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Sum of Powers of Mersenne Numbers as Perfect Squares and Powers of Two

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Abstract. This mathematical research investigates whether the sum of two powers of Mersenne numbers can be expressed as a square. It also determines whether the sum of powers of these special numbers can be a power of two. Diophantine analysis using elementary methods and well-established results in number theory served as the basis for the work. The results show that there are infinitely many Mersenne numbers of the form $2^{2\alpha} - 1$ whose sum of powers can be written as a perfect square and that the sum of the zeroth power and the first power of Mersenne numbers can be written as powers of 2. Moreover, the sum of two zeroth powers of Mersenne numbers always yields 2. Similarly, the sum of any two positive powers of 1, the first positive Mersenne number, is always equal to 2.

2020 Mathematics Subject Classifications: 11D61, 11D72

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1. Introduction

Number theory is a fundamental branch of mathematics that is concerned with the study of integers and their properties. One key area within number theory is the study of integer sequences. Integer sequences have numerous applications across different fields of science that include mathematics, computer science, engineering, and physics, among others. For instance, number sequences appear in algebraic structures, such as groups and rings. In cryptography, sequences secure cryptographic protocols. They also appear as patterns in nature. Because of their applications on various fields, number theorists have conducted many studies on different integer sequences.

One of the most studied and popular integer sequences is the Mersenne sequence $\{M_n\}_{n\geq 1}$. This sequence is defined by the initial values $M_1 = 1$ and $M_2 = 3$, and the

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recurrence relation $M_n = 3M_{n-1} - 2M_{n-2}$ for $n \geq 3$. Moreover, the Mersenne sequence can be generated using the formula $M_n = 2^n - 1$ for positive integer n. The numbers in the Mersenne sequence are called Mersenne numbers. The Mersenne numbers less than 100 are $1, 3, 7, 15, 31$ and 63. There are infinitely many of these numbers, as the sequence is infinite.

Several authors have extensively studied Mersenne numbers. For instance, Gayo [1] studied an exponential Diophantine equation involving Mersenne numbers, where he showed that some of the sums of a power of Mersenne number and a power of the sum of the same Mersenne number and 1 are expressible as a perfect square. Also, Alan and Alan [2] researched Mersenne numbers that can be represented as the product of two arbitrary Pell numbers. Moreover, Emin [3] performed investigation of the Mersenne numbers that can be expressible as summation of two Fibonacci numbers. Additionally, Emin [4] proved which Pell numbers can be represented as the sum of two arbitrary Mersenne numbers. Moreover, he also determined which Mersenne numbers can be expressed as the sum of two arbitrary Pell numbers. Mersenne numbers that are at the same time are primes were explored by Gayo and Bacani [5–7] in their studies on exponential Diophantine equations. Other studies involving specific Mersenne numbers that are part of exponential Diophantine equation can be seen in the works of Ashtana [8], Rabago [9] and Sroysang [10, 11]. In cryptography, Mersenne plays a vital role. A proposed new public-key cryptosystem whose security is based on the computational intractability of numbers which includes Mersenne numbers of the form $2^n - 1$, where n is a prime [12].

Despite the abundance of literature and studies on Mersenne numbers, there is none that covers the study of Mersenne numbers in which the sum of its powers can be expressed as a perfect square or a power of two. This deficiency serves as the motivation behind our study that we seek to address in this research undertaking. Thus, in this research paper, we focus on the sum of two powers of the same Mersenne number and explore conditions under which the sums can be expressed as perfect squares or powers of two. Specifically, we investigate the integer solutions to the Diophantine equations $M_n^x + M_n^y = z^2$ and $M_n^x + M_n^y = 2^\gamma$, where M_n^x and M_n^y are positive powers of Mersenne number and z^2 and 2^γ are perfect square and positive power of two, respectively.

2. Preliminaries

This section presents the fundamental concepts, essential definitions, key lemmas, and relevant theorems that are necessary to analyze and solve the Diophantine equations $M_n^x + M_n^y = z^2$ and $M_n^x + M_n^y = 2^\gamma$. Specifically, it sets the groundwork needed to investigate integer solutions to the equations. These also serve as the foundation for the subsequent proofs and discussions of the main findings.

2.1. Mersenne Numbers

We start by defining Mersenne number which is the very core of this research. Mersenne numbers are denoted by M_n.

Definition 1. *Mersenne numbers are numbers of the form* $2^n - 1$*, where* n *is a positive integer.*

The next lemma gives us a property of Mersenne number. Its proof that utilizes modulo can be found in [1].

