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Approximating the Fixed-Point Using Generalized (γ, κ)-Nonexpansive Mappings with Application

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Abstract. This study introduces a fast-converging iterative method for approximating fixed points of a broad class of generalized nonexpansive (γ, κ) mappings in uniformly convex Banach spaces. We establish both weak and strong convergence theorems for the proposed method. Several numerical examples are presented to illustrate its effectiveness and computational efficiency. Finally, the method is applied to obtain approximate solutions of a two-dimensional Volterra integral equation.

2020 Mathematics Subject Classifications: 47H09, 47H10

Key Words and Phrases: Modified iteration, fixed point, nonexpansive mapping, convergence, convex Banach space.

1. Introduction

The study of fixed points is a fundamental area of mathematics, particularly within mathematical analysis. This field plays a crucial role in addressing a wide range of problems in the mathematical sciences, especially those arising from complex computations or situations in which standard analytical techniques are insufficient. Fixed point theory has proven to be an effective framework for resolving such challenges. Over time, the search for fixed points has led to the development of various mathematical approaches, among which iterative algorithms have emerged as some of the most powerful techniques. The Banach contraction mapping theorem and the Picard iteration are two foundational theoretical contributions that established the basis for iterative fixed point computations [1]. Although the Picard iterative method is straightforward to implement, Krasnoselskii and Hussain [2, 3] noted that it may fail to converge for nonexpansive mappings.

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Nonexpansive mappings, which generalize contractive mappings, play a fundamental role in various fields such as data processing, convex optimization, and functional analysis, as they ensure the stability and persistence of fixed points. In connection with the Banach contraction principle, the Picard iterative method is defined as follows:

$$\chi_{a+1} = T\chi_a. \tag{1}$$

However, this approach generally fails to produce a fixed point when applied to nonexpansive mappings. To overcome this limitation, Mann [4] and Ishikawa [5] introduced two-step iterative methods. Subsequently, several other iterative schemes for approximating fixed points of nonexpansive mappings were proposed by Noor [6], Agarwal [7], Nazir and Abbas [8], Thakur et al. [9], and Arshad and Ullah [10, 11]. These studies emphasize the distinctions among results related to generalized nonexpansive mappings of Suzuki type. In what follows, the closed convex subset is denoted by Γ, while the Banach space is denoted by $\mathcal{B}$. Let $T : \Gamma \to \Gamma$ be a mapping. If $T(\mu) = \mu$, then μ is called a fixed point of T. If T has at least one fixed point, then the set of all fixed points of T is $F_{ix}(T) = \{\mu \in \Gamma : T\mu = \mu\}$. Contraction mapping occurs when a relation T has some $\alpha \in [01)$ such that $||T\tau - T\sigma|| \leq \alpha||\tau - \sigma||$. $||T\tau - T\sigma|| \leq \alpha||\tau - \sigma||$ shows that the mappings are nonexpansive.

Piri et al. [12] introduced a distinct three-step iterative process that achieves significantly faster convergence than the methods previously proposed by other authors, and Tehreem et al. also proposed iterative schemes that converges faster. [13, 14] For two sequences of real numbers, the sequence χ_a generated by Piri et al. is defined as follows m_a and n_a in (0, 1):

$$\begin{cases} \chi_1 \in \Gamma, \\ \chi_{a+1} = (1 - m_a)T\xi_a + m_aT\nu_a, \\ \nu_a = T\xi_a, \\ \xi_a = T(1 - n_a)\chi_a + n_aT\chi_a. \end{cases} \tag{2}$$

Where $0 < n_a, m_a < 1$.

By motivating the refinement of preceding schemes, we developed a new, significantly improved algorithm that converges faster, which is given below.

$$\begin{cases} \chi_1 \in \Gamma, \\ \nu_a = T((1 - m_a)T\chi_a + m_aT\chi_a), \\ \xi_a = T((1 - n_a)\nu_a + n_aT\nu_a). \\ \chi_{a+1} = T(T^2\xi_a), \end{cases} \tag{3}$$

Where $0 < n_a, m_a < 1$.

The focus of this study is the extension of the iterative process (1.3) from the case of generalized γ-nonexpansive mappings to generalized (γ, κ)-nonexpansive mappings. This study seeks to provide a thorough understanding of the theoretical underpinnings of this

extension and its practical applications. Furthermore, we present numerical examples along with the results for both weak and strong convergence to clearly demonstrate the efficacy of the proposed method.

2. Preliminaries

To remember some preliminary information and to understand our primary discoveries in this particular part, we need the following fundamental ideas and concepts.

Definition 1. *A mapping $T : \Gamma \longrightarrow \Gamma$ is called quasi-nonexpansive if* $\mathrm{Fix}(T) \neq \emptyset$ *and*

$$\|T\chi - \mu\| \leq \|\chi - \mu\|, \quad \forall\, \chi \in \Gamma,\ \mu \in \mathrm{Fix}(T).$$

Kirk [15] established in 1965 that every nonexpansive mapping on a bounded, closed, and convex subset of a uniformly convex Banach space has a fixed point. Similar results were independently obtained by Ditrich Göhde [16] and Felix E. Browder [17] in the same year.

