EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2026, Vol. 19, Issue 2, Article Number 7266
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Fibonacci Sequences in the Möbius Gyrovector Space

Rasimate Maungchang[1], Tarid Suwansri[1], Teerapong Suksumran[2,*]

[1] *School of Science, Walailak University, Nakhon Si Thammarat 80160, Thailand*
[2] *Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand*

Abstract. Let G be a non-empty set, and let $\star$ be a binary operation on G. In $(G, \star)$ with $a, b \in G$, the right-Fibonacci sequence generated by a and b is the sequence $F_0, F_1, \ldots, F_n, \ldots$ defined by the recurrence relation $F_0 = a, F_1 = b$, and $F_n = F_{n-2} \star F_{n-1}$ for all integers $n \geq 2$. In this article, we focus on right-Fibonacci sequences in the Möbius gyrovector space $(\mathbb{D}, \oplus_M, \otimes_M)$, with particular attention to the convergence of sequences generated by a and b, where a and b are on the same diameter of the open unit disk $\mathbb{D}$ in the complex plane. It turns out that such sequences converge to $a/|a|$, or $b/|b|$, or 0, depending on relationships among a, b, and the golden ratio.

2020 Mathematics Subject Classifications: 11B39, 20N05, 51M10

Key Words and Phrases: Fibonacci sequence, gyrogroup, gyrovector space, möbius addition, hyperbolic geometry

1. Introduction

The Möbius gyrovector space [1–4] provides a rich setting in which classical geometric objects, such as lines and triangles, can be reinterpreted in a hyperbolic geometry framework. Unlike the Euclidean vector space, the Möbius gyrovector space preserves essential features of non-Euclidean geometry while maintaining algebraic structures that resemble vector spaces, as indicated in Section 6.14 of [5]. From a theoretical perspective, the geometric structure of the Möbius gyrovector space may provide an alternative setting for studying optimization principles in non-Euclidean contexts [6]. In the present article, we aim to study the convergence of right-Fibonacci sequences in the open unit disk $\mathbb{D}$ of the complex plane under Möbius addition $\oplus_M$ and Möbius scalar multiplication $\otimes_M$. We then find the limits of right-Fibonacci sequences generated by a and b, where a and b lie on the same diameter in $\mathbb{D}$. We also exhibit concrete examples of Fibonacci sequences in the Möbius gyrovector space that converge to distinct limits depending on the interplay between the initial generators and the golden ratio. These examples illustrate how

*Corresponding Author.
DOI: https://doi.org/10.29020/nybg.ejpam.v19i2.7266

Email addresses: mate105@gmail.com (R. Maungchang)
taridsuwansri@gmail.com (T. Suwansri), teerapong.suksumran@cmu.ac.th (T. Suksumran)

gyrovector spaces can extend familiar mathematical phenomena into new geometrical contexts, thereby enriching both the theory of gyrovector spaces and the study of sequence convergence.

2. Background

2.1. Gyrogroups and gyrovector spaces

In this subsection, we provide the formal definitions of gyrogroups and gyrovector spaces. Let $(G, \oplus)$ be a groupoid. We will simply call it G if there is no ambiguity in the operation. An *automorphism* f of G is a bijection from G to itself with the property that $f(g_1 \oplus g_2) = f(g_1) \oplus f(g_2)$ for all $g_1, g_2 \in G$. The set of all automorphisms on G is denoted by $\mathrm{Aut}(G, \oplus)$.

Definition 1 (Definition 2.7 of [5]). Let $(G, \oplus)$ be a non-empty groupoid. We say that G is a *gyrogroup* if the following properties hold:

(i) There is a unique identity element $e \in G$ such that

$$e \oplus x = x = x \oplus e \qquad \text{for all } x \in G.$$

(ii) For each $x \in G$, there exists a unique *inverse* element $\ominus x \in G$ such that

$$\ominus x \oplus x = e = x \oplus (\ominus x).$$

(iii) For all $x, y \in G$, there exists an automorphism $\mathrm{gyr}[x, y] \in \mathrm{Aut}(G, \oplus)$ such that

$$x \oplus (y \oplus z) = (x \oplus y) \oplus \mathrm{gyr}[x, y](z) \qquad \text{(left gyroassociative law)}$$

for all $z \in G$.