Lemma 1. *For* $n > 1$*,* $M_n \equiv 3 \pmod 4$*.*

Lemma 1 shows that Mersenne numbers greater than 1 are congruent to $3 \pmod 4$.

2.2. Perfect Squares

As observed in the equation $M_n^x + M_n^y = z^2$, a perfect square is involved. Since one of the approaches we will employ is the modular arithmetic method, it is appropriate to first examine the modular properties of perfect squares. One property of perfect square is given in the next lemma.

Lemma 2. *A perfect square is congruent to* 0 *or* $1 \pmod 4$*.*

2.3. The Mihailescu's Theorem

A very important theorem that led to the solvability of many Diophantine equations and will also be used to prove our results is the Mihailescu's Theorem.

Theorem 1 (Mihailescu's Theorem [13])**.** *The triple* $(3, 2, 2, 3)$ *is the unique solution* (a, b, x, y) *for the Diophantine equation* $a^x - b^y = 1$*, where* a, b, x *and* y *are integers with* $\min\{a, b, x, y\} > 1$*.*

3. Results

This section formally presents the theorems concerning the nonnegative solutions of the Diophantine equations $M_n^x + M_n^y = z^2$ and $M_n^x + M_n^y = 2^\gamma$. In these equations, M_n denotes a Mersenne number.

3.1. The Equation $M_n^x + M_n^y = z^2$

In this part, we solve the $M_n^x + M_n^y = z^2$, where the sum of two powers of a Mersenne number results in a perfect square. Its solutions are given by the following theorem.

Theorem 2. *Every nonnegative integer solution* (M_n, x, y, z) *of the Diophantine equation* $M_n^x + M_n^y = z^2$ *is of the form* $(M_{2\alpha}, 2\beta+1, 2\beta, 2^\alpha M_{2\alpha}^\beta)$*, where* α *and* β *are positive integers.*

Proof. Let M_n be a Mersenne number and x, y, z be nonnegative inetegers such that $M_n^x + M_n^y = z^2$. Without loss of generality, assume $x \geq y$.

If $x = y$, then $2M_n^x = z^2$. If $n = 1$, then $M_n = 1$. This leads to $z^2 = 2$, which has no integer solution. If $n > 1$, then $M_n \equiv 3 \pmod 4$. Consequently, for any positive integer x, M_n^x is either congruent to 1 or 3 (mod 4). Moreover, $2M_n^x \equiv 2 \pmod 4$ and so $z^2 \equiv 2 \pmod 4$. However, this is impossible because $z^2 \equiv 0, 1 \pmod 4$ by Lemma 2.

If $x > y$, then $M_n^y(M_n^{x-y}+1) = z^2$. Note that M_n^y and $M_n^{x-y}+1$ are coprime and z^2 has a factor of M_n and so does z. Now, let $z = 2^\alpha M_n^\beta k$, where $\alpha \geq 1$ and k is an odd integer not divisible by M_n. So, we have

$$M_n^y(M_n^{x-y}+1) = 2^{2\alpha} M_n^{2\beta} k^2.$$

This equation implies the system

$$\begin{cases} M_n^y = M_n^{2\beta} \\ M_n^{x-y}+1 = 2^{2\alpha}k^2. \end{cases}$$

The first equation in the system implies that $y = 2\beta$. The second equation becomes

$$M_n^{x-2\beta}+1 = 2^{2\alpha}k^2.$$

If $x - 2\beta = 1$ or equivalently $x = 2\beta + 1$, then $M_n + 1 = 2^{2\alpha}k^2$. This equation is equivalent to $2^n = 2^{2\alpha}k^2$. This implies that $k = 1$ and $n = 2\alpha$. It further implies that $z = 2^\alpha M_{2\alpha}^\beta$. Hence, the quadruples of the form $(M_{2\alpha}, 2\beta+1, 2\beta, 2^\alpha M_{2\alpha}^\beta)$ are solutions to the Diophantine equation $M_n^x + M_n^y = z^2$.

If $x - 2\beta > 1$, the equation has no solution by Mihailescu's Theorem.

By Theorem 2, the Diophantine equation $M_n^x + M_n^y = z^2$ has infinitely many solutions of the form $(M_{2\alpha}, 2\beta+1, 2\beta, 2^\alpha M_{2\alpha}^\beta)$, where α and β are positive integers. Some of these solutions are presented in Table 1.

