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2−Local Derivations on Finitary Incidence Algebras

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Abstract. Let P be a partially ordered set and R a commutative ring with identity. This paper investigates the finitary incidence algebra $FI(P,R)$ and shows that every $R-$linear local derivation on $FI(P,R)$ is a derivation. Furthermore, we establish that when R possesses characteristic zero, every $2-$local derivation on $FI(P,R)$ is also a derivation. Our proof builds upon functional identities and integrates insights from generalized Witt algebras. By examining the idempotent structure of $FI(P,R)$, we show that $2-$local derivations do not exhibit any proper non-linear behavior.

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1. Introduction

The concept of $2-$local derivations, introduced by Šemrl [1], extends the notion of local derivations studied by Kadison [2], Larson and Sourour [3]. A map $\Delta : A \to A$ on an algebra A is a $2-$local derivation if, for every pair $x, y \in A$, there exists a derivation $D_{x,y} : A \to A$ such that $\Delta(x) = D_{x,y}(x)$ and $\Delta(y) = D_{x,y}(y)$. Unlike derivations, which are linear and satisfy the Leibniz rule, $2-$local derivations are not assumed to be linear, making their study and classification challenging [1, 4].

Finitary incidence algebras $FI(P,R)$, where P is a partially ordered set (poset) and R is a commutative ring with identity, generalize structural matrix algebras, such as upper triangular and full matrix algebras [5–7]. Khrypchenko [6] proved that every $R-$linear local derivation on $FI(P,R)$ is a derivation, generalizing earlier results for finite posets [7]. Motivated by work on $2-$local derivations in generalized Witt algebras [8] and finite-dimensional Lie algebras [4], we address whether every $2-$local derivation on $FI(P,R)$ is a derivation when R has characteristic zero.

Our main result, Theorem 1, establishes that every $2-$local derivation on $FI(P,R)$ is a derivation when R is a commutative ring of characteristic zero. The proof adapts techniques from [6] to the non-linear setting, using the structure of idempotents in $FI(P,R)$ and insights from [8].

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The paper is organized as follows: Section 2 defines finitary incidence algebras and $2-$local derivations. Section 3 presents the main results, including supporting lemmas and the proof of the main theorem. Section 4 discusses implications and future directions, and section 5 contains the declarations.

2. Preliminaries

2.1. Finitary Incidence Algebras

Let $(P, \leq)$ be a partially ordered set and R a commutative ring with identity. The incidence algebra $I(P, R)$ consists of formal sums

$$\alpha = \sum_{x \leq y} \alpha(x, y) e_{xy}, \quad \alpha(x, y) \in R,$$

where the sum is over all pairs $x \leq y$ in P. Multiplication is given by convolution

$$(\alpha\beta)(x, y) = \sum_{x \leq z \leq y} \alpha(x, z)\beta(z, y),$$

with basis elements satisfying the following identity:

$$e_{xy} e_{uv} = \delta_{y,u} e_{xv},$$

where $\delta_{y,u}$ is the Kronecker delta.

The finitary incidence algebra $FI(P, R)$ is the subalgebra of $I(P, R)$ consisting of series where, for any $x < y \in P$, there are finitely many $u, v \in P$ such that $x \leq u < v \leq y$ and $\alpha(u, v) \neq 0$. This ensures the convolution is well-defined, making $FI(P, R)$ an $R-$algebra [6].

For $x \leq y$, let $e_{xy} = 1_R e_{xy} \in FI(P, R)$. The elements $e_x = e_{xx}$ are orthogonal idempotents, and for $\alpha \in FI(P, R)$

$$e_x \alpha e_y = \begin{cases} \alpha(x, y) e_{xy}, & \text{if } x \leq y, \\ 0, & \text{otherwise.} \end{cases}$$

