EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2026, Vol. 19, Issue 2, Article Number 7221
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Lie-Backlund Symmetry Analysis and New Explicit Traveling Wave Solutions to the Generalized $(2+1)$-Dimensional Kundu-Mukherjee-Naskar Equation

Rawya Al-deiakeh[1,*], Marwan Alquran[2,3], Abdullahi Yusuf[4,5], Fadia Al-Bashaireh[6]

[1] *Applied Science Department, Ajloun University College, Al-Balqa Applied University, Ajloun, Jordan*
[2] *College of Integrative Studies, Abdullah Al Salem University, Firdous Street, Block 3, Khaldiya, Kuwait*
[3] *Department of Mathematics and Statistics, Jordan University of Science & Technology, Irbid 22110, Jordan*
[4] *Department of Mathematics, Firat University, Elazig, Turkey*
[5] *Department of Mathematics, Federal University, Dutse, Nigeria*
[6] *Jordan Ministry of Education, Tabaria School, Irbid, Jordan*

Abstract. Conservation Laws serve as a validation tool for the correctness and physical relevance of nonlinear models. Their presence ensures that the model preserves essential physical quantities such as energy, momentum, or mass during evolution, which is especially important in realistic and applied systems. In this work, the generalized $(2+1)$-dimensional Kundu-Mukherjee-Naskar equation is investigated thoroughly. First, the Lie-Backlund symmetries analysis is performed to obtain the corresponding conservation Laws using the sense of Ibragimov. Then, we solve explicitly the model by adapting the Riccati-Bernoulli sub-ODE method. In addition, graphical illustrations are provided to gain some physical insight. We believe that the conservation Laws together with the obtained exact solutions supports the structural properties of the model, and provide important information regarding its dynamical behavior.

2020 Mathematics Subject Classifications: 35C08, 76M60.

Key Words and Phrases: Kundu-Mukherjee-Naskar equation, lie-Backlund symmetries analysis, conservation Laws, explicit traveling wave solutions, riccati-Bernoulli method.

*Corresponding Author.
DOI: https://doi.org/10.29020/nybg.ejpam.v19i2.7221

Email addresses: R.deiakeh@bau.edu.jo (R. Al-deiakeh), marwan04@just.edu.jo (M. Alquran) ayusuf@firat.edu.tr (A. Yusuf), Fadia_Bashaireh@yahoo.com (F. Al-Bashaireh)

1. Introduction

Nonlinear partial differential equations constitute the mathematical formulations for many applications and phenomena. Studying such models play key role in understanding the depth of practical physical phenomena. An important example of such models is the Kundu-Mukherjee-Naskar equation (GKMN) which describes of wave propagation and interaction in different nonlinear media. The GKMN is constructed to describe stable solitary wave solutions that maintain their shape while traveling at constant velocities and it is relevant in various disciplines such as optical fiber pulses, fluid dynamics, medical imaging, and ocean rogue waves. Moreover, the GKMN is introduced as further extension of the nonlinear Schrödinger equation to depict soliton pulses by incorporating several types of nonlinearity terms classified as Kerr and non Kerr-law nonlinearities [1].

The mathematical formulation to the GKMN is:

$$iw_t + aw_{xy} + ibw(ww_x^* - w^*w_x) = 0, \tag{1}$$

where $w(x, y, t)$ is the wave-envelope function, and its conjugate is defined as $w^*(x, y, t)$. The scalar a represents the wave dispersion term, and b guarantees the existence of various cases of nonlinear media.

Many researchers have explored the GKMN equation using various techniques to gain insights into its behavior , explain it, and gain fruitful information about the model. For example, in [2], singular periodic solutions were obtained by employing the novel rational sinh-Gordon expansion. Applying the Jacobi elliptic function method, extended algebraic method, and the modified trial function method, a collection of solutions including bright dark-solitons and singular solutions were derived [3–5]. Furthermore, additional solutions were reported for the GKMN by implementing other approaches such as the Riccati function and F-expansion principle [6], Csch-function method and extended Tanh–Coth method [7], exp-function method [8], and Sardar sub-equation method [9]. Recently, in [10], by adopting rational form functions that incorporate both sine and cosine variables or sinh and cosh variables, novel types of solutions exhibiting elliptic-periodic, convex-periodic, cusp-soliton, and quasi-periodic patterns were derived.

Although the traveling wave solutions found in this work are at times similar to those found in the literature, based on classic Riccati or tanh-based methods, the solutions are fundamentally different in terms of their derivation and related structure. Also, the model conservation Laws based on these symmetries give greater information on the physical structure of the model that has not been reported elsewhere. In this way, as certain profiles of individual solutions can be formally similar to traditional prior work, the mechanism of their derivation and dependence on the parameters is a brand new piece of the current work, as well as the conserved quantities.

The majority of available literature is based mainly on the creation of solution families with no direct relation provided between the symmetry structures and the conservation Laws of the governing equations. Conversely, his study contributes by employing the Lie-Backlund symmetry approach to derive conservation Laws associated with the GKMN equation. This method offers a novel and effective framework for uncovering the underlying structure of nonlinear systems and understanding complex physical phenomena. In addition, the analysis is extended by applying the Riccati-Bernoulli sub-ODE method to obtain further exact solutions of the model. The availability of explicit solutions plays a crucial role in gaining deeper insight into the system's dynamics, which has not hitherto been considered in the same light by other analyses based on this model.

2. Conservation Laws associated with Lie-backlund symmetries

In this section, we present a brief overview of the Lie-Backlund symmetries for PDEs and some formulas in order to construct conservation Laws associated. For more information, one can refer to [11–13].