Definition 2. *A Banach space $\mathcal{B}$ is said to be a uniformly convex Banach space if, for every $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $\nu, \xi \in \mathcal{B}$ with*

$$\|\nu\| \leq 1, \quad \|\xi\| \leq 1, \quad \|\nu - \xi\| \geq \epsilon,$$

we have

$$\left\| \frac{\nu + \xi}{2} \right\| \leq 1 - \delta.$$

Definition 3 (Opial Condition)**.** *[18] A Banach space $\mathcal{B}$ satisfies the Opial condition if for every sequence $\{\chi_a\} \subset \mathcal{B}$ weakly converging to χ, we have*

$$\limsup_{a\to\infty} ||\chi_a - \chi|| < \limsup_{a\to\infty} ||\chi_a - \xi||, \quad \forall \xi \in \mathcal{B},\ \xi \neq \chi.$$

Definition 4. *[19] Let $\mathcal{B}$ be a Banach space with a closed convex subset Γ, and let $\{\chi_a\}$ be a bounded sequence in $\mathcal{B}$. The asymptotic radius of $\{\chi_a\}$ with respect to Γ is defined as*

$$R(\Gamma, \{\chi_a\}) = \inf_{\chi\in\Gamma} \limsup_{a\to\infty} ||\chi_a - \chi||,$$

and the asymptotic center is defined as

$$A(\Gamma, \{\chi_a\}) = \left\{ \chi \in \Gamma : \limsup_{a\to\infty} ||\chi_a - \chi|| = R(\Gamma, \{\chi_a\}) \right\}.$$

Lemma 1. *[23] Let $\mathcal{B}$ be a Banach space and $0 < t_a < 1$. If sequences $\{\chi_a\}$ and $\{\nu_a\}$ satisfy*

$$\limsup_{a\to\infty} ||\chi_a|| \leq \varphi, \quad \limsup_{a\to\infty} ||\nu_a|| \leq \varphi, \quad \limsup_{a\to\infty} ||t_a\chi_a + (1 - t_a)\nu_a|| = \varphi,$$

for some $\varphi \geq 0$, then

$$\lim_{a\to\infty} ||\chi_a - \nu_a|| = 0.$$

Definition 5. *[24] A mapping $T : \Gamma \to \Gamma$ is a Suzuki generalized nonexpansive mapping if for all $\chi, \nu \in \Gamma$,*

$$\frac{1}{2}||\chi - T\chi|| \leq ||\chi - \nu|| \implies ||T\chi - T\nu|| \leq ||\chi - \nu||.$$

Mappings satisfying this property are called Suzuki generalized nonexpansive mappings. Most Suzuki mappings are also nonexpansive, but the convergence results for Suzuki mappings are broader than those for standard nonexpansive mappings [23].

Definition 6. *Let $\alpha : (0, \infty) \to (0, \infty)$ be an increasing function with $\alpha(0) = 0$. A mapping $T : \Gamma \to \Gamma$, with $\Gamma \subset \mathcal{B}$ closed and convex, satisfies Condition I if*

$$d(\chi, T\chi) \geq \alpha(d(\chi, T\chi)), \quad \text{where } d(\chi, T\chi) = \inf\{||\chi - \tau|| : \tau \in T\chi\}.$$

In 2011, Koji Ayama and Fumaki Kohska [27] introduced $\sim$-nonexpansive mappings, while Pantt and Shokla [28] in 2017 defined *generalized nonexpansive mappings* satisfying Condition C.

Definition 7. *A mapping $T : \Gamma \to \Gamma$, with $\Gamma \subset \mathcal{B}$ closed and convex, is a generalized γ-nonexpansive mapping if for all $\chi, \nu \in \Gamma$,*

$$\frac{1}{2}||\chi - T\chi|| \leq ||\chi - \nu|| \implies ||T\chi - T\nu|| \leq \gamma||\chi - T\nu|| + \gamma||\nu - T\chi|| + (1 - 2\gamma)||\chi - \nu||.$$

Reich-Suzuki-type nonexpansive mappings form a broader class containing Suzuki's generalization, as shown by Pandy et al. [29] in 2019.
Generalized (γ, κ)-nonexpansive mappings are a broader class of mappings that satisfy Suzuki's generalized γ nonexpansive, generalized *kappa*-nonexpansive, and Reich-Suzuki type nonexpansive mappings. Ulah et al.[30] presented this type of mapping in 2020.

Definition 8. *Assume that a map $T : \Gamma \to \Gamma$ and $\Gamma \subset \mathcal{B}$ is known as generalized (γ, κ)-nonexpansive mapping as if for all $\nu, \chi \in \Gamma$ and $\gamma, \kappa \in [0, 1)$ having $\gamma + \kappa \leq 1$ therefore it follows:*

$$\frac{1}{2}||\chi - T\chi|| \leq ||\chi - \nu|| \Rightarrow ||T\chi - T\nu|| \leq \gamma||\chi - T\nu|| + \gamma||\nu - T\chi|| + \kappa||\chi - T\chi|| + \kappa||\nu - T\nu|| + (1 - 2\gamma - 2\kappa)||\chi - \nu||.$$

Proposition 1. *[30] Assume that a map $T : \Gamma \to \Gamma$ and $\Gamma \subset \mathcal{B}$ we get,*

- *All mappings with Requirement C are nonexpansive generalized (0, 0) mapping.*
- *There exists a generalized (γ,0)-nonexpansive projection for every generalized γ nonexpansive mappings.*
- *Generalized (0,κ)-nonexpansive mappings are all κ-Reich-Suzuki-nonexpansive mappings.*

The opposite of the proposition (2.10) does not hold.

Lemma 2. *[30] Suppose that a map $T : \Gamma \to \Gamma$ is generalized (γ, κ)-nonexpansive having a fixed point μ and $\Gamma \subset \mathcal{B}$, then T is referred to as quasi-nonexpansive.*

Lemma 3. *[30] Assume that a map $T : \Gamma \to \Gamma$ is generalized (γ, κ)-nonexpansive having a fixed point μ and $\Gamma \subset \mathcal{B}$ contains the following inequality*

$$||\chi - T\nu|| \leq (\frac{\gamma + \kappa + 3}{1 - \gamma - \kappa})||\chi - T\chi|| + ||\chi - \nu||.$$

3. Main results

We obtain particular convergence results for nonexpansive mappings (1.3) using our approach. This section shows faster results compared to the iterative method in (1.2). For a given UCBS, we now write $\mathcal{B}$ as the Banach space.

