\left(\sigma,R_M,M\right)+\lim{}_{\eta\to\infty}\ \left(\frac{\left(mes\left\{t\in[0,\eta]:R\left(t,\sigma,X\right)\subseteq X-M\right\}\right)}{\eta}\right)\leq 1,$$

we obtain estimate (15):

$$\begin{aligned}\wp^*\left(\sigma,R_M,M\right)&\leq 1-\lim_{\eta\to\infty}\left(\frac{\left(mes\left\{t\in[0,\eta]:R\left(t,\sigma,X\right)\subseteq X-M\right\}\right)}{\eta}\right)\\&\leq 1-lim_{n\to\infty}\left(\frac{\sum_{i=1}^{l}\sum_{k=1}^{ni}\eta_{ki}}{\sum_{k=1}^{n}\eta_{k}}\right)\\&\leq 1-\frac{1}{E_{\eta}}\sum_{i}^{\lambda_i^2<\infty}\pi_i\int_{\lambda_i^2}^{\infty}\left(tdG\left(t\right)-\lambda_i^2\left(1-G\left(\lambda_i^2\right)\right)\right).\end{aligned}$$

Example 1. *We will take the following system:*

$$\dot{x}=A(h^t\sigma)x+B(h^t\sigma)u, \tag{19}$$

where:

- $\sigma\in\Sigma=\Sigma_1\times\Sigma_2$ *is the switching parameter,*

$$\Sigma_1=[a,b]^{\mathbb{N}},\quad \theta_k\sim Uniform(a,b)$$

$$\Sigma_2=\{\psi_1,\psi_2\},\quad 0<a<b$$

- $u\in U=[-1,1]$ *is the control input,* $x\in\mathbb{R}^2$ *is the state vector*
- *The parameter* σ *is generated by a metric dynamical system* $(\Sigma,\mathcal{P},\mu,\{h^t\}_{t\in\mathbb{R}})$.
- *The system modes are defined by the matrix pairs:*

$$\begin{aligned}\psi_1&=(A_1,B_1),\\\psi_2&=(A_2,B_2)\end{aligned}$$

For subsystem ψ_1:

$$A_1 = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{20}$$

For subsystem ψ_2:

$$A_2 = \begin{bmatrix} -1 & 2 \\ 0 & -2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{21}$$

We assume the Markov chain transition matrix that

$$P = \begin{pmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{pmatrix} \tag{22}$$

,

and the stationary distribution π satisfies:

$$\pi P = \pi, \implies \pi_1 = \frac{5}{9}, \quad \pi_2 = \frac{4}{9}$$

.

For system (19) *we associate the differential inclusion $x \in F(h^t\sigma, x)$, which was defined in* (3) *so*

$$g(h^t\sigma, x, U) = A(h^t\sigma)x + B(h^t\sigma)U. \tag{23}$$

Now let the Lyapunov function be

$$V(\sigma, x) = 2x_1^2 + x_2^2 - \frac{1}{4} \tag{24}$$

We will calculate the upper derivative analysis for Lyapunov function:

For $\sigma \in \Sigma_1$:

$$\dot{V}_{max} = \sup_{u \in U} \dot{V} \leq -6x_1^2 - 2x_2^2 + 2|x_1| \tag{25}$$

For $\sigma \in \Sigma_2$:

$$\dot{V}_{max} \leq -2x_1^2 - 4x_2^2 + 4|x_2|$$

For subsystem ψ_1

$$\dot{V}_{max} \leq -6x_1^2 - 2x_2^2 + 2|x_1| \leq -3V + 0.5$$

$$\implies a_1 = 3, \quad b_1 = 0.5$$

For Subsystem ψ_2

$$\dot{V}_{max} \leq -2x_1^2 - 4x_2^2 + 4|x_2| \leq -2V + 1$$

$$\implies a_2 = 2, \quad b_2 = 1$$

Assuming exponentially distributed dwell times η_k with rate parameter λ:

$$G(t) = 1 - e^{-\lambda t}, \quad \textit{density} \quad g(t) = \lambda e^{-\lambda t}$$

and we note expected Value;

$$\mathbb{E}[\eta] = \int_0^\infty t g(t) dt = \frac{1}{\lambda}. \tag{26}$$

And Integral Computations

$$\int_{a_i}^\infty t dG(t) = \int_{a_i}^\infty t\lambda e^{-\lambda t} dt = e^{-\lambda a_i}\left(a_i + \frac{1}{\lambda}\right) \tag{27}$$

$$a_i(1 - G(a_i)) = a_i e^{-\lambda a_i}$$

$$\int_{a_i}^\infty t dG(t) - a_i(1 - G(a_i)) = \frac{e^{-\lambda a_i}}{\lambda} \tag{28}$$