(iv) For all $x, y \in G$, $\mathrm{gyr}[x \oplus y, y] = \mathrm{gyr}[x, y]$. (left loop property)

For all elements a, b, c in G, the gyroautomorphism $\mathrm{gyr}[a, b]$ is indeed given by the following identity:

$$\mathrm{gyr}[a, b](c) = \ominus(a \oplus b) \oplus (a \oplus (b \oplus c)). \qquad \text{(gyrator identity)}$$

Definition 2 (Definition 2.8 of [5]). A gyrogroup $(G, \oplus)$ is *gyrocommutative* if

$$a \oplus b = \mathrm{gyr}[a, b](b \oplus a)$$

for all $a, b \in G$.

Definition 3 (Definition 6.2 of [5]). A *real inner product gyrovector space* $(G, \oplus, \otimes)$ (*gyrovector space*, in short) is a gyrocommutative gyrogroup $(G, \oplus)$ that obeys the following axioms:

(i) G is a subset of a real inner product vector space $\mathbb{V}$ called the carrier of G, $G \subset \mathbb{V}$, from which it inherits its inner product, $\cdot$, and norm, $\|\cdot\|$, which are invariant under gyroautomorphisms, that is,

(V0) $\mathrm{gyr}[\mathbf{u},\mathbf{v}](\mathbf{a}) \cdot \mathrm{gyr}[\mathbf{u},\mathbf{v}](\mathbf{b}) = \mathbf{a} \cdot \mathbf{b}$ (Inner Product Gyroinvariance)
for all points $\mathbf{a},\mathbf{b},\mathbf{u},\mathbf{v} \in G$.

(ii) G admits a scalar multiplication, $\otimes$, possessing the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points $\mathbf{a} \in G$:

(V1) $1 \otimes_M \mathbf{a} = \mathbf{a}$ (Identity Scalar Multiplication)

(V2) $(r_1 + r_2) \otimes_M \mathbf{a} = (r_1 \otimes_M \mathbf{a}) \oplus_M (r_2 \otimes_M \mathbf{a})$ (Scalar Distributive Law)

(V3) $(r_1 r_2) \otimes_M \mathbf{a} = r_1 \otimes_M (r_2 \otimes_M \mathbf{a})$ (Scalar Associative Law)

(V4) $\frac{|r| \otimes_M \mathbf{a}}{\|r \otimes_M \mathbf{a}\|} = \frac{\mathbf{a}}{\|\mathbf{a}\|}, \mathbf{a} \neq 0, r \neq 0$ (Scaling Property)

(V5) $\mathrm{gyr}[\mathbf{u},\mathbf{v}](r \otimes_M \mathbf{a}) = r \otimes_M \mathrm{gyr}[\mathbf{u},\mathbf{v}](\mathbf{a})$ (Gyroautomorphism Property)

(V6) $\mathrm{gyr}[r_1 \otimes_M \mathbf{v}, r_2 \otimes_M \mathbf{v}] = I$ (Identity Gyroautomorphism)

(iii) Real, one-dimensional vector space structure $(\|G\|, \oplus, \otimes)$ for the set $\|G\|$ of one-dimensional vectors

(VV) $\|G\| = \{\pm\|\mathbf{a}\| : \mathbf{a} \in G\} \subset \mathbb{R}$ (Vector Space)
with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in G$,

(V7) $\|r \otimes_M \mathbf{a}\| = |r| \otimes_M \|\mathbf{a}\|$ (Homogeneity Property)

(V8) $\|\mathbf{a} \oplus_M \mathbf{b}\| \leq \|\mathbf{a}\| \oplus_M \|\mathbf{b}\|$ (Gyrotriangle Inequality)

2.2. The Möbius gyrovector space

In this subsection, we provide the definition of Möbius addition $\oplus_M$ and Möbius scalar multiplication $\otimes_M$ on the open unit disk $\mathbb{D}$ in the complex plane. The real inner product gyrovector space structure of $(\mathbb{D}, \oplus_M, \otimes_M)$ was introduced by Ungar and discussed in Section 6.14 of [5].