Conservation Laws play a critical role in revealing the intrinsic properties and underlying symmetries of nonlinear systems. Through these symmetries, one can derive conservation Laws which, when satisfied, indicate that the system under consideration reflects physical reality and adheres to natural principles. Before attempting to solve any nonlinear PDE, especially in physical applications, it is essential to verify whether the system respects such fundamental Laws.

Consider the following system of PDEs:

$$\triangle = \mathcal{F}_{\sigma}(, U, U^{(1)}, U^{(2)}, ..., U^{(s)}). \ \sigma = 1, 2, ..., n. \tag{2}$$

The symbol $U^{(s)}$ means the s-times derivative with respect to t, x or y.

Theorem 1. *[12, 13] Each Lie-Backlund symmetry of system (2) results in conservation Laws. The components $\mathcal{T}^i = (x, y, t, U, ...)$ for the conserved vectors $(\mathcal{T}^1, \mathcal{T}^2, .., \mathcal{T}^m)$ can be constructed by the formula:*

$$\begin{aligned} \mathcal{T}^r = \tau^r + C^{\sigma}\left[\frac{\partial \mathcal{L}}{\partial U^{\sigma}_r} - D_r\left(\frac{\partial \mathcal{L}}{\partial U^{\sigma}_{rl}}\right) + D_l D_k\left(\frac{\partial \mathcal{L}}{\partial U^{\sigma}_{rlk}}\right) - ...\right] \\ + D_l(C^{\sigma})\left[\frac{\partial \mathcal{L}}{\partial U^{\sigma}_{rl}} - D_k\left(\frac{\partial \mathcal{L}}{\partial U^{\sigma}_{rlk}}\right) + ...\right] + ... \end{aligned} \tag{3}$$

where D_r and $\mathcal{L}$ are the total derivative with respect to r, and the formal Lagrangian, respectively. The formal Lagrangian $\mathcal{L}$ is defined by

$$\mathcal{L} =: \upsilon^{\sigma} F_{\sigma}(x, y, t, U, U^{(1)}, U^{(2)}, ..., U^{(s)}), \tag{4}$$

Lemma 1. *[14] The components $\mathcal{T}^1, \mathcal{T}^2, ..$ and $\mathcal{T}^m$ constitute a conserved vector $(\mathcal{T}^1, \mathcal{T}^2, .., \mathcal{T}^m)$ of (2), if each satisfies $\sum_{i=0}^{m} D_r \mathcal{T}^i = 0$. Where, the adjoint system for the PDEs can be defined as below:*

$$F^* = \frac{\delta \mathcal{L}}{\delta U} = 0. \tag{5}$$

Theorem 2. *[12, 13] Each Lie-Backlund symmetries generator*

$$\mathcal{Q} = \eta^U(x, y, t, U, U_t, U_x, U_y, U_{xx}, U_{xy}, ...). \tag{6}$$

can produce conservation Laws. These conservation Laws is for the system and its associated adjoint system. In other words, Lie–Backlund symmetry generators (transformations under which the PDEs remain invariant) may give rise to conserved quantities that stay constant along solutions of the system and its adjoint system.

The Lie-Backlund transformation group is an extension of group transformations which are one-parameter continuous symmetry groups that transform one equation into another in the context of partial differential equations [15–18]. It can be regarded as a tangent transformation group that generates Lie-Backlund symmetries. For the Lie-Backlund symmetry group, these symmetries will be generated as vector field characterized. The Lie-Backlund symmetry generators are solutions of the determining equations in the invariance condition of the generalized fractional KMN equation. To simplify algebraic computations, Maple was used to carry out symbolic computations. The generators obtained are non-equivalent and contain higher order symmetry elements over classical point symmetries.

To construct the Lie-Backlund symmetries, we convert the GKMN equation into imaginary and real parts by applying transformation $w = u + iv$.

$$\begin{aligned} &a u_{\mathrm{xy}} + 2bu^2 v_x - 2buvu_x - v_t = 0, \\ &a v_{\mathrm{xy}} - 2bv^2 u_x + 2buvv_x + u_t = 0, \end{aligned} \tag{7}$$

As a result, the vector field associated with the GKMN equation is

$$\mathcal{Q} = \eta^u(x, y, t, u, u_x, u_y, u_{xx}, u_{yy}, v, v_x, v_{xy}) + \eta^v(x, y, t, u, u_x, u_y, u_{xy}, v, v_x, v_{xx}, v_{yy}), \tag{8}$$

where $\mathcal{Q}$ satisfy the 2^{nd} prolongation

$$\mathcal{Q}^2(\triangle_i)|_{\triangle_i = 0} = 0, i = 1, 2.$$

The system depicted in (8) is overdetermined. Solving it using computational software such as Maple yields the following solutions:

$$\begin{aligned} \eta^u =\ & actu_x + ac_4tv_{\mathrm{xy}} + ac_3v_{\mathrm{xy}} - 2bc_4tv^2u_x + 2bc_4tuvv_x - 2bc_3v^2u_x + 2bc_3uvv_x + c_5u_x - \\ & 2c_2u_{\mathrm{xx}} - c_4u_{\mathrm{xx}} + c_6u_y + 2c_2u_{\mathrm{yy}} + c_2u + 2cvy + c_7v, \\ \eta^v =\ & -ac_4tu_{\mathrm{xy}} + actv_x - ac_3u_{\mathrm{xy}} - 2bc_4tu^2v_x + 2bc_4tuvu_x - 2bc_3u^2v_x + 2bc_3uvu_x - cuy \end{aligned}$$

$$+ c_7 u + c_5 v_x - 2c_2 v_{\text{xx}} - c_4 v_{\text{xx}} + c_6 v_y + 2c_2 v_{\text{yy}} + c_2 v. \tag{9}$$