Lemma 4. *A closed convex subset $\Gamma \neq \phi$ of a Banach space $\mathcal{B}$ and a generalized (γ, κ)-nonexpansive mapping $T : \Gamma \to \Gamma$ with Fix(T) $\neq \phi$. Here $\lim_{k\to\infty} ||\chi_a - \mu||$ exists for a new modified ishi iterative scheme* (3) *and for all $\mu \in Fix(T)$.*

Proof. Suppose that $\chi \in \Gamma$ and $\mu \in F_{ix}(T)$. By Lemma (2.11), T is known as quasi-nonexpansive.

$$||T\chi - \mu|| \leq ||\chi - \mu||.$$

By using (1.3), we get

$$\begin{aligned} ||\nu_a - \mu|| &\leq ||T((1 - m_a)T\chi_a + m_aT\chi_a) - \mu|| \\ &\leq ||T(1 - m_a)\chi_a + m_aT\chi_a - \mu|| \\ &\leq (1 - m_a)||\chi_a - \mu|| + m_a||T\chi_a - \mu||. \end{aligned} \tag{4}$$

Let, T is (γ, κ)-nonexpansive generalized map and already defined $\mu \in F_{ix}(T) \Longrightarrow T\mu = \mu$, then we obtain,

$$\begin{aligned} ||T\chi_a - \mu|| &\leq ||T\chi_a - T\mu|| \\ &\leq \gamma||\chi_a - T\mu|| + \gamma||\mu - T\chi_a|| + \kappa||\chi_a - T\chi_a|| + \kappa||\mu - T\mu| + (1 - 2\gamma - 2\kappa)||\chi_a - \mu|| \\ &\leq \gamma||\chi_a - \mu|| + \gamma||T\chi_a - \mu|| + \kappa||T\chi_a - \mu|| + \kappa||\chi_a - \mu| + \kappa||\mu - \mu|| \\ &+ (1 - 2\gamma - 2\kappa)||\chi_a - \mu|| \\ &\leq \gamma||T\chi_a - \mu|| + \kappa||T\chi_a - \mu|| + (1 - \gamma - \kappa)||\chi - \mu|| \\ &\leq ||\chi_a - \mu||. \end{aligned} \tag{5}$$

By applying (3.1) in conjunction with (3.2), we have

$$\begin{aligned} ||\nu_a - \mu|| &\leq (1 - m_a)||\chi_a - \mu|| + m_a||\chi_a - \mu|| \\ &\leq ||\chi_a - \mu|| \end{aligned} \tag{6}$$

Now,

$$\begin{aligned} ||\xi_a - \mu|| &\leq ||T((1 - n_a)\nu_a + n_a T\nu_a) - \mu|| \\ &\leq ||(1 - n_a)\nu_a + m_a T\nu_a - \mu|| \\ &\leq (1 - n_a)||\nu_a - \mu|| + n_a||\nu_a - \mu|| \\ = ||\nu_a - \mu|| &\leq ||\chi_a - \mu||. \end{aligned} \tag{7}$$

We have,

$$\begin{aligned} ||\chi_{a+1} - \mu|| &\leq ||T(T^2\xi_a) - \mu|| \\ &\leq ||T(T\xi_a) - \mu|| \\ &\leq ||T\xi_a) - \mu|| \\ &\leq ||\chi_a - \mu||. \end{aligned} \tag{8}$$

As a result, the sequence $\{||\chi_{a+1} - \mu||\}$ is bounded and decreasing for each $\mu \in Fix(M)$. Therefore, a limit exists.
We now show the sufficient and necessary requirement for the existence of a fixed point in the following theorem for generalized (γ, κ)-nonexpansive mappings on a compact convex subset of a Banach space.

Theorem 1. *A closed convex subset $\Gamma \neq \phi$ of space $\mathcal{B}$ and a generalized (γ, κ)-nonexpansive mapping $T : \Gamma \to \Gamma$ with Fix(T) $\neq \phi$. Here $\lim_{k\to\infty} ||\chi_a - \mu||$ exists iff χ_a is bounded for a new modified iterative scheme of ishi*(3)*.*

Proof. Let $\mu \in Fix(M)$ along with $F_{ix}(T) \neq 0$, then by Lemma (4.1) limit exists and it is a bounded sequence. we take

$$\lim_{a\to\infty} ||\chi_a - \mu|| = u. \tag{9}$$

Lemma (2.11) is used to obtain,

$$\limsup_{a\to\infty} ||T\chi_a - \mu|| \leq \limsup_{a\to\infty} ||T\chi_a - \mu|| \leq u. \tag{10}$$

Inequality (3.5) is used. therefore,

$$||\chi_{a+1} - \mu|| \leq ||T(T^2\xi_a) - \mu||$$

$$\begin{aligned} &\leq ||T(T\xi_a) - \mu|| \\ &\leq ||T\xi_a) - \mu|| \\ &\leq ||\nu_a - \mu|| \\ &\leq ||\chi_a - \mu||. \end{aligned} \tag{11}$$

It implies that,

$$||\chi_{a+1} - \mu|| - ||\chi_a - \mu|| \leq \frac{||\chi_{a+1} - \mu|| - ||\chi_a - \mu||}{n_a} \leq ||\nu_a - \mu|| - ||\chi_a - \mu||. \tag{12}$$