Let $\mathbb{D}$ be the complex open unit disk, that is, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and let $\mathbb{R}$ be the set of real numbers. Define *Möbius addition* $\oplus_M : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ and *Möbius scalar multiplication* $\otimes_M : \mathbb{R} \times \mathbb{D} \to \mathbb{D}$ by the formula

$$a \oplus_M b = \frac{a + b}{1 + \bar{a}b} \tag{1}$$

for all $a, b \in \mathbb{D}$ and by the formula

$$r \otimes_M z = \frac{(1+|z|)^r - (1-|z|)^r}{(1+|z|)^r + (1-|z|)^r} \frac{z}{|z|} \tag{2}$$

for all $r \in \mathbb{R}$ and for all $z \in \mathbb{D}$, where $z \neq 0$; and $r \otimes_M 0 = 0$. Möbius addition is not associative in general. Hence, $(\mathbb{D}, \oplus_M)$ is not a group. However, Ungar found that $(\mathbb{D}, \oplus_M)$ is a gyrogroup with the gyroautomorphisms or gyrations defined by the formula

$$\operatorname{gyr}[a, b](c) = \frac{1 + a\bar{b}}{1 + \bar{a}b} c \tag{3}$$

for all $a, b, c \in \mathbb{D}$. Furthermore, the space $(\mathbb{D}, \oplus_M, \otimes_M)$ is shown to be a gyrovector space; see Theorem 6.84 of [5].

3. Results

In this section, we determine the convergence of right-Fibonacci sequences whose generators lie on the same diameter of $\mathbb{D}$. Taking advantage of the fact that a diameter of $\mathbb{D}$ is the intersection of a one-dimensional real vector subspace of $\mathbb{C}$ with $\mathbb{D}$. (see, for instance, Section 3 of [1]), we find that convergence depends on relationships between the two generators and the golden ratio.

Definition 4. Let G be a non-empty set, and let $\star$ be a binary operation on G. In $(G, \star)$ with $a, b \in G$, *the right-Fibonacci sequence* generated by a and b is the sequence (a_n), where $a_0 = a, a_1 = b$, and $a_n = a_{n-2} \star a_{n-1}$ for all integers $n \geq 2$.

Using the exponential forms of complex numbers, one can easily see that a diameter of $(\mathbb{D}, \oplus_M, \otimes_M)$ is the intersection of a one-dimensional real vector subspace of $\mathbb{C}$ with $\mathbb{D}$. The following theorem, where the operation $\star$ in Definition 4 is taken to be Möbius addition $\oplus_M$, can be proved directly using mathematical induction.

Theorem 1. *If a and b are on the same diameter of $\mathbb{D}$, then the right-Fibonacci sequence generated by a and b is given by*

$$a_0 = a, a_1 = b, \text{ and } a_n = (F_{n-1} \otimes_M a) \oplus_M (F_n \otimes_M b) \text{ for } n \geq 2,$$

where F_i is the i-th term in the classical Fibonacci sequence generated by $F_0 = 0$ and $F_1 = 1$.

The previous theorem suggests that dealing with right-Fibonacci sequences with the two generators on the same diameter should be an easier case.

Definition 5. Let $a \in (-1, 1)\backslash\{0\}$. The *sign* of a, denoted by $\operatorname{sgn}(a)$, is 1 if $a > 0$; and -1 if $a < 0$.

The following two lemmas are essential for establishing the main theorem and are of importance in their own right.

Lemma 1. *Let a and b be in the interval $(0, 1) \subset \mathbb{D}$. Then, $a \oplus_M b > \max\{a, b\}$.*

Proof. Since $1 > a^2$ and $b > 0$, we obtain that $b > a^2 b$. Then, $a+b > a+a^2 b = a(1+ab)$, and so $a \oplus_M b = \dfrac{a+b}{1+ab} > a$. A similar argument shows that $a \oplus_M b > b$. $\square$

On page 97 of [7], the authors mentioned a version of the following lemma in terms of Fibonacci vector sequences in $\mathbb{R}^n$ without giving a proof. Here, we present a version adapted to our context, together with its proof.