Now, we let $c, c_2, c_3, ..., c_7$ be arbitrary constants. Thus, the resulting seven Lie-backlund symmetries are

$$\begin{aligned}
\mathcal{Q}_1 &= v_x \frac{\partial}{\partial v} + u_x \frac{\partial}{\partial u},\\
\mathcal{Q}_2 &= v_y \frac{\partial}{\partial v} + u_y \frac{\partial}{\partial u},\\
\mathcal{Q}_3 &= v \frac{\partial}{\partial u} - u \frac{\partial}{\partial v},\\
\mathcal{Q}_4 &= (atu_x + v_y)\frac{\partial}{\partial u} + (atv_x - u_y)\frac{\partial}{\partial v},\\
\mathcal{Q}_5 &= (-2u_{\text{xx}} + 2u_{\text{yy}} - u)\frac{\partial}{\partial u} + (-2v_{\text{xx}} + 2v_{\text{yy}} + v)\frac{\partial}{\partial v},\\
\mathcal{Q}_6 &= (av_{\text{xy}} - 2bv^2 u_x + 2buvv_x)\frac{\partial}{\partial u} + \left(-au_{\text{xy}} - 2bu^2 v_x + 2buvu_x\right)\frac{\partial}{\partial v},\\
\mathcal{Q}_7 &= (atv_{\text{xy}} - 2btv^2 u_x + 2btuvu_x - u_{\text{xx}})\frac{\partial}{\partial u} + \left(-atu_{\text{xy}} - 2btu^2 v_x + 2btuvu_x - v_{\text{xx}}\right)\frac{\partial}{\partial v},
\end{aligned} \tag{10}$$

Definition 1. *A system (7) is defined as quasi-self-adjoint if the adjoint system of its becomes the same as the original system under the substitutions* $v = \lambda(u)$ *and* $u = \xi(v)$ *where* $\lambda(u), \xi(v) \neq 0$.

Theorem 3. *[15] Every symmetry related to Lie point, Lie–Backlund, and nonlocal symmetries of a strictly or quasi-self-adjoint system leads to a conservation law.*

Theorem 4. *The GKMN equation is quasi-self-adjoint.*

Proof. *The formal Lagrangian for the GKMN can be presented by*

$$\mathcal{L} = \lambda(x, y, t)(au_{xy} + 2bu^2 v_x - 2buvu_x - v_t) + \xi(x, y, t)(av_{xy} - 2bv^2 u_x + 2buvv_x + u_t), \tag{11}$$

and the adjoint system is:

$$\begin{aligned}
a\lambda_{xy} + 2buv\lambda_x + 6bu\lambda v_x + 2bv^2\xi_x + 6b\xi vv_x - \xi_t &= 0,\\
-a\xi_{xy} + 2bu^2\lambda_x + 6b\xi vu_x + 2buv\lambda_x + 6bu\lambda u_x - \lambda_t &= 0,
\end{aligned} \tag{12}$$

where $\lambda(x, y, t)$ *and* $\xi(x, y, t)$ *are new introduced independent variables. To ensure that the GKMN equation is self-adjoint, we substitute* $\lambda = v$ *and* $\xi = -u$ *in (12) to get (7) and complete the proof.*

Now, we are able to find the t-component, x-component and y-component of the conservation Laws for (2) based on formula (3). This can be constructed by

$$\mathcal{T}^t = C^1 \left[\frac{\partial \mathcal{L}}{\partial u_t}\right] + C^2 \left[\frac{\partial \mathcal{L}}{\partial v_t}\right],$$

$$\begin{aligned}
\mathcal{T}^x &= C^1\left[\frac{\partial\mathcal{L}}{\partial u_x} - D_y\left(\frac{\partial\mathcal{L}}{\partial u_{\text{xy}}}\right)\right] + D_y(C^1)\left[\frac{\partial\mathcal{L}}{\partial u_{\text{xy}}}\right] + C^2\left[\frac{\partial\mathcal{L}}{\partial v_x} - D_y\left(\frac{\partial\mathcal{L}}{\partial v_{\text{xy}}}\right)\right] \\
&\quad + D_y(C^2)\left[\frac{\partial\mathcal{L}}{\partial v_{\text{xy}}}\right], \\
\mathcal{T}^y &= -C^1\left[D_x\left(\frac{\partial\mathcal{L}}{\partial u_{\text{xy}}}\right)\right] + D_x(C^1)\left[\frac{\partial\mathcal{L}}{\partial u_{\text{xy}}}\right] - C^2\left[D_x\left(\frac{\partial\mathcal{L}}{\partial v_{\text{xy}}}\right)\right] + D_x(C^2)\left[\frac{\partial\mathcal{L}}{\partial v_{\text{xy}}}\right] \qquad (13)
\end{aligned}$$

Now, the conservation Laws of the GKMN equation will be examined by the following cases:

Case 1. By means of (11)-(13), through the Lie-backlund $\mathcal{Q}_1$, the Lie characteristic equations are $C^1 = u_x,\ C^2 = v_x$, and the conserved vectors are

$$\begin{aligned}
\mathcal{T}^t &= \xi u_x - \lambda v_x, \\
\mathcal{T}^x &= -au_x\lambda_y + a\lambda u_{\text{xy}} - av_x\xi_y + a\xi v_{\text{xy}} + 2bu^2\lambda v_x - 2b\xi v^2 u_x + 2b\xi uvv_x - 2buv\lambda u_x, \\
\mathcal{T}^y &= a\left(-u_x\lambda_x + \lambda u_{\text{xx}} - v_x\xi_x + \xi v_{\text{xx}}\right). \qquad (14)
\end{aligned}$$