Table 1: Solutions $(M_{2\alpha}, 2\beta+1, 2\beta, 2^\alpha M_{2\alpha}^\beta)$ of the Diophantine Equation $M_n^x + M_n^y = z^2$ for $\alpha, \beta \leq 3$

α	β	$M_n = 2^{2\alpha}-1$	$x = 2\beta+1$	$y = 2\beta$	$z = 2^\alpha M_{2\alpha}^\beta$	Solution
1	1	3	3	2	6	$(3, 3, 2, 6)$
1	2	3	5	4	18	$(3, 5, 4, 18)$
1	3	3	7	6	54	$(3, 7, 6, 54)$
2	1	15	3	2	60	$(15, 3, 2, 60)$
2	2	15	5	4	900	$(15, 5, 4, 900)$
2	3	15	7	6	13500	$(15, 7, 6, 13500)$
3	1	63	3	2	504	$(63, 3, 2, 504)$
3	2	63	5	4	31752	$(63, 5, 4, 31752)$
3	3	63	7	6	2000376	$(63, 7, 6, 2000376)$

3.2. The Diophantine Equation $M_n^x + M_n^y = 2^\gamma$

In this part, we find the solutions of the Diophantine equation $M_n^x + M_n^y = 2^\gamma$, where the sum of two powers of a Mersenne number equals a power of two. Its solutions are given in the following theorem.

Theorem 3. *The nonnegative integer solutions* (M_n, x, y, γ) *of the Diophantine equation* $M_n^x + M_n^y = 2^\gamma$ *are* $(M_n, 0, 0, 1)$, $(1, x, y, 1)$ *and* $(M_n, 1, 0, n)$, *where* n *is a positive integer.*

Proof. Let M_n be a Mersenne number and x, y, z be nonnegative inetegers such that $M_n^x + M_n^y = 2^\gamma$. Note that since $\min\{x, y\} = 0$, it follows that $2^\gamma \geq M_n^0 + M_n^0$ or equivalently $2^\gamma \geq 2$. Hence, $\gamma \geq 1$. Now, without loss of generality, assume $x \geq y$. Because M_n and 2 are coprime, the equation results to the system

$$\begin{cases} M_n^y = 1 \\ M_n^{x-y} + 1 = 2^\gamma \end{cases}.$$

The first equation in the system holds if $y = 0$ or $M_n = 1$.

If $M_n = 1$, then the second equation in the system becomes $1^{x-y} + 1 = 2^\gamma$. This equation holds for any nonnegative values of x, y, which results in $2 = 2^\gamma$. This suggests that $\gamma = 1$. Hence, we have the solution $(M_n, x, y, 1) = (1, x, y, 1)$.

If $y = 0$, then the second equation in the system becomes $M_n^x + 1 = 2^\gamma$, which is equivalent to $2^\gamma - M_n^x = 1$. If $\gamma, x > 1$, by Mihailescu's Theorem, there is no solution. If $\gamma = 1$, then $M_n^x = 1$, which gives $x = 0$ for any Mersenne number M_n. Hence, $(M_n, x, y, \gamma) = (M_n, 0, 0, 1)$. If $x = 0$, then $2^\gamma = 2$, which gives the value $\gamma = 1$. This gives the same solution as the previous case. If $x = 1$, then $2^\gamma = M_n + 1$. Let $M_n = 2^n - 1$. Then $2^\gamma = 2^n$. This results to $\gamma = n$. Hence, $(M_n, x, y, \gamma) = (M_n, 1, 0, n)$.

Theorem 3 ensures that the Diophantine equation $M_n^x + M_n^y = 2^\gamma$ admits infinitely many nonnegative integer solutions of the forms $(1, x, y, 1)$, $(M_n, 0, 0, 1)$ and $(M_n, 1, 0, n)$, where n is a positive integer. Table 2 presents several solutions corresponding to the form $(1, x, y, 1)$.

Table 2: Some Solutions of the Diophantine Equation $M_n^x + M_n^y = 2^\gamma$ of the Form $(1, x, y, 1)$