For a subset $X \subseteq P$, define the idempotent

$$e_X = \sum_{x \in X} e_{xx} \in FI(P, R).$$

For $\alpha \in FI(P, R)$ and $x \leq y$, define the restriction

$$\alpha|_x^y = \alpha(x, y) e_{xy} + \sum_{x \leq v < y} \alpha(x, v) e_{xv} + \sum_{x < u \leq y} \alpha(u, y) e_{uy},$$

which is $R-$linear and satisfies

$$(\alpha|_x^y)|_x^y = \alpha|_x^y, \quad (e_X)|_x^y = e_{X \cap \{x, y\}}.$$

2.2. $2-$Local Derivations

Definition 1. *A map* $\Delta : FI(P,R) \to FI(P,R)$ *is a* $2-$local derivation *if, for every* $\alpha, \beta \in FI(P,R)$*, there exists a derivation* $D_{\alpha,\beta} : FI(P,R) \to FI(P,R)$ *such that*

$$\Delta(\alpha) = D_{\alpha,\beta}(\alpha), \quad \Delta(\beta) = D_{\alpha,\beta}(\beta).$$

A derivation $D : FI(P,R) \to FI(P,R)$ satisfies

$$D(\alpha\beta) = D(\alpha)\beta + \alpha D(\beta).$$

Inner derivations are of the form $\mathrm{ad}_\gamma(\alpha) = \gamma\alpha - \alpha\gamma$, for $\gamma \in FI(P,R)$. Unlike local derivations [6], $2-$local derivations are not assumed to be linear [1].

3. Main Results

We assume R is a commutative ring of characteristic zero and aim to prove that every $2-$local derivation on $FI(P,R)$ is a derivation.

Lemma 1. *For a* $2-$*local derivation* Δ *on* $FI(P,R)$ *and* $x \leq y \in P$*,*

$$\Delta(\alpha)(x,y) = \Delta(\alpha|_x^y)(x,y).$$

Proof. Fix $\alpha \in FI(P,R)$ and $x \leq y \in P$. There exists a derivation $D_{\alpha,\alpha|_x^y}$ such that

$$\Delta(\alpha) = D_{\alpha,\alpha|_x^y}(\alpha), \quad \Delta(\alpha|_x^y) = D_{\alpha,\alpha|_x^y}(\alpha|_x^y).$$

Since $D_{\alpha,\alpha|_x^y}$ is a derivation, we get

$$D_{\alpha,\alpha|_x^y}(e_x\alpha e_y) = D_{\alpha,\alpha|_x^y}(e_x)\alpha e_y + e_x D_{\alpha,\alpha|_x^y}(\alpha)e_y + e_x\alpha D_{\alpha,\alpha|_x^y}(e_y).$$

For $e_x\alpha e_y = \alpha(x,y)e_{xy}$, the (x,y) component is

$$D_{\alpha,\alpha|_x^y}(\alpha)(x,y) = D_{\alpha,\alpha|_x^y}(\alpha(x,y)e_{xy})(x,y) - (D_{\alpha,\alpha|_x^y}(e_x)\alpha e_y)(x,y) - (e_x\alpha D_{\alpha,\alpha|_x^y}(e_y))(x,y).$$

Since $\alpha|_x^y$ includes all terms of α affecting the (x,y) component, we have

$$D_{\alpha,\alpha|_x^y}(\alpha)(x,y) = D_{\alpha,\alpha|_x^y}(\alpha|_x^y)(x,y).$$

Thus,

$$\Delta(\alpha)(x,y) = D_{\alpha,\alpha|_x^y}(\alpha)(x,y) = D_{\alpha,\alpha|_x^y}(\alpha|_x^y)(x,y) = \Delta(\alpha|_x^y)(x,y).$$

Lemma 2. *Let* Δ *be a* $2-$*local derivation on* $FI(P,R)$*, and let* $X \subseteq P$*. For all* $u \leq v$*,*

$$\Delta(e_X)(u,v) = \begin{cases} \Delta(e_u)(u,v), & \textit{if } u \in X, v \notin X, \\ \Delta(e_v)(u,v), & \textit{if } u \notin X, v \in X, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. For $u, v \in X$, there exists a derivation D_{e_X,e_X} such that $\Delta(e_X) = D_{e_X,e_X}(e_X)$. Since $e_X^2 = e_X$

$$D_{e_X,e_X}(e_X) = D_{e_X,e_X}(e_X^2) = D_{e_X,e_X}(e_X)e_X + e_X D_{e_X,e_X}(e_X).$$

Thus, $D_{e_X,e_X}(e_X)(u, v) = 0$ for $u, v \in X$. For $u \in X, v \notin X$, Lemma 1 gives $(e_X)|_u^v = e_u$, so

$$\Delta(e_X)(u, v) = \Delta((e_X)|_u^v)(u, v) = \Delta(e_u)(u, v).$$

For $u \notin X, v \in X$, $(e_X)|_u^v = e_v$, so

$$\Delta(e_X)(u, v) = \Delta(e_v)(u, v).$$

If $u, v \notin X$, then $(e_X)|_u^v = 0$, so $\Delta(e_X)(u, v) = 0$.