Hence it is,

$$||\chi_{a+1} - \mu|| \leq ||\nu_a - \mu||. \tag{13}$$

Now equation (3.6) is used,

$$u \leq \liminf_{a\to\infty} ||\nu_a - \mu||. \tag{14}$$

We get from equations (3.7) and (3.1),

$$u = \lim_{a\to\infty} ||\nu_a - \mu||. \tag{15}$$

Thu,s from the inequality (3.6), we obtain

$$\begin{aligned} u = \lim_{a\to\infty} ||\nu_a - \mu|| &= \lim_{k\to\infty} ||T\nu_a - \mu|| \\ = \lim_{a\to\infty} ||T((1-m_a)\chi_a + m_a T\chi_a) - \mu|| \\ = \lim_{a\to\infty} ||(1-m_a)\chi_a + m_a T\chi_a - \mu|| \\ \leq \lim_{a\to\infty} (1-m_a)||\chi_a - \mu|| + m_a||\chi_a - \mu|| \\ \leq \lim_{a\to\infty} (1-m_a)||\chi_a - \mu|| + \lim_{a\to\infty} m_a||\chi_a - \mu|| \\ \leq u. \end{aligned} \tag{16}$$

Therefore,

$$\lim_{a\to\infty} (1-m_a)||\chi_a - \mu|| + m_a||\chi_a - \mu|| = u. \tag{17}$$

We concluded that $\lim_{a\to\infty} ||T\chi_a - \chi|| = 0$.
We shall now demonstrate this in reverse that $\chi_a is bounded and ||T\chi_a - \chi|| = 0$

$$\gamma(T\mu, \chi_a) = \limsup_{a\to\infty} ||\chi_a - T\mu||$$

$$\leq (\frac{\gamma + \kappa + 3}{1 - \gamma - \kappa}) \limsup_{a\to\infty} ||T\chi_a - \chi_a|| + \limsup_{a\to\infty} ||\chi_a - \mu||$$

$$= \limsup_{a\to\infty} ||\chi_a - \mu||$$
$$= \gamma(\mu, \chi_a). \tag{18}$$

It proves that it possesses only a singleton set and Γ is UCB $\Longrightarrow T\mu = \mu$.

Theorem 2. *A closed convex subset $\Gamma \neq \phi$ of space $\mathcal{B}$ and a nonexpansive generalized (γ, κ) mapping $T : \Gamma \to \Gamma$ with Opial property and Fix(T) $\neq \phi$ for a new modified Ishi iterative scheme* (3)*. Then $\{\chi_a\}$ converges weakly to a fixed point of T.*

Proof. Assume that $\mu zin F_{ix}(T)$. Then, $lim||T\chi_a - \chi_a|| = 0$ and the resulting sequence χ_a is bounded according to Theorem 3.2. Because $\mathcal{B}$ is reflexive, it is convex. Therefore, it has a subsequence χ_{a_i} such that χ_a weakly converges to some $t_1 \in \Gamma$ .It Is $T(t_1) = t_1$ by Lemma 2.11. At this point, demonstrating that the iteration χ_a meets weakly to $t_1 1$, however, to note that the sequence χ_a fails to converge weakly to t_1. There is a subsequence χ_{a_i} of χ_a having $t_2 \in \Gamma$ that weakly converges and $t_1 \neq t_2$.
We prove this by opial property that $t_1 = t_2$.

$$\begin{aligned}
\lim_{a\to\infty} ||\chi_a - t_1|| &= \lim_{r\to\infty} ||\chi_{a_i} - t_1|| \\
&< \lim_{i\to\infty} ||\chi_{a_i} - t_2|| \\
&= \lim_{a\to\infty} ||\chi_a - t_2|| \\
&= \lim_{i\to\infty} ||\chi_{a_i} - t_2|| \\
&< \lim_{i\to\infty} ||\chi_{a_i} - t_1|| \\
&= \lim_{a\to\infty} ||\chi_a - t_1||.
\end{aligned}$$

There is an inconsistency here. Thus, we get $t_1 = t_2$. As a result,χ_a weakly converges to Fix(T).
The subsequent theorem presents the necessary and sufficient conditions under which the convergence becomes stationary.

Theorem 3. *A closed convex subset $\Gamma \neq \phi$ of Banach space $\mathcal{B}$ and a generalized (γ, κ)-nonexpansive mapping $T : \Gamma \to \Gamma$ with Fix(T) $\neq \phi$ for a new modified Ishi iteration scheme* (3)*. Then $\{\chi_a\}$ converges to q fixed point of T iff* $\lim_{a\to\infty}(\chi_a, F_{ix}(T)) = 0$.

Proof. Assume that a sequence χ_a converges to fixed point of T for $\mu \in F_{ix}(T)$, $\chi_a \to \mu$ as $a \to \infty$. Then

$$\lim_{a\to\infty}(\chi_a, F_{ix}(T)) = 0.$$

Assume that its converse $\lim_{a\to\infty}(\chi_a, F_{ix}(T)) = 0$. Limit exists by the given proof of Lemma (4.1) for every $\mu \in F_{ix}(T)$. Therefore for a given ϵ there must be exist $x_0 \in N$, then for $x \geq x_0$ $\lim_{a\to\infty}(\chi_a, F_{ix}(T)) \leq \frac{\epsilon}{2}$. Hence, we obtain

$$||\chi_{a+1} - \chi_a|| \leq ||\chi_{a+1} - \mu|| + ||\chi_a - \mu||$$

$$\leq ||\chi_{a_0} - \mu|| + ||\chi_{a_0} - \mu||$$
$$= 2||\chi_{a_0} - \mu||$$
$$= \epsilon$$

As a result, we deduced that it is a Cauchy-type sequence in Γ. It contains a point t since Γ is an entirely closed section of a Banach space $\mathcal{B}$. such that $\lim_{a\to\infty}(\chi_a = t$.