Next, we find frequency estimation using Theorem (1).

$$\wp_*(\sigma, R_M, M) \geq \frac{1}{\mathbb{E}[\eta]} \sum_{i=1}^2 \pi_i \left(\frac{e^{-\lambda a_i}}{\lambda}\right) \tag{29}$$

Substituting $\mathbb{E}[\eta] = \frac{1}{\lambda}$:

$$\wp_* \geq \lambda \sum_{i=1}^2 \pi_i \left(\frac{e^{-\lambda a_i}}{\lambda}\right) = \sum_{i=1}^2 \pi_i e^{-\lambda a_i} \tag{30}$$

so the numerical result with $\lambda = 0.5$, $a = 1$, $b = 3$:

$$\wp_* \geq \frac{5}{9} e^{(-0.5)\cdot 3} + \frac{4}{9} e^{(-0.5)\cdot 2} = \frac{5}{9} e^{-1.5} + \frac{4}{9} e^{-1} \approx 0.084. \tag{31}$$

The distribution function $G(t)$ is not just a statistical parameter, but a tool to model the behavior of time in dynamic systems, and its impact on stability estimates can exceed the impact of the stability coefficients themselves. In the second part, we will study the impact of the distribution function on the dynamic system for more than one distribution.

6. The role of the distribution function in the dynamic system

Next, we will choose two different distributions to illustrate the role of the distribution function on the stability of random dynamic systems: the exponential distribution and the Pareto distribution. We will make some modifications to the matrices, Lipunov function, and intervals from the example above to simplify the calculations.

6.1. Sample Space

$$\Sigma = \Sigma_1 \times \Sigma_2, \quad \Sigma_1 = [0.5, 2]^{\mathbb{N}}, \quad \theta_k \sim \text{Uniform}(0.5, 2) \tag{32}$$
$$\Sigma_2 = \{\psi_1, \psi_2\}$$

- *Subsystem Matrices*:

  For subsystem ψ_1:
  $$A_1 = \begin{pmatrix} -4 & 1 \\ 0 & -2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
  For subsystem ψ_2:
  $$A_2 = \begin{pmatrix} -1 & 0 \\ 3 & -3 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- Control Set
  $$U = [-0.8, 0.8]$$

- Markov Transition Matrix
  $$P = \begin{pmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{pmatrix}$$
  -The stationary distribution π:
  $$\pi_1 = 0.75, \quad \pi_2 = 0.25$$

- Novel Lyapunov Function
  $$V(\sigma, x) = 3x_1^2 + 2x_2^2 - \frac{1}{9} \tag{33}$$

- Stability Constants
  $$\begin{aligned} &\text{Subsystem } \psi_1: \quad a_1 = 2.5, \quad b_1 = 0.3 \\ &\text{Subsystem } \psi_2: \quad a_2 = 1.8, \quad b_2 = 0.6 \end{aligned}$$

6.2. Mode-Specific Distribution Functions

- Exponential Distribution for ψ_1
  $$G_1(t) = 1 - e^{-\lambda_1 t}, \quad g_1(t) = \lambda_1 e^{-\lambda_1 t}, \quad \lambda_1 = 0.6 \tag{34}$$
  Mean dwell time: $\mathbb{E}[\eta_1] = \lambda_1^{-1} \approx 1.67$ time units

- Pareto Distribution for ψ_2
  $$G_2(t) = 1 - \left(\frac{x_m}{t}\right)^{\alpha} \text{ for } t \geq x_m, \quad g_2(t) = \frac{\alpha x_m^{\alpha}}{t^{\alpha+1}} \tag{35}$$
  Parameters: $x_m = 0.7$, $\alpha = 2.5$ (heavy-tailed distribution)

6.3. Integral Computations

- Exponential Case (ψ_1)

$$\begin{aligned}\int_{a_1}^{\infty} t dG_1(t) - a_1(1 - G_1(a_1)) &= \frac{e^{-\lambda_1 a_1}}{\lambda_1} \\ &= \frac{e^{-0.6\times 2.5}}{0.6} \approx 0.3717\end{aligned}$$