M_n	x	y	γ	$M_n^x + M_n^y = 2^\gamma$	Solution
1	0	0	1	$1^0 + 1^0 = 2^1$	$(1, 0, 0, 1)$
1	1	0	1	$1^1 + 1^0 = 2^1$	$(1, 1, 0, 1)$
1	1	1	1	$1^1 + 1^1 = 2^1$	$(1, 1, 1, 1)$
1	2	0	1	$1^2 + 1^0 = 2^1$	$(1, 2, 0, 1)$
1	2	1	1	$1^2 + 1^1 = 2^1$	$(1, 2, 1, 1)$
1	2	2	1	$1^2 + 1^2 = 2^1$	$(1, 2, 2, 1)$
1	3	0	1	$1^3 + 1^0 = 2^1$	$(1, 3, 0, 1)$
1	3	1	1	$1^3 + 1^1 = 2^1$	$(1, 3, 1, 1)$
1	3	2	1	$1^3 + 1^2 = 2^1$	$(1, 3, 2, 1)$
1	3	3	1	$1^3 + 1^3 = 2^1$	$(1, 3, 3, 1)$

Table 3 lists the first ten solutions of the form $(M_n, 0, 0, 1)$. These solutions corresponds to the cases where $n \geq 10$, illustrating the pattern predicted by Theorem 3 for this specific family of solutions.

Table 3: First Ten Solutions of the Diophantine Equation $M_n^x + M_n^y = 2^\gamma$ of the Form $(M_n, 0, 0, 1)$

n	$M_n = 2^n - 1$	x	y	γ	$M_n^x + M_n^y = 2^\gamma$	Solution
1	1	1	0	1	$1^0 + 1^0 = 2^1$	$(1, 0, 0, 1)$
2	3	0	0	1	$3^0 + 3^0 = 2^1$	$(3, 0, 0, 1)$
3	7	0	0	1	$7^0 + 7^0 = 2^1$	$(7, 0, 0, 1)$
4	15	0	0	1	$15^0 + 15^0 = 2^1$	$(15, 0, 0, 1)$
5	31	0	0	1	$31^0 + 31^0 = 2^1$	$(31, 0, 0, 1)$
6	63	0	0	1	$63^0 + 63^0 = 2^1$	$(63, 0, 0, 1)$
7	127	0	0	1	$127^0 + 127^0 = 2^1$	$(127, 0, 0, 1)$
8	255	0	0	1	$255^0 + 255^0 = 2^1$	$(255, 0, 0, 1)$
9	511	0	0	1	$511^0 + 511^0 = 2^1$	$(511, 0, 0, 1)$
10	1023	0	0	1	$1023^0 + 1023^0 = 2^1$	$(1023, 0, 0, 1)$

Table 4 presents the first ten solutions of the form $(M_n, 1, 0, n)$, where n is a positive integer. These solutions confirms the existence of infinitely many solutions of this type.

Table 4: First Ten Solutions of the Diophantine Equation $M_n^x + M_n^y = 2^\gamma$ of the Form $(M_n, 1, 0, n)$

n	$M_n = 2^n - 1$	x	y	$\gamma = n$	Solution
1	1	1	0	1	$(1, 1, 0, 1)$
2	3	1	0	2	$(3, 1, 0, 2)$
3	7	1	0	3	$(7, 1, 0, 3)$
4	15	1	0	4	$(15, 1, 0, 4)$
5	31	1	0	5	$(31, 1, 0, 5)$
6	63	1	0	6	$(63, 1, 0, 6)$
7	127	1	0	7	$(127, 1, 0, 7)$
8	255	1	0	8	$(255, 1, 0, 8)$
9	511	1	0	9	$(511, 1, 0, 9)$
10	1023	1	0	10	$(1023, 1, 0, 10)$

4. Conclusions

In this research study, we investigated if sum of two powers of Mersenne number can be expressed as a perfect square and as a power of two using elementary methods in number theory. Results of the Diophantine analysis on the equations $M_n^x + M_n^y = z^2$ and $M_n^x + M_n^y = 2^\gamma$ show the existence of solutions. It was revealed that infinitely many powers of the Mersenne numbers of the form $2^{2\alpha} - 1$ can be expressed as a perfect square. Moreover, all powers of two can be represented by a sum of powers of Mersenne number. These findings contribute to a broader understanding of exponential Diophantine equations and the intricate behavior of Mersenne numbers in algebraic and arithmetic settings.

5. Open Problem

This research study solved the two exponential Diophantine equations $M_n^x + M_n^y = z^2$ and $M_n^x + M_n^y = 2^\gamma$. In these two equations, the bases of the powers M_n^x and M_n^y are the same Mersenne numbers. It is still unknown whether the sum of powers of different Mersenne numbers can also be expressed as a perfect square or a power of two. Hence, we propose an open problem on finding the solutions of the exponential Diophantine equations $M_n^x + M_k^y = z^2$ and $M_n^x + M_k^y = 2^\gamma$, where M_n and M_k are different Mersenne numbers.

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