Case 2. Lie-backlund $\mathcal{Q}_2$, the Lie characteristic equations are $C^1 = u_y,\ C^2 = v_y$, and the conserved vectors are

$$\begin{aligned}
\mathcal{T}^t &= \xi u_y - \lambda v_y, \\
\mathcal{T}^x &= a\lambda u_{\text{yy}} + a\xi v_{\text{yy}} - 2bvu_y(u\lambda + \xi v) + 2buv_y(u\lambda + \xi v), \\
\mathcal{T}^y &= a\left(\lambda u_{\text{xy}} + \xi v_{\text{xy}}\right). \qquad (15)
\end{aligned}$$

Case 3. Lie-backlund $\mathcal{Q}_3$, the Lie characteristic equations are $C^1 = v,\ C^2 = -u$, and the conserved vectors are

$$\begin{aligned}
\mathcal{T}^t &= u\lambda + \xi v, \\
\mathcal{T}^x &= -a\xi u_y + a\lambda v_y - 2bu^3\lambda - 2b\xi u^2 v - 2buv^2\lambda - 2b\xi v^3, \\
\mathcal{T}^y &= a\lambda v_x - a\xi u_x. \qquad (16)
\end{aligned}$$

Case 4. Lie-backlund $\mathcal{Q}_4$, the Lie characteristic equations are $C^1 = atu_x + v_y,\ C^2 = -atv_x - u_y$, and the conserved vectors are

$$\begin{aligned}
\mathcal{T}^t &= at\left(\xi u_x - \lambda v_x\right) + \lambda u_y + \xi v_y, \\
\mathcal{T}^x &= -2bu(u\lambda + \xi v)\left(u_y - atv_x\right) - 2bv(u\lambda + \xi v)\left(atu_x + v_y\right) + a\xi\left(atv_{\text{xy}} - u_{\text{yy}}\right) \\
&\quad + a\lambda\left(atu_{\text{xy}} + v_{\text{yy}}\right), \\
\mathcal{T}^y &= a\left(at\left(\lambda u_{\text{xx}} + \xi v_{\text{xx}}\right) - \xi u_{\text{xy}} + \lambda v_{\text{xy}}\right). \qquad (17)
\end{aligned}$$

Case 5. Lie-backlund $\mathcal{Q}_5$, the Lie characteristic equations are $C^1 = -2u_{\text{xx}} + 2u_{\text{yy}} - u,\ C^2 = -2v_{\text{xx}} + 2v_{\text{yy}} + v$, and the conserved vectors are

$$\mathcal{T}^t = -\xi\left(2u_{\text{xx}} - 2u_{\text{yy}} + u\right) - \lambda\left(-2v_{\text{xx}} + 2v_{\text{yy}} + v\right),$$

$$
\begin{aligned}
\mathcal{T}^x &= -a\lambda\left(2u_{\text{xxy}} + u_y - 2u_{\text{yyy}}\right) + a\xi\left(-2v_{\text{xxy}} + v_y + 2v_{\text{yyy}}\right) \\
&\quad + 2bv\left(2u_{\text{xx}} - 2u_{\text{yy}} + u\right)\left(u\lambda + \xi v\right) + 2bu\left(-2v_{\text{xx}} + 2v_{\text{yy}} + v\right)\left(u\lambda + \xi v\right), \\
\mathcal{T}^y &= a\xi\left(v_x - 2v_{\text{xxx}} + 2v_{\text{xyy}}\right) - a\lambda\left(u_x + 2u_{\text{xxx}} - 2u_{\text{xyy}}\right).
\end{aligned} \tag{18}
$$

Case 6. Lie-backlund $\mathcal{Q}_6$, the Lie characteristic equations are $C^1 = av_{\text{xy}} - 2bv^2u_x + 2buvv_x$, $C^2 = -au_{\text{xy}} - 2bu^2v_x + 2buvu_x$, and the conserved vectors are

$$
\begin{aligned}
\mathcal{T}^t &= a\lambda u_{\text{xy}} + a\xi v_{\text{xy}} + 2bu^2\lambda v_x - 2b\xi v^2u_x + 2buv\left(\xi v_x - \lambda u_x\right), \\
\mathcal{T}^x &= a^2\left(\lambda v_{\text{xyy}} - \xi u_{\text{xyy}}\right) - 2bu^2\left(a\left(\lambda u_{\text{xy}} + \xi v_{\text{xy}}\right) + 2bv^2\left(\lambda v_x - \xi u_x\right)\right) \\
&\quad + 2bu\left(u_x\left(a\xi v_y + 2bv^3\lambda\right)\right) + 2bu\left(v_x\left(a\lambda v_y - 2\xi\left(au_y + bv^3\right)\right)\right) - 2abv^2\left(\lambda u_{\text{xy}} + \xi v_{\text{xy}}\right) \\
&\quad + 2abv\left(u_x\left(\xi u_y - 2\lambda v_y\right) + \lambda u_y v_x\right) - 4b^2u^4\lambda v_x + 4b^2u^3v\left(\lambda u_x - \xi v_x\right) + 4b^2\xi v^4u_x, \\
\mathcal{T}^y &= a\left(-a\xi u_{\text{xxy}} + a\lambda v_{\text{xxy}} - 2b\xi u^2v_{\text{xx}} - 2bv^2\lambda u_{\text{xx}} + 2buv_x\left(\lambda v_x - \xi u_x\right)\right) \\
&\quad + 2abv\left(\left(-\lambda u_x v_x + u\left(\xi u_{\text{xx}} + \lambda v_{\text{xx}}\right) + \xi u_x^2\right)\right).
\end{aligned} \tag{19}
$$