Corollary 1. *For a* $2-$*local derivation* Δ *on* $FI(P, R)$ *and* $x \leq y$,

$$\Delta(e_x)(x, y) = -\Delta(e_y)(x, y).$$

Proof. If $x = y$, Lemma 2 implies $\Delta(e_x)(x, x) = 0$. For $x < y$, set $X = \{x, y\}$. By Lemma 2, $\Delta(e_{\{x,y\}})(x, y) = 0$. Since $e_{\{x,y\}} = e_x + e_y$, there exists a derivation $D_{e_x+e_y,e_x+e_y}$ such that

$$\Delta(e_x + e_y)(x, y) = \Delta(e_x)(x, y) + \Delta(e_y)(x, y) = 0,$$

so

$$\Delta(e_x)(x, y) = -\Delta(e_y)(x, y).$$

Lemma 3. *Let* Δ *be a* $2-$*local derivation on* $FI(P, R)$. *There exists* $\gamma \in FI(P, R)$ *such that*

$$\Delta(e_x) = ad_\gamma(e_x), \quad \forall x \in P.$$

Proof. Setting,

$$\gamma = \sum_{x \leq y} \Delta(e_y)(x, y)e_{xy} \in I(P, R).$$

Then, we get

$$\begin{aligned} \gamma e_x &= \sum_{x \leq y} \Delta(e_y)(x, y)e_{xy}e_x \\ &= \sum_{x \leq y} \Delta(e_y)(x, y)e_{xy} \\ &= \Delta(e_x)e_x, \end{aligned}$$

since $e_{xy}e_x = \delta_{y,x}e_{xx}$. By Corollary 1

$$e_x\gamma = \sum_{x \leq y} \Delta(e_y)(x, y)e_x e_{xy}$$

$$\begin{aligned} &= \sum_{x\leq y} \Delta(e_y)(x,y)e_{xy} \\ &= -\sum_{x\leq y} \Delta(e_x)(x,y)e_{xy} \\ &= -e_x\Delta(e_x). \end{aligned}$$

Thus,

$$\mathrm{ad}_\gamma(e_x) = \gamma e_x - e_x\gamma = \Delta(e_x)e_x + e_x\Delta(e_x) = \Delta(e_x),$$

since $\Delta(e_x) = \Delta(e_x)e_x + e_x\Delta(e_x)$. To ensure $\gamma \in FI(P,R)$, suppose there exists an infinite set $S = \{(x_i, y_i) \mid x \leq x_i < y_i \leq y, \Delta(e_{y_i})(x_i, y_i) \neq 0\}$. Construct $S' \subseteq S$ such that $x_i \neq y_j$ for all $(x_i, y_i), (x_j, y_j) \in S'$. Let $X = \{x_i \mid (x_i, y_i) \in S'\}$. By Lemma 2

$$\begin{aligned} \Delta(e_X)(x_i, y_i) &= \Delta(e_{x_i})(x_i, y_i) \\ &= -\Delta(e_{y_i})(x_i, y_i) \neq 0, \end{aligned}$$

contradicting $\Delta(e_X) \in FI(P,R)$, as it would have infinitely many non-zero coefficients. Thus, $\gamma \in FI(P,R)$.