Theorem 4. *A closed convex subset* $\Gamma \neq \phi$*of space* $\mathcal{B}$ *and a generalized* (γ, κ)*-nonexpansive mapping* $T : \Gamma \to \Gamma$ *having Fix(T)* $\neq \phi$ *for a new modified ishi iteration* (3)*. Then* $\{\chi_a\}$ *converges strongly to fixed point of T*

Proof. it is given that $F_{ix}(T) \neq \phi$, and by the theorem (4.1) we get

$$\lim_{a\to\infty} ||T\chi_a - \chi_a|| = 0 \tag{19}$$

As we know that Γ is convex, there exists a subsequence χ_{a_i} that converges to t. Then we obtain

$$||\chi_{a_i} - T(t)|| \leq (\frac{\gamma + \kappa + 3}{1 - \gamma - \kappa}) \leq ||\chi_{a_i} - T\chi_{a_i}|| + ||\chi_{a_i} + t||$$

By applying the limit, we obtain $\chi_{a_i} \to T(t)$ as $i \to \infty$. Therefor,e the given sequence strongly converges.

Theorem 5. *A closed convex subset* $\Gamma \neq \phi$ *of space* $\mathcal{B}$ *and a generalized* (γ, κ)*-nonexpansive mapping* $T : \Gamma \to \Gamma$ *fulfill the condition I [27] having Fix(T)* $\neq \phi$ *for a new modifieIshihi iterative scheme* (3)*. Then* χ_a *converges strongly to fixed point of T*

Proof. As it is proven in Theorem (3.2)

$$\lim_{a\to\infty} ||T\chi_a - \chi_a|| = 0.$$

By taking condition I, we get

$$0 \leq \lim_{a\to\infty} \alpha d(\chi_a, F_{ix}(T)) \leq \lim_{a\to\infty} ||T\chi_a - \chi_a|| = 0.$$

$$\lim_{a\to\infty} \alpha d(\chi_a, F_{ix}(T)) = 0.$$

Since a function α is an increaing function. So, we have

$$\lim_{a\to\infty} d(\chi_a, F_{ix}(T)) = 0.$$

Thus, all the required conditions are satisfied, and the sequence converges strongly.

Theorem 3.7. Let T be a self-map on any subset Γ that is closed and convex in B. If T forms a generalized (γ, κ)-nonexpansive map that has a nonempty fixed point set $F(T)$,

then, for the sequence of iterations defined in (1.3), the sequence $\{\chi_a\}$ converges weakly to a fixed point of T, provided T satisfies condition (I).

Proof. As shown in Theorem 3.2, we have

$$\lim_{a\to\infty} \|\chi_a - T(\chi_a)\| = 0.$$

From Condition (I) and equation (4.11), we obtain

$$0 \leq \lim_{a\to\infty} \Upsilon(d(\chi_a, F(T))) \leq \lim_{a\to\infty} \|\chi_a - T(\chi_a)\| = 0,$$

which implies

$$\lim_{a\to\infty} \Upsilon(d(\chi_a, F(T))) = 0.$$

Since $\Upsilon : [0,1] \to [0,1]$ is an increasing function with $\Upsilon(0) = 0$ and $\Upsilon(t) > 0$ for all $t > 0$, we deduce that

$$\lim_{a\to\infty} d(\chi_a, F(T)) = 0.$$

Hence, by Theorem 3.4, the sequence $\{\chi_a\}$ converges to a fixed point of T. □

4. Summary

Recent studies in fixed point approximation for generalized nonexpansive mappings have introduced enhanced iterative schemes that demonstrate both theoretical convergence and improved numerical efficiency. For instance, the PC iteration process for Suzuki's generalized nonexpansive mappings has been shown to achieve faster convergence than many classical algorithms, supported by numerical illustrations. Similarly, the HK-iteration scheme for generalized α nonexpansive mappings establishes convergence and improved accuracy compared to existing approaches. In comparison, our proposed method for generalized (γ, κ)-nonexpansive mappings not only establishes both weak and strong convergence in uniformly convex Banach spaces but also exhibits superior convergence speed in numerical experiments. [31] These results align with and extend recent advances in the literature, confirming that our approach offers competitive or improved performance relative to other state-of-the-art iterative schemes.

5. Example

Koji Ayama and Fumaki Kohska [27] introduced a new class of mappings called - nonexpansive mappings in 2011.
A new class of mappings known as generalized-nonexpansive mappings was created in 2017 by Pantt and Shokla [29]. Their structure satisfies Condition C.

Example 1. *Assume that a mapping* $T : [0,\infty) \to [0,\infty)$ *defined as:*

$$T\chi = \begin{cases} 7 & \text{if } \chi \in [0,2) \\ \frac{10\chi+11}{11} & \text{if } \chi \in [2,\infty). \end{cases}$$

Here, Condition C does not meet with the mapping T; therefore the map T is (γ, κ) *generalized non expansive mapping. Let* $\chi = \frac{7}{3}$, $\nu = \frac{17}{6}$ *and* $T\chi = \frac{103}{33}$. *Thus*

$$\frac{1}{2}|\chi - T\chi| = \frac{1}{2}|\frac{7}{3} - \frac{103}{33}| = \frac{1}{2}|\frac{26}{33}| = \frac{13}{33}.$$

And

$$|\chi - \nu| = |\frac{7}{3} - \frac{17}{6}| = \frac{1}{2} \Longrightarrow \frac{1}{2}|\chi - T\chi| \leq |\chi - \nu|.$$

However,

$$|T\chi - T\nu| = |\frac{260}{33},$$
$$|T\chi - T\nu| \geq |\chi - \nu|.$$

A mapping T is not fulfilling the requirement C.
Now we observe the following cases for $\gamma = \frac{10}{21}$ *and* $\kappa = \frac{1}{42}$ *having* $\gamma + \kappa = \frac{1}{2} \leq 1$.
Case 1: If $\chi, \nu \in [0, 2)$, *then*

$$\frac{10}{21}|\chi - T\nu| + \frac{10}{21}|\nu - T\chi| + \frac{1}{42}|\chi - T\chi| + \frac{1}{42}|\nu - T\nu| \geq 0 \geq |T\chi - T\nu|.$$