Case 7. Lie-backlund $\mathcal{Q}_7$, the Lie characteristic equations are $C^1 = atv_{\text{xy}} - 2btv^2u_x + 2btuvu_x - u_{\text{xx}}$, $C^2 = -atu_{\text{xy}} - 2btu^2v_x + 2btuvu_x - v_{\text{xx}}$, and the conserved vectors are

$$
\begin{aligned}
\mathcal{T}^t &= at\xi v_{\text{xy}} + a\lambda u_{\text{xy}} + 2btu^2\lambda v_x - 2bt\xi v^2u_x + 2btuv\left(\xi v_x - \lambda u_x\right) - \xi u_{\text{xx}} + \lambda v_{\text{xx}}, \\
\mathcal{T}^x &= a\lambda\left(atv_{\text{xyy}} + 2bt\left(-v^2u_{\text{xy}} + v\left(u_y v_x - 2u_x v_y + uv_{\text{xy}}\right) + uv_x v_y\right) - u_{\text{xxy}}\right) - \\
&\quad 2bu(u\lambda + \xi v)\left(au_{\text{xy}} + 2btu\left(uv_x - vu_x\right) + v_{\text{xx}}\right) + \\
&\quad 2bv(u\lambda + \xi v)\left(-atv_{\text{xy}} + 2btv\left(vu_x - uv_x\right) + u_{\text{xx}}\right) \\
&\quad - a\xi\left(au_{\text{xyy}} - 2btv\left(u_x u_y + uu_{\text{xy}}\right) + 2btu\left(2u_y v_x - u_x v_y + uv_{\text{xy}}\right) + v_{\text{xxy}}\right), \\
\mathcal{T}^y &= a\left(at\lambda v_{\text{xxy}} - a\xi u_{\text{xxy}} - 2bt\xi u^2v_{\text{xx}} - 2btv^2\lambda u_{\text{xx}} + 2btuv_x\left(\lambda v_x - \xi u_x\right)\right) \\
&\quad + a\left(2btv\left(-\lambda u_x v_x + u\left(\xi u_{\text{xx}} + \lambda v_{\text{xx}}\right) + \xi u_x^2\right) - \lambda u_{\text{xxx}} - \xi v_{\text{xxx}}\right).
\end{aligned} \tag{20}
$$

For example, The quantity conserved from $\mathcal{Q}_3$ can be interpreted as a mass-like invariant associated with the squared amplitude of the wave's envelope in the GKMN equation. This invariant reflects the balance between dispersion and nonlinearity during the propagation of waves, and it demonstrates that the overall amplitude content of the wave's envelope remains invariant along the direction of evolution. From a physical perspective, this conservation property promotes the stability of the associated traveling wave solutions and highlights the resilience of local and periodic structures of waves, consistent with the behaviors observed in Section 4 (see Figures 4 and 5). The Lie-Backlund symmetry generators are solutions of the determining equations in the invariance condition of the GKMN equation. To simplify algebraic computations, Maple was used to carry out symbolic computations. The generators obtained are non-equivalent and contain higher order symmetry elements over classical point symmetries.

3. Description and Application of the Riccati-Bernoulli Method

In this section we present and apply, in detail, the steps implementation of the proposed methodology [19–21] to derive explicit solutions of the GKMN model. Consider a general

PDE in the form Let us consider the GKMN equation with three variables as

$$P(w, w_x, w_y, w_t, w_{xx}, ...) = 0, \quad w = w(x, y, t). \tag{21}$$

Step One: Convert (21) using the transformation $z = kx \pm py \pm vt$ to an ODE

$$Q(w, w', w'', , ...) = 0, \quad w = w(z). \tag{22}$$

Step Two: The solution of (22) obeys the auxiliary Sub-ODE

$$w' = \alpha w^{2-m} + \varsigma w^m + \beta w, \tag{23}$$

where α, ς, β and m are constants that will be determined later. Based on the discriminant of $\triangle = \beta^2 - 4\alpha\varsigma$ and the constraints on α, β, and m. the solution of (23) are given by the following cases:

Case 1. If $m = 1$, the solution is

$$w(z) = Je^{z*(\alpha+\varsigma+\beta)}. \tag{24}$$

Case 2. If $m \neq 1, \varsigma = 0$ and $\beta = 0$, the solution is

$$w(z) = (\alpha(m-1)(J+z))^{\frac{1}{m-1}}. \tag{25}$$

Case 3. If $m \neq 1, \varsigma = 0$ and $\beta \neq 0$, the solution is

$$w(z) = \left(Je^{\beta(m-1)z} - \frac{\alpha}{\beta}\right)^{\frac{1}{m-1}}. \tag{26}$$

Case 4. If $m \neq 1$, $\triangle > 0$ and $\alpha \neq 0$, two solutions are available and defined by:

$$w(z) = \left(-\frac{\beta}{2\alpha} - \frac{\sqrt{\beta^2 - 4\alpha\varsigma}\tanh\left(\frac{1}{2}(J+z)\left((1-m)\sqrt{\beta^2 - 4\alpha\varsigma}\right)\right)}{2\alpha}\right)^{\frac{1}{1-m}}, \tag{27}$$

and

$$w(z) = \left(-\frac{\beta}{2\alpha} - \frac{\sqrt{\beta^2 - 4\alpha\varsigma}\coth\left(\frac{1}{2}(J+z)\left((1-m)\sqrt{\beta^2 - 4\alpha\varsigma}\right)\right)}{2\alpha}\right)^{\frac{1}{1-m}}. \tag{28}$$

Case 5. If $m \neq 1$, $\triangle < 0$ and $\alpha \neq 0$, the solutions are:

$$w(z) = \left(-\frac{\beta}{2\alpha} - \frac{\sqrt{\beta^2 - 4\alpha\varsigma}\tan\left(\frac{1}{2}(J+z)\left((1-m)\sqrt{\beta^2 - 4\alpha\varsigma}\right)\right)}{2\alpha}\right)^{\frac{1}{1-m}}, \tag{29}$$

and

$$w(z)=\left(-\frac{\beta}{2\alpha}-\frac{\sqrt{4\alpha\varsigma-\beta^2}\cot\left(\frac{1}{2}(J+z)\left((1-m)\sqrt{4\alpha\varsigma-\beta^2}\right)\right)}{2\alpha}\right)^{\frac{1}{1-m}}. \tag{30}$$