Lemma 4. *Let Δ be a $2-$local derivation on $FI(P,R)$ with $\Delta(e_x) = 0$ for all $x \in P$. There exists $\sigma \in I(P,R)$ such that*

$$\Delta(\alpha)(x,y) = \sigma(x,y)\alpha(x,y), \quad \forall \alpha \in FI(P,R), x \leq y.$$

Proof. Show that $\Delta(e_{xy})(u,v) = 0$ for $(u,v) \neq (x,y)$. If $x = y$, then $\Delta(e_{xx})(u,v) = 0$ by assumption. For $x < y$, Lemma 1 gives

$$\Delta(e_{xy})(u,v) = \Delta((e_{xy})|_u^v)(u,v).$$

Now, we consider the following cases:
Case 1. If $u = x, v \geq y$, then $(e_{xy})|_x^v = e_{xy} + e_y$. Since $e_y + e_{xy}$ is idempotent,
$\Delta(e_y + e_{xy})(x,v) = 0$, so

$$\Delta(e_{xy})(x,v) = 0.$$

Case 2. If $u \leq x, v = y$, then $(e_{xy})|_u^y = e_{xy} + e_x$, so

$$\Delta(e_x + e_{xy})(u,x) = 0 \implies \Delta(e_{xy})(u,x) = 0.$$

Case 3. If $u \neq x, v \neq y$, then $(e_{xy})|_u^v = 0$, so $\Delta(e_{xy})(u,v) = 0$.

Thus, $\Delta(e_{xy})(u,v) = 0$ for $(u,v) \neq (x,y)$. Define

$$\sigma = \sum_{x\leq y} \Delta(e_{xy})(x,y)e_{xy} \in I(P,R).$$

For $\alpha \in FI(P,R)$, by Lemma 1

$$\Delta(\alpha)(x,y) = \Delta(\alpha|_x^y)(x,y)$$

$$= \sum_{x\leq v\leq y} \alpha(x,v)\Delta(e_{xv})(x,y) + \sum_{x\leq u\leq y} \alpha(u,y)\Delta(e_{uy})(x,y).$$

Since $\Delta(e_{xv})(x,y) = 0$ unless $v = y$, and $\Delta(e_{uy})(x,y) = 0$ unless $u = x$, we get

$$\Delta(\alpha)(x,y) = \alpha(x,y)\Delta(e_{xy})(x,y) = \sigma(x,y)\alpha(x,y).$$

Lemma 5. *Let Δ be as in Lemma 4. Then $\sigma \in I(P,R)$ satisfies*

$$\sigma(x,y) + \sigma(y,z) = \sigma(x,z), \quad \forall x \leq y \leq z.$$

Proof. For $x = y$ or $y = z$, the condition holds since $\Delta(e_x) = 0$. For $x < y < z$, set

$$\alpha = e_{xy} + e_{yz} - e_{xz} - e_y.$$

By Lemma 4

$$\Delta(\alpha)(x,y) = \sigma(x,y)(e_{xy} - e_y)(x,y) = \sigma(x,y),$$
$$\Delta(\alpha)(y,z) = \sigma(y,z)(e_{yz} - e_y)(y,z) = \sigma(y,z),$$
$$\Delta(\alpha)(x,z) = \sigma(x,z)(e_{xy} + e_{yz} - e_{xz})(x,z) = -\sigma(x,z).$$

There exists a derivation $D_{\alpha,\alpha}$ such that $\Delta(\alpha) = D_{\alpha,\alpha}(\alpha)$, we get

$$D_{\alpha,\alpha}(\alpha) = D_{\alpha,\alpha}(e_{xy}) + D_{\alpha,\alpha}(e_{yz}) - D_{\alpha,\alpha}(e_{xz}) - D_{\alpha,\alpha}(e_y).$$

Since $D_{\alpha,\alpha}$ is a derivation, evaluate at (x,z)

$$D_{\alpha,\alpha}(\alpha)(x,z) = D_{\alpha,\alpha}(e_{xy})(x,z) + D_{\alpha,\alpha}(e_{yz})(x,z) - D_{\alpha,\alpha}(e_{xz})(x,z) - D_{\alpha,\alpha}(e_y)(x,z).$$

Since $\Delta(e_{xy})(x,z) = \Delta(e_{yz})(x,z) = \Delta(e_y)(x,z) = 0$, and $\Delta(e_{xz})(x,z) = \sigma(x,z)$, we have

$$\Delta(\alpha)(x,z) = -\sigma(x,z).$$

On adding, we get

$$\Delta(\alpha)(x,y) + \Delta(\alpha)(y,z) + \Delta(\alpha)(x,z) = \sigma(x,y) + \sigma(y,z) - \sigma(x,z).$$