Case 2: If $\nu \in [0, 2)$ *and* $\chi \in [2, \infty)$, *then*

$$\begin{aligned}
\frac{10}{21}|\chi - T\nu| + \frac{10}{21}|\nu - T\chi| + \frac{1}{42}|\chi - T\chi| + \frac{1}{42}|\nu - T\nu| \\
\frac{10}{21}|\chi - 11| + \frac{10}{21}|\nu - \frac{10\chi + 11}{11}| + \frac{1}{42}|\chi - \frac{10\chi + 11}{11}| + \frac{1}{42}|\nu - 11| \\
\frac{10}{21}|\chi - 11| + \frac{10}{21}|\nu - \frac{10\chi + 11}{11}| + \frac{1}{42}|\frac{\chi - 11}{11}| + \frac{1}{42}|\nu - 11| \\
\geq \frac{10}{11}|\chi - 11| \\
= |T\chi - T\nu|
\end{aligned}$$

Case 3: If $\chi, \nu \in [0, 2)$, *then*

$$\begin{aligned}
\frac{10}{21}|\chi - T\nu| + \frac{10}{21}|\nu - T\chi| + \frac{1}{42}|\chi - T\chi| + \frac{1}{42}|\nu - T\nu| \\
\frac{10}{21}|\chi - \frac{10\chi + 11}{11}| + \frac{10}{21}|\nu - \frac{10\chi + 11}{11}| + \frac{1}{42}|\chi - \frac{10\chi + 11}{11}| + \frac{1}{42}|\nu - \frac{10\chi + 11}{11}| \\
\geq \frac{20}{42}|\frac{21\chi - 21\nu}{11}| + \frac{1}{42}|\frac{\chi - \nu}{11}|
\end{aligned}$$

$$\geq \frac{421}{42}|\frac{\chi-\nu}{11}|$$
$$= |T\chi - T\nu|$$

Thus, it can be concluded that the iterative approach (1.3) gives faster accuracy than the Thakur version, the Abbas and Nazir version, the Noor iteration, the Maan iteration, and the S iteration. The initial point is 16.35312 with $n_a = 0.87$ *and* $m_a = 0.56$*. Figure 1,2, and Table 1,2 present the comparison of results.*

Table 1: Convergence analysis of the proposed modified scheme (3) in Example 1

k	ishi(3)	Piri	Thakur	Abbas	S	Noor	Maan
1	16.35312	16.35312	16.35312	16.35312	16.35312	16.35312	16.35312
2	14.04021	14.86675	15.22812	15.50545	15.65093	15.80823	16.08060
3	12.58037	13.79309	14.33955	14.79201	15.04085	15.31881	15.82195
4	11.72152	13.01755	13.63772	14.19154	14.51080	14.87920	15.57647
5	11.37510	12.45735	13.08338	13.68616	14.05028	14.48434	15.34348
6	11.19498	12.05270	12.64554	13.26080	13.65016	14.12968	15.12236
7	11.10134	11.76040	12.29972	12.90280	13.30253	13.81111	14.91250
8	11.10134	11.54926	12.02657	12.60149	13.0005	13.52497	14.71331
9	11.05265	11.39675	11.81083	12.34790	12.73809	13.26796	14.52427
10	11.01273	11.28659	11.64043	12.13446	12.51009	13.03711	14.34486
11	11.00658	11.20701	11.50583	11.95481	12.31201	12.82975	14.17457
12	11.00341	11.14953	11.39953	11.80362	12.13991	12.6435	14.01296
13	11.00341	11.10801	11.31556	11.67636	11.99038	12.47621	13.85957
14	11.00176	11.07802	11.24925	11.56926	11.86047	12.32595	13.71399
15	11.00094	11.05636	11.19687	11.47912	11.74760	12.19098	13.57583
16	11.00046	11.04071	11.15549	11.40325	11.64953	12.06976	13.44469
17	11.00023	11.02941	11.12282	11.33940	11.56433	11.96087	13.32024
18	11.00010	11.02124	11.09700	11.28565	11.49031	11.86306	13.20211
19	11.00005	11.01534	11.07662	11.24042	11.42599	11.77251	13.09001
20	11.00002	11.01108	11.06052	11.20235	11.37011	11.69630	12.98361
21	11.00000	11.00801	11.04780	11.17031	11.32156	11.62543	12.88262
22	11.00000	11.00578	11.03775	11.14334	11.27938	11.56177	12.78678
23	11.00000	11.00418	11.02982	11.12064	11.24274	11.50458	12.69582
24	11.00000	11.00302	11.02355	11.10154	11.21090	11.45322	12.60948
25	11.00000	11.00218	11.01860	11.08546	11.18323	11.40709	12.44978
26	11.00000	11.00157	11.01469	11.07193	11.15920	11.40565	12.37597
27	11.00000	11.00114	11.01161	11.06054	11.13831	11.32843	12.30592