Case 6. If $m\neq 1$, $\triangle=0$, $\beta\neq 0$ and $\alpha\neq 0$, the solution is given by

$$w(z)=\left(\frac{1}{\alpha(m-1)(J+z))}-\frac{\alpha}{\beta}\right)^{\frac{1}{m-1}}. \tag{31}$$

Step Three: Substitute (23) and its necessary derivatives into (22), and collect the terms of $w^i(i=0,1,2,...)$. By setting each coefficient to zero, we obtain a system of algebraic equations in terms of the unknowns α,ς,β and m. By solving the resulting system, the solutions of (21) can be recognized.

Now, we apply the above illustrative steps of Riccati-Bernoulli sub-ODE method to find some traveling wave solutions for the GKMN equation. Consider the complex wave transform as below.

$$w(x,y,t)=w(z)\exp(i\psi), \tag{32}$$

where

$$z=-ft+dx-ey, \tag{33}$$

and

$$\psi=nt-gx-hy+r. \tag{34}$$

Substitute (32), (33), and (34) into (1) yields

$$0 \;=\; -w\left(agh+n\right)+adew''-2bgw^3, \tag{35}$$

which is restricted to the constraint relation:

$$f \;=\; -a\left(eg+dh\right). \tag{36}$$

Applying the steps of the method yields the following results, taking into account the fulfillment of the conditions of the cases for each solution as follows.

Result 1:

$$\alpha=\pm\frac{\sqrt{b}\sqrt{g}}{\sqrt{a}\sqrt{d}\sqrt{e}},\beta=0,n=-agh-2\alpha de\varsigma. \tag{37}$$

Accordingly, the resulting traveling wave solutions are:

$$u_{1,2}(x,y,t)=ze^{\pm\frac{\sqrt{b}\sqrt{g}}{\sqrt{a}\sqrt{d}\sqrt{e}}+i\psi+\varsigma}, \tag{38}$$

$$w_{3,4}(x,y,t)=\pm\frac{e^{i\psi}\sqrt{a}\sqrt{d}\sqrt{e}}{\sqrt{b}\sqrt{g}(J+z)}, \tag{39}$$

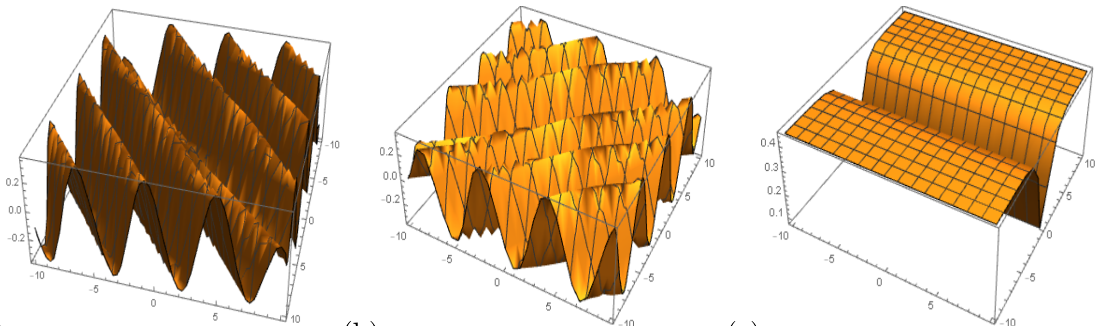
$$w_{5,6}(x,y,t) = -\frac{\sqrt{a}\sqrt{d}\sqrt{e}\sqrt{\pm\frac{\sqrt{b}\sqrt{g}\varsigma}{\sqrt{a}\sqrt{d}\sqrt{e}}}\coth\left((J+z)\sqrt{\pm\frac{\sqrt{b}\sqrt{g}\varsigma}{\sqrt{a}\sqrt{d}\sqrt{e}}}\right)}{\sqrt{b}\sqrt{g}}, \tag{40}$$

$$w_{7,8}(x,y,t) = -\frac{\sqrt{a}\sqrt{d}\sqrt{e}\sqrt{\pm\frac{\sqrt{b}\sqrt{g}\varsigma}{\sqrt{a}\sqrt{d}\sqrt{e}}}\tanh\left((J+z)\sqrt{\pm\frac{\sqrt{b}\sqrt{g}\varsigma}{\sqrt{a}\sqrt{d}\sqrt{e}}}\right)}{\sqrt{b}\sqrt{g}}, \tag{41}$$

$$w_{9,10}(x,y,t) = -\frac{\sqrt{a}\sqrt{d}\sqrt{e}\sqrt{\pm\frac{\sqrt{b}\sqrt{g}\varsigma}{\sqrt{a}\sqrt{d}\sqrt{e}}}\cot\left((J+z)\sqrt{\pm\frac{\sqrt{b}\sqrt{g}\varsigma}{\sqrt{a}\sqrt{d}\sqrt{e}}}\right)}{\sqrt{b}\sqrt{g}}, \tag{42}$$