- Pareto Case (ψ_2)

$$\begin{aligned}\int_{a_2}^{\infty} t dG_2(t) &= \frac{\alpha x_m^{\alpha}}{(\alpha-1)a_2^{\alpha-1}} \\ &= \frac{2.5 \cdot 0.7^{2.5}}{1.5 \cdot 1.8^{1.5}} \approx 0.892\end{aligned}$$

$$\begin{gathered}a_2(1 - G_2(a_2)) = \\ 1.8 \cdot \left(\frac{0.7}{1.8}\right)^{2.5} \approx 0.0846\end{gathered}$$

Required value & $\approx$ 0.892 - 0.0846 = 0.8074

6.4. Frequency Estimation

We compute the expected dwelling time.

$$\begin{aligned}&\mathbb{E}[\eta_1] = \lambda_1^{-1} = 1.67, \quad \mathbb{E}[\eta_2] = \frac{\alpha x_m}{\alpha - 1} = \frac{2.5 \cdot 0.7}{1.5} \approx 1.1667 \\ &\mathbb{E}[\eta] = \pi_1 \mathbb{E}[\eta_1] + \pi_2 \mathbb{E}[\eta_2] = 0.75 \times 1.67 + 0.25 \times 1.1667 \approx 1.5442\end{aligned} \tag{36}$$

we note that Frequency bound

$$\begin{aligned}\wp_* &\geq \frac{1}{\mathbb{E}[\eta]} \left[\pi_1 \cdot 0.3717 + \pi_2 \cdot 0.8074\right] \\ &= \frac{0.75 \times 0.3717 + 0.25 \times 0.8074}{1.5442} \\ &\approx \frac{0.4807}{1.5442} \approx 0.3113\end{aligned} \tag{37}$$

Now we analyze the impact of the distribution in Table 1.

6.5. Theoretical and Practical Implications

Next, we note the heavy-tail effect analysis.

Table 1: Comparative impact of distribution tails on stability

Criterion	**Subsystem ψ_1**	**Subsystem ψ_2**
Stability constant a_i	$a_1 = 2.5$ (higher)	$a_2 = 1.8$ (lower)
Distribution type	Exponential ($\lambda_1 = 0.6$)	Pareto ($x_m = 0.7, \alpha = 2.5$)
Probability tail	Light (fast decay)	Heavy (slow decay)
Integral contribution	0.3717	0.8074
Frequency impact (stability vs dwell time)	Lower-higher stability	Higher-lower stability

(i) **Pareto distribution advantage**: Yields higher $\int_{a_i}^{\infty} tdG(t)$ values due to greater probability of long dwell times, compensating for ψ_2's lower stability ($a_2 < a_1$)

(ii) **Distribution shape comparison**:

- *Exponential*:

$$\int_{a_i}^{\infty} tdG(t) - a_i(1 - G(a_i)) = \frac{e^{-\lambda a_i}}{\lambda} \text{ (rapid decay)} \tag{38}$$

- *Pareto*:

$$\int_{a_i}^{\infty} tdG(t) - a_i(1 - G(a_i)) \approx \frac{\alpha x_m^{\alpha}}{(\alpha - 1)a_i^{\alpha-1}} - a_i \left(\frac{x_m}{a_i}\right)^{\alpha} \text{ (slow decay)} \tag{39}$$

(iii) **Stability-dwell time tradeoff**: Using exponential for ψ_2 would yield: Value $= \frac{e^{-0.6\times1.8}}{0.6} \approx 0.5667 \implies \wp_* \approx 0.2852 < 0.3113$

6.6. Practical applications

6.6.1. Power grid control systems

- ψ_1: Normal operation (high stability, short-term faults)
- ψ_2: Fault mode (lower stability, persistent faults)
- Pareto distribution increases safe operation estimate by 9.1%

6.6.2. Autonomous vehicles

- ψ_1: Smooth road conditions (high stability)
- ψ_2: Rough terrain (lower stability but longer dwell times)

6.7. Choosing a distribution based on usage

- **Use heavy-tailed distributions (Pareto, logistic) for systems**:
  - (i) Subject to long-term failures
  - (ii) With unpredictable operating environments
- **Use exponential distributions for systems:**
  - (i) Subject to short-term failures
  - (ii) With controlled environments.

7. Implications of the study

The findings of the investigation have several significant implications for controlled linear systems with random parameters theory as well as real-world applications:

7.1. Enhanced understanding of reachability sets

The research offers a more profound comprehension of reachability sets' statistical behavior in systems susceptible to stochastic impacts. This knowledge is essential for developing control techniques in complex systems, where stochastic components must be considered, and deterministic models are inadequate.

7.2. Improved control strategy design

New tools for control system designers are provided by the statistical invariance analysis and the introduction of relative absorption frequency. With these tools, more resilient control techniques that guarantee system performance may be developed even in the face of uncertainty. It is imperative in domains where unpredictability is prevalent, such as robotics, aircraft, and networked control systems.