Table 2: Convergence analysis of our modified scheme (3) for $m_a = 0.56$ and $n_a = 0.87$ in Example 1

k	Ishi (3)	Piri	Thakur	Abbas	S	Noor
1	9.751230	9.751230	9.75123	9.751230	9.751230	9.751230
2	10.40985	10.09780	10.01553	10.00883	9.917079	9.921108
3	10.69321	10.34819	10.22389	10.21329	10.06090	10.06788
4	10.83709	10.52908	10.38815	10.37558	10.18562	10.19468
5	10.91519	10.65978	10.51764	10.50439	10.29378	10.30423
6	10.95590	10.75420	10.61973	10.60662	10.38757	10.39888
7	10.97707	10.82242	10.70021	10.68777	10.46891	10.48065
8	10.99150	10.87170	10.76366	10.75218	10.53944	10.55130
9	10.99477	10.90731	10.81368	10.8033	10.60061	10.61234
10	10.99827	10.93303	10.85312	10.84388	10.65365	10.66508
11	10.99908	10.95162	10.88420	10.87608	10.69965	10.71064
12	10.99965	10.96505	10.90871	10.90164	10.73954	10.75000
13	10.99976	10.97475	10.92803	10.92193	10.77413	10.78401
14	10.99987	10.98175	10.94326	10.93804	10.80413	10.81339
15	10.99995	10.98682	10.95527	10.95082	10.83014	10.83878
16	10.99997	10.99048	10.96474	10.96096	10.85270	10.86071
17	10.99998	10.99312	10.97220	10.96902	10.87226	10.87966
18	10.99999	10.99503	10.97808	10.97541	10.88923	10.89603
19	10.99999	10.99641	10.98272	10.98048	10.90394	10.91017
20	10.99999	10.99741	10.98638	10.98451	10.91670	10.92239
21	10.99999	10.99813	10.98926	10.98770	10.92776	10.93295
22	10.99999	10.99865	10.99153	10.99024	10.93736	10.94207
23	11.00000	10.99903	10.99333	10.99225	10.94568	10.94995
24	11.00000	10.99929	10.99474	10.99385	10.95289	10.95676
25	11.00000	10.99949	10.99585	10.99512	10.95915	10.96264
26	11.00000	10.99963	10.99673	10.99613	10.96457	10.96772

The results in the tables and graphs clearly demonstrate that the proposed iterative scheme outperforms existing methods, achieving faster convergence and higher accuracy in approximating fixed points of generalized (γ, κ) nonexpansive mappings. These findings confirm the efficiency and effectiveness of our approach compared to other iterative schemes.

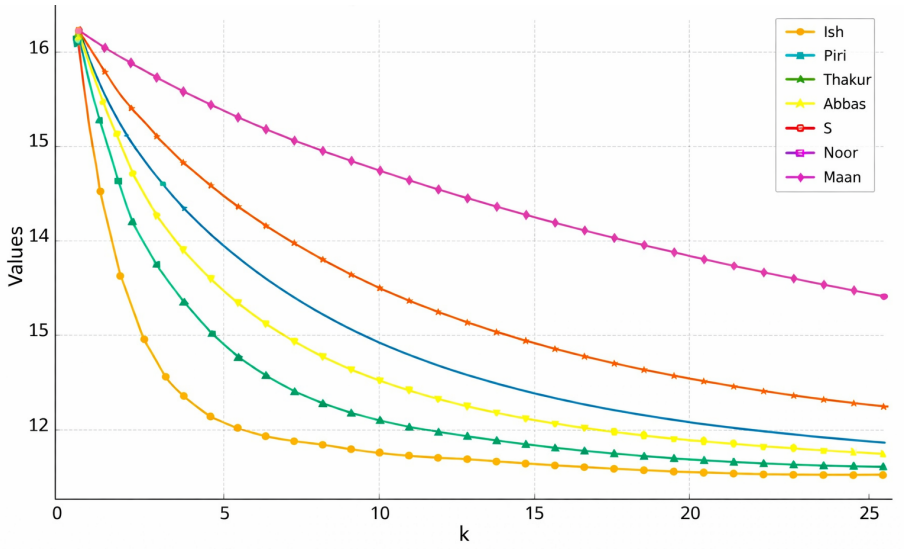

Figure 1: Graphical comparison illustrating the convergence behavior of the modified iterative scheme for Table 1

(3)

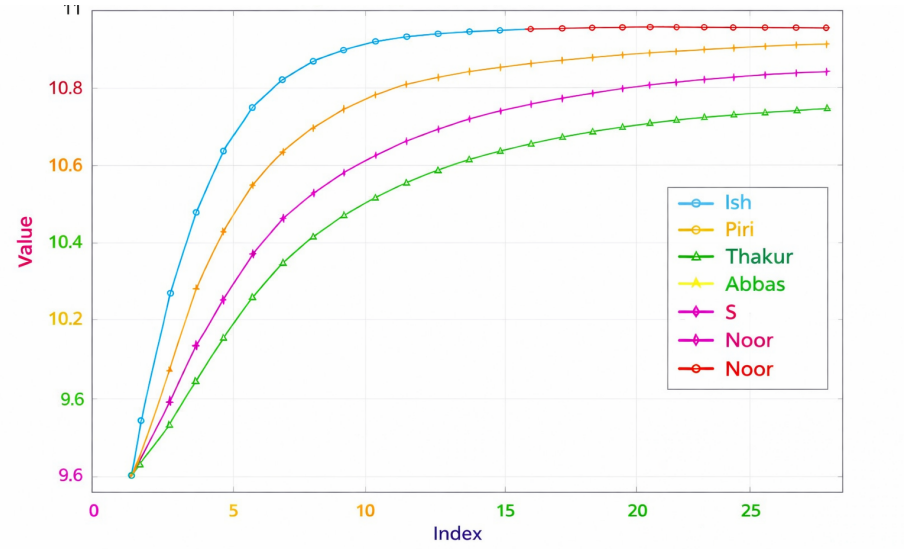

Figure 2: Graphical representation of the convergence of our scheme for table 2

(3)

6. Anaylysis

The proposed iterative scheme for generalized (γ, κ)-nonexpansive mappings demonstrates faster performance in both theoretical and numerical aspects. In numerical exper-

iments, the scheme exhibits faster convergence compared to existing iterative methods, consistently achieving higher accuracy in approximating fixed points. From a computational perspective, each iteration primarily involves evaluating the mapping T and performing a finite number of algebraic operations on elements of the Banach space, resulting in a per-iteration computational cost of $O(n)$, where n denotes the dimensionality of the problem. Considering k iterations required to reach a desired level of accuracy, the overall computational complexity can be expressed as $O(kn)$. This formal analysis complements the empirical results, confirming that the proposed iterative method is both efficient and scalable.