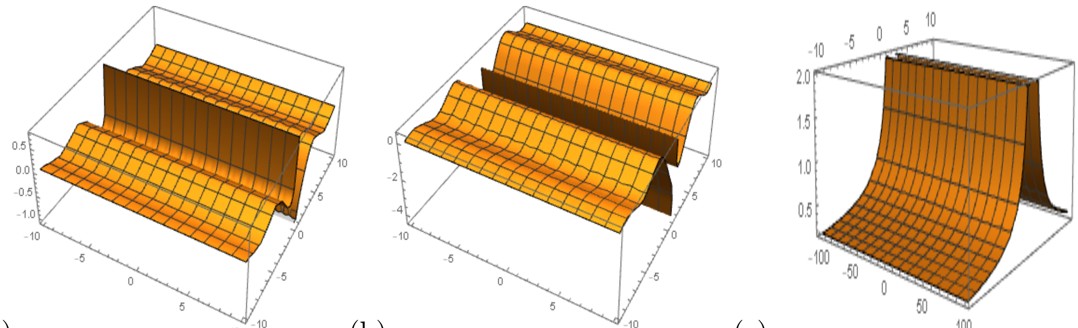
$$w_{11,12}(x,y,t) = -\frac{\sqrt{a}\sqrt{d}\sqrt{e}\sqrt{\pm\frac{\sqrt{b}\sqrt{g}\varsigma}{\sqrt{a}\sqrt{d}\sqrt{e}}}\tan\left((J+z)\sqrt{\pm\frac{\sqrt{b}\sqrt{g}\varsigma}{\sqrt{a}\sqrt{d}\sqrt{e}}}\right)}{\sqrt{b}\sqrt{g}}, \tag{43}$$

where $z = -ft + dx - ey$, $\psi = nt + (-g)\,x - hy + r$, and $ade \neq 0$.



(a) (b) (c)

Figure 1: 3D plots of the solution $w_3(x,0,t)$: (a) real part, (b) imaginary part, (c) modulus, by selecting the arbitrary parameters as $(a = 0.4; b = 1; J = 0.4; d = 1.01; e = 1; g = 0.4; h = -3.5; r = 1.4; \varsigma = -0.2)$



(a) (b) (c)

Figure 2: 3D plots of the solution $w_9(x,0,t)$: (a) real part, (b) imaginary part, (c) modulus, by selecting the arbitrary parameters as $(a = 0.8; b = 1; J = 0.4; d = 1.01; e = 1; g = 0.4; h = -3.5; r = 1.4; \varsigma = 0.2)$

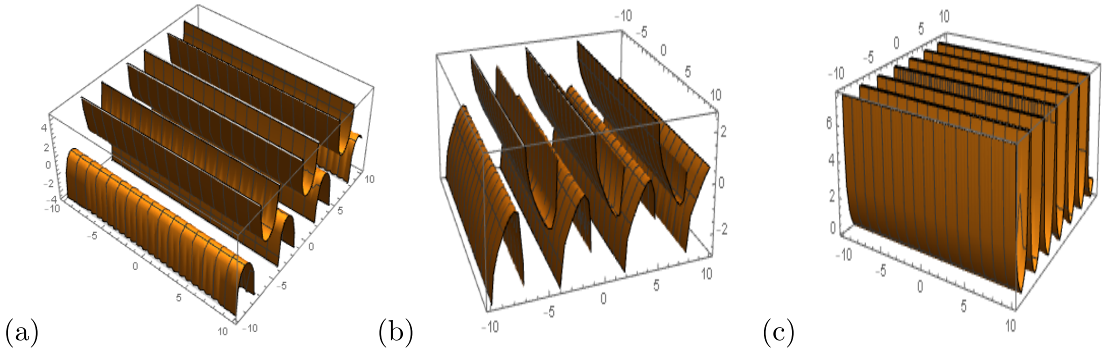


Figure 3: 3D plots of the solution $w_{17}(x, 0, t)$: (a) real part, (b) imaginary part, (c) modulus, by selecting the arbitrary parameters as ($\alpha = 1; a = 0.3; J = 0.4; e = 0.4; g = 0.4; h = -3.5; r = 1.4; \varsigma = -1$)

Result 2:

$$d = 0, \beta = 0, n = -agh. \tag{44}$$

Consequently, the resulting traveling wave solutions are:

$$w_{13}(x, y, t) = (e(-y) - ft)\, e^{i(-aght-gx-hy+r)+\alpha+\varsigma}, \tag{45}$$

$$w_{14}(x, y, t) = -\frac{e^{i(-aght-gx-hy+r)}}{\alpha\,(J - ft - ey)}, \tag{46}$$

$$w_{15}(x, y, t) = -\frac{\sqrt{\alpha\varsigma}\tan\left(\sqrt{\alpha\varsigma}\,(J - ft - ey)\right)}{\alpha}, \tag{47}$$

$$w_{16}(x, y, t) = -\frac{\sqrt{\alpha\varsigma}\cot\left(\sqrt{\alpha\varsigma}\,(J - ft - ey)\right)}{\alpha}, \tag{48}$$

$$w_{17}(x, y, t) = -\frac{\sqrt{\alpha\varsigma}\tanh\left(\sqrt{\alpha\varsigma}\,(J - ft - ey)\right)}{\alpha}, \tag{49}$$

$$w_{18}(x, y, t) = -\frac{\sqrt{\alpha\varsigma}\coth\left(\sqrt{\alpha\varsigma}\,(J - ft - ey)\right)}{\alpha}, \tag{50}$$

where $g \neq 0$ and $e \neq 0$.

Result 3:

$$\alpha, \varsigma = 0, n = a\beta^2 de - agh, g \neq 0. \tag{51}$$

Thus, two more solutions are obtained and defined as:

$$w_{19}(x, y, t) = (-ft + dx - ey)\exp\left(\beta + i\left(t\left(a\beta^2 de - agh\right) - gx - hy + r\right)\right), \tag{52}$$

$$w_{20}(x, y, t) = \frac{\exp\left(\beta\,(-ft + dx - ey) + i\left(t\left(a\beta^2 de - agh\right) - gx - hy + r\right)\right)}{J}, \tag{53}$$

where $g \neq 0$ for the existence of the solutions.