7.3. Applications in randomized systems

The research bridges the gap between deterministic and stochastic control theory by transforming the controlled system into one with random coefficients. This method may be used for various real-world systems where the parameters, such as financial, biological, and climatic models, are unknown by nature and change over time.

7.4. Ergodic Theorem and Markov processes in system analysis

Utilizing the ergodic theorem and Markov process features in this particular situation creates novel opportunities for examining the long-term behavior of systems. Better forecasts and optimization in systems that change over time can result from this, increasing the likelihood that intended outcomes will be realized.

7.5. Stationarity and consistency in restoration processes

The results of the restoration process suggest that some random processes can stabilize over time, which is necessary to guarantee consistency and dependability in system functionality. It has real-world implications for the upkeep and functioning of systems like electricity grids and communication networks that must recover from disruptions or disturbances.

7.6. Broader applicability to stochastic control

This study's ideas and techniques can be widely applied to various fields of stochastic control. These discoveries have the potential to be used by scholars and professionals in a variety of domains where control under uncertainty is crucial, including robotics, environmental management, and energy systems.

Beyond the particular system under analysis, this study's implications are broad and helpful, providing tools and insights for various applications where control under uncertainty is a primary issue.

8. Conclusions

The study looked at the reachability set $R(t, \sigma, X)$ statistical properties for a family of linear systems under control that were parameterized by a metric dynamical system. The investigation concentrated on the system that the differential inclusion depicted, and it produced several significant findings that may be summed up as follows:

(i) Relative Absorption Frequency: The study introduced and analyzed the concept of relative absorption frequency and established conditions under which the reachability set $R(t, \sigma, X)$ is absorbed by a given set M. The study also defined and computed the upper and lower relative absorption frequencies for cases where the limit does not exist.

(ii) Statistical Invariance: The study provided enough requirements for a set $M(\sigma)$ to be statistically invariant concerning the controlled system. The study proved that if the relative frequency of absorption is equal to one, the set $M(\sigma)$ is statistically invariant, meaning that the reachability set is very certainly absorbed by $M(\sigma)$.

(iii) Probability Space and Random Processes: The study changed the controlled system to one with random coefficients by establishing a probability space $(\Sigma\ , P, \mu)$ as the direct product of two probability spaces. Given the irreducibility and positive recurrence criteria, it was demonstrated that the system may be described as a stable random process in the narrow sense. We next looked at the characteristics of this random process.

(iv) Ergodic Theorem and Law of Large Numbers: The study obtained significant equalities and relations for the Markov chain ϖ by utilizing the ergodic theory and the strong law of large numbers. These were especially relevant for the ratios of random variables that correspond to the durations of intervals in which the system is in a particular state.

(v) Restoration Process: The analysis of the restoration process linked to the random variable ζ_0 revealed that it functions as a stationary process with a steady recovery rate. This feature was essential for determining the system's statistical makeup and guaranteeing the reachability set's long-term stability.

The results have implications for designing and analyzing systems where uncertainty and stochastic elements play a significant role, especially in assuring the robustness and reliability of the control strategies employed. The study offers a rigorous framework for understanding the statistical behavior of reachability sets in controlled linear systems with random parameters.

9. Limitations of the study

Although this work has some limitations, it substantially contributes to analyzing control systems with random parameters. The main drawback is that the controlled systems' assumed linearity may not adequately represent the complexity of nonlinear systems present in many real-world applications. Furthermore, even though the work uses Markov processes, ergodic theorems, and novel metrics, these techniques might need to be improved to handle situations with more complicated or unpredictable unpredictability. Furthermore, the theoretical framework's actual application could be limited by the requirement for significant processing power, mainly when working with high-dimensional systems or when parameter randomness deviates from the idealized circumstances in this study.

10. Future scope

Building on our earlier work on BCH and alternant codes over Gaussian, Eisenstein, and quaternion integer structures [14–17], several promising directions emerge. The framework can be extended to nonlinear control systems and more complex forms of randomness, broadening its applicability to real-world scenarios. Integrating machine learning techniques may lead to adaptive control strategies suitable for dynamic environments with evolving uncertainties. Further exploration of efficient computational methods is important for handling high-dimensional systems without excessive cost. Finally, practical validation across industrial and engineering applications is essential to demonstrate the robustness and usefulness of the proposed theoretical results for advanced control and communication systems.

Declaration of competing interest

The authors declare no conflict of interest.

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Data availability

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