7. Application

7.1. Application to 2D Volterra Integral Equations

We now consider the solution of two-dimensional Volterra integral equations in the framework of generalized (γ, κ)-nonexpansive mappings. Instead of using other iterative algorithms, we apply the ishi-Iterative Algorithm to approximate the solution of the following integral equation:

$$h(x,y) = \kappa(x,y) + \int_0^x \int_0^y \delta_1(\lambda, r, h(\lambda, r))\, d\lambda dr + \Lambda \int_0^x \delta_2(y, r, h(x, r))\, dr + \gamma \int_0^y \delta_3(x, \lambda, h(y, \lambda))\, d\lambda,$$

for all $x, y, \lambda, r \in [0,1]$, where $h \in U \times U$, $\kappa : [0,1] \times [0,1] \to \mathbb{R}^2$, and $\Lambda_i (i = 1,2,3) : [0,1] \times [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$. Here $\Lambda, \gamma \geq 0$, and $M = C[0,1]$ is a Banach space equipped with the maximum norm

$$\|u - v\| \infty = \max \omega \in [0,1] |u(\omega) - v(\omega)|, \quad \forall u, v \in C[0,1].$$

Theorem 7.1. Consider Ω as a closed and convex subset of M such that $T : \Omega \to \Omega$ is defined by

$$T(h(x,y)) = \kappa(x,y) + \int_0^x \int_0^y \delta_1(\lambda, r, h(\lambda, r))\, d\lambda dr + \Lambda \int_0^x \delta_2(y, r, h(x, r))\, dr + \gamma \int_0^y \delta_3(x, \lambda, h(y, \lambda))\, d\lambda,$$

Assume the following assertions hold:

(A_1) The function $h : U \times U \to \mathbb{R}^2$ is continuous.

(A_2) The functions $\Lambda_i (i = 1,2,3) : [0,1] \times [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$ are continuous, and there exist positive constants l_1, l_2, l_3 such that for all $\sigma_1, \sigma_2 \in \mathbb{R}^2$,

$$\begin{aligned} |\delta_1(\lambda, v, \sigma_1) - \delta_1(\lambda, v, \sigma_2)| &\leq l_1 \|\sigma_1 - \sigma_2\|, \\ |\delta_2(\lambda, v, \sigma_1) - \delta_2(\lambda, v, \sigma_2)| &\leq l_2 \|\sigma_1 - \sigma_2\|, \\ |\delta_3(\lambda, v, \sigma_1) - \delta_3(\lambda, v, \sigma_2)| &\leq l_3 \|\sigma_1 - \sigma_2\|. \end{aligned}$$

(A_3) For $\Lambda, \gamma \geq 0$, $l_1 + \Lambda l_2 + \gamma l_3 \leq \phi$, where $\phi \in (0,1)$.

Therefore, the ishi algorithm converges to a solution as long as the sequence of iterates t $\liminf_{na\to\infty} d(\chi_a, T) = 0$ gets arbitrarily close to the solution set.

Proof. Let $u, v \in U \times U$. Then

$$\|u - T(v)\|_\infty = \max_{\omega\in[0,1]} |u(x,y)(\omega) - T(v(x,y))(\omega)|.$$

Following similar arguments as in the earlier Theorems, we obtain

$$\|u - T(v)\|_\infty \leq (l_1 + \Lambda l_2 + \gamma l_3)\|u - v\|_\infty.$$

Therefore, T is a generalized (γ, κ)-nonexpansive mapping since it satisfies

$$\|x - T(y)\| \leq \frac{3\gamma + \kappa}{1 - \gamma - \kappa}\|x - T(x)\| + \|x - y\|,$$

for $\frac{3\gamma+\kappa}{1-\gamma-\kappa} = 1$.

Hence, by Theorem 3.2, all conditions are satisfied, and the ishi-Iterative Algorithm converges to the solution.

8. Conclusion

This paper carried out the following new findings.

- In this study, we used an iterative method that was suggested by Ishi (1.3) to estimate fixed points related to nonexpansive generalized (γ, κ)-mappings. Some convergence results are shown using the iterative approach. We proved both weak and strong convergence results for the mappings in Banach spaces that are uniformly convex and show generalized (γ, κ)- nonexpansiveness.
- A few numerical studies have been conducted to validate the paper's primary findings. A numerical example shows that the ishi-iterative scheme for generalized (γ, κ)-nonexpansive mappings possess faster convergence rates than some of the existing algorithms. We presented convergence conditions for generalized non-expansive mappings by an Ishi-iterative approach. Finally, sufficient convergence conditions for generalized nonexpansive mappings were formulated within the framework of the Ishi-type iterative approach, highlighting the effectiveness and applicability of the proposed method.

Declarations

Conceptualization and methodology, T.I.; software, A.B. and T.I.; investigation, T.I., A.B ,and A.H; Formal Analysis, H.A. and A.H.; All authors; writing—original draft preparation, All authors writing—review and editing. All authors have read and agreed to the published version of the manuscript.

Ethics Approval

This is my doctoral research proposal that was approved by BASAR of FJWU.

Availability of Data and Materials

No data were generated or analysed for the objective of this inquiry.

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Competing interests

The writers claim to have no conflicting agendas.

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