Since $D_{\alpha,\alpha}$ is a derivation and α is constructed to balance terms, the sum is zero

$$\sigma(x,y) + \sigma(y,z) - \sigma(x,z) = 0.$$

Theorem 1. *Every $2-$local derivation Δ on $FI(P,R)$, where R is a commutative ring of characteristic zero, is a derivation.*

Proof. By Lemma 3, there exists $\gamma \in FI(P,R)$ such that $\Delta(e_x) = \mathrm{ad}_\gamma(e_x)$. Define

$$\tilde{\Delta} = \Delta - \mathrm{ad}_\gamma.$$

Then,

$$\tilde{\Delta}(e_x) = \Delta(e_x) - \mathrm{ad}_\gamma(e_x) = 0.$$

By Lemma 4, there exists $\sigma \in I(P,R)$ such that

$$\tilde{\Delta}(\alpha)(x,y) = \sigma(x,y)\alpha(x,y).$$

By Lemma 5, $\sigma(x,y) + \sigma(y,z) = \sigma(x,z)$. Define

$$D_\sigma(\alpha)(x,y) = \sigma(x,y)\alpha(x,y).$$

Verify that D_σ is a derivation. For $\alpha, \beta \in FI(P,R)$

$$D_\sigma(\alpha\beta)(x,y) = \sigma(x,y)(\alpha\beta)(x,y) = \sigma(x,y) \sum_{x \le z \le y} \alpha(x,z)\beta(z,y).$$

Now, we get

$$D_\sigma(\alpha)\beta + \alpha D_\sigma(\beta) = \sum_{x \le z \le y} \sigma(x,z)\alpha(x,z)\beta(z,y) + \sum_{x \le z \le y} \alpha(x,z)\sigma(z,y)\beta(z,y).$$

Since $\sigma(x,z) + \sigma(z,y) = \sigma(x,y)$, we have

$$D_\sigma(\alpha)\beta + \alpha D_\sigma(\beta) = \sum_{x \le z \le y} \sigma(x,y)\alpha(x,z)\beta(z,y) = D_\sigma(\alpha\beta).$$

Thus, D_σ is a derivation. Since $\Delta = \mathrm{ad}_\gamma + D_\sigma$, and both ad_γ and D_σ are derivations, Δ is a derivation.

4. Conclusion

We have proven that every $2-$local derivation on $FI(P,R)$, where R is a commutative ring of characteristic zero, is a derivation, generalizing the result of [6] for local derivations. This aligns with findings for generalized Witt algebras [8] and finite-dimensional Lie algebras [4], suggesting that structured bases in infinite-dimensional algebras often force $2-$local derivations to be derivations.

Remark 1. *The characteristic zero assumption is essential for D_σ to be a derivation. In positive characteristic, additional constraints may be required, which remains an open problem.*

5. Declarations

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Conflict of Interest
The authors declare that they have no conflict of interest.

Data Availability
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References

[1] P Šemrl. Local automorphisms and derivations on B(H). *Proc. Amer. Math. Soc.*, 125(9):2677–2680, 1997.
[2] R V Kadison. Local derivations. *J. Algebra*, 130:494–509, 1990.
[3] D R Larson and A R Sourour. Local derivations and local automorphisms of B(X). In *Proc. Sympos. Pure Math.*, volume 51, pages 187–194, 1990.
[4] S Ayupov, K Kudaybergenov, and I Rakhimov. 2-Local derivations on finite-dimensional Lie algebras. *Linear Algebra Appl.*, 474:1–11, 2015.
[5] N S Khripchenko and B V Novikov. Finitary incidence algebras. *Comm. Algebra*, 37(5):1670–1676, 2009.
[6] M Khrypchenko. Local derivations of finitary incidence algebras. *Acta Math. Hungar.*, 154(1):48–55, 2018.
[7] A Nowicki and I Nowosad. Local derivations of subrings of matrix rings. *Acta Math. Hungar.*, 105(1-2):145–150, 2004.
[8] S Ayupov, K Kudaybergenov, and B Yusupov. 2-Local derivations on generalized Witt algebras. *Linear Multilinear Algebra*, 69(16):3130–3140, 2021.