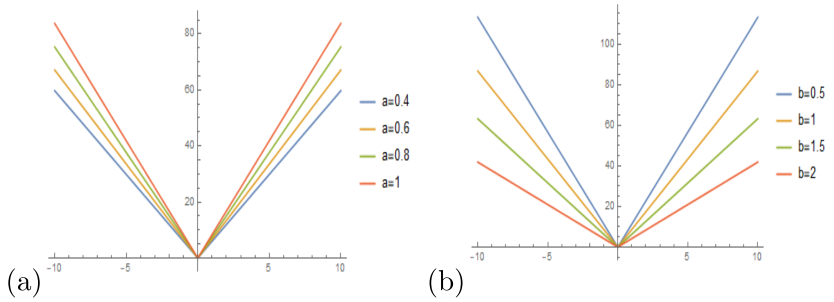


Figure 4: 2D plot of the modulus solution $w_1(x,0,1)$: (a)for $a = 0.4, 0.6, 0.8, 1$ and $b = 1$ (b)for $b = 0.5, 1, 1.5, 2$ and $a = 1$, by selecting the arbitrary parameters as $(J = 0.4; d = 1.01; e = 1; g = 1; h = -3.5; r = 1.4; \varsigma = 0.2)$

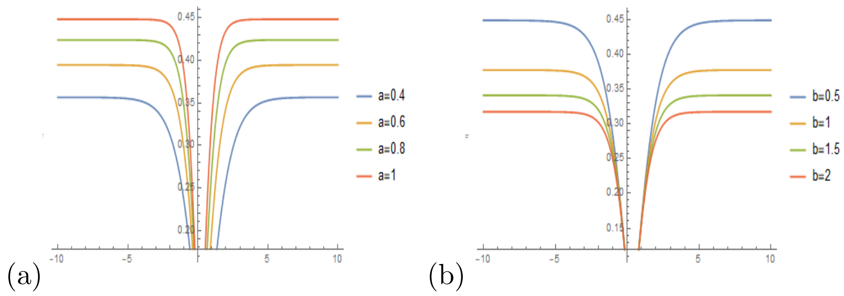


Figure 5: 2D plot of the modulus solution $w_7(x,0,1)$: (a) for $a = 0.4, 0.6, 0.8, 1$ and $b = 1$ (b) for $b = 0.5, 1, 1.5, 2$ and $a = 1$, by selecting the arbitrary parameters as $(J = 0.4; d = 1.01; e = 1; g = 1; h = -3.5; r = 1.4; \varsigma = -0.2)$

For clarity and physical interpretation, the obtained exact solutions can be classified into four main categories according to their qualitative wave behavior. The first category consists of soliton-type solutions, characterized by localized and stable wave profiles that preserve their shape during propagation, as illustrated by solutions such as w_7, w_{17}, and w_{20}. The second category includes periodic solutions, which exhibit oscillatory behavior in space and time and are associated with trigonometric or elliptic functional forms, such as solutions w_9 - w_{12}. These waves describe repeating patterns. The third category comprises singular solutions, which display sharp gradients or blow-up behavior at specific points, as observed in solutions derived from cotangent and hyperbolic cotangent forms. Finally, rational solutions are obtained in cases where the solution structure is expressed as algebraic functions of the traveling variable, such as w_3, w_4, and w_{19}.

To better understand the physical characteristics of the GKMN equation, we present 2D and 3D visualizations of several obtained solutions. In Figure 1, the evolution of the real and imaginary parts of $w_3(x,0,t)$ exhibits quasi-periodic waveforms, while the corresponding norm function displays an anti-cusp propagation profile. Figure 2 illustrates the damped wave behavior for w_9, with noticeable fluctuations centered around the origin. Lastly,In Figure 3, the solution w_{17} demonstrates a periodic convex-concave pattern in its real and imaginary parts, while its norm function reveals a convex-periodic structure. The interplay between dispersion and nonlinearity is a fundamental characteristic of soliton dynamics, as it governs the formation and stability of localized wave structures. In the context of the GKMN equation, this balance can be quantitatively examined through the combined effect of the parameters a (dispersion coefficient) and b (nonlinearity coefficient) on the wave amplitude.

Remark 1. *As illustrated in Figures 4 and 5, both the dispersion and nonlinearity coefficients play a critical role in affecting the propagation of waves described by of the GKMN: wave amplitude increases with stronger dispersion (a), while it decreases as the nonlinearity coefficient grows (b). This reveals a balance between dispersion and nonlinearity that maintains the stability of the soliton profile.*

4. Conclusion

Our present work was a Lie-Backlund symmetry analysis of the GKMN model in the framework of Ibragimov to obtain the conservation Laws associated with it and have an idea of the dynamical behavior of the resulting wave structures. Moreover, the Riccati-Bernoulli technique was successfully used to discover new solutions of solitary waves. The results that will be discussed here are especially relevant, because the conservation Laws obtained point out the importance of dynamic information that is conserved during the evolution of waves. The explicit solutions also improve our knowledge of the impact of the parameters that are contained in the model itself [22–24]. It is also worth mentioning at this point, that symmetries, like Lie or Lie-Backlund symmetries, provide more information on the structure of the system in question and in some cases allow one to reduce the PDE to an ODE, which greatly simplifies the solution process.

In addition to deriving explicit traveling wave solutions, the present study highlights the structural richness of the GKMN equation through Lie-Backlund symmetries. Unlike classical Lie point symmetries, the Lie-Backlund framework enables the construction of higher-order conservation Laws, thereby revealing deeper invariant structural properties of the model. The obtained solutions comprise soliton-type, periodic, singular, and rational wave structures, which extend previously reported results in the literature. Future research directions may includes stability and sensitivity analyses of these solutions, as well as extensions to fractional or variable-coefficient versions of the model.

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