Lemma 2. *Let a and b be in the interval $(-1,1)\backslash\{0\} \subseteq \mathbb{D}$, and let α be the golden ratio $\dfrac{1+\sqrt{5}}{2}$. We write $a = c \otimes_M b$ for some $c \in \mathbb{R}$. Let (a_n) be the right-Fibonacci sequence generated by a and b. Then, (a_n) converges in the closure $\overline{\mathbb{D}}$ of $\mathbb{D}$ with the following conditions.*

(i) If $\operatorname{sgn}(a) = \operatorname{sgn}(b)$, then $\lim_{n\to\infty}(a_n) = \operatorname{sgn}(a) = \operatorname{sgn}(b)$.

(ii) If $\operatorname{sgn}(a) \neq \operatorname{sgn}(b)$, then

$$\lim_{n\to\infty} a_n = \begin{cases} \operatorname{sgn}(a), & |c| > \alpha; \\ \operatorname{sgn}(b), & |c| < \alpha; \\ 0, & |c| = \alpha. \end{cases}$$

Proof. Suppose that $a, b > 0$. Then, by Lemma 1, (a_n) is an increasing sequence. The sequence (a_n) is bounded above by 1 due to the closure property of Möbius addition. Hence, by the Monotone Convergence Theorem, (a_n) converges in $(0,1] \subset \overline{\mathbb{D}}$. Recall that

$$a_n = a_{n-2} \oplus_M a_{n-1} = \frac{a_{n-2} + a_{n-1}}{1 + a_{n-2}a_{n-1}}.$$

Thus,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{a_{n-2} + a_{n-1}}{1 + a_{n-2}a_{n-1}} \right) = \frac{\lim_{n\to\infty} a_{n-2} + \lim_{n\to\infty} a_{n-1}}{1 + \lim_{n\to\infty} a_{n-2} \lim_{n\to\infty} a_{n-1}}. \tag{4}$$

Let $\lim_{n\to\infty} a_n = L$. From Equation (4), we have

$$\begin{aligned} L &= \frac{L+L}{1+L\cdot L} \\ L(1+L^2) &= 2L \\ L + L^3 - 2L &= 0 \\ L(L-1)(L+1) &= 0. \end{aligned}$$

Hence, $L = -1$ or 0 or 1. Since (a_n) is increasing and $a, b > 0$, we conclude that $L = 1 = \operatorname{sgn}(a)$.

With a similar argument, it can be shown that if $a, b < 0$, then (a_n) converges to $-1 = \operatorname{sgn}(a)$. This completes the proof of part 1.

To prove part 2, we divide the proof into three cases.

Case $|c| > \alpha$. Consider $\lim_{n\to\infty} a_n$. Using the fact that $((-1,1), \oplus_M, \otimes_M)$ is a vector space, we have

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} ((F_{n-1} \otimes_M a) \oplus_M (F_n \otimes_M b)) \\
&= \lim_{n\to\infty} ((F_{n-1} \otimes_M (c \otimes_M b)) \oplus_M (F_n \otimes_M b)) \\
&= \lim_{n\to\infty} (((F_{n-1}c) \otimes_M b) \oplus_M (F_n \otimes_M b)) \\
&= \lim_{n\to\infty} ((F_{n-1}c + F_n) \otimes_M b) \\
&= \left(\lim_{n\to\infty} (F_{n-1}c + F_n)\right) \otimes_M b \\
&= \left(\lim_{n\to\infty} \left(F_{n-1}\left(c + \frac{F_n}{F_{n-1}}\right)\right)\right) \otimes_M b \\
&= -\operatorname{sgn}(b) \\
&= \operatorname{sgn}(a).
\end{aligned}$$

To see that the seventh equality is true, we note that the limit in the sixth equality is $-\infty$, and, by definition, $r \otimes_M b = \dfrac{(1+|b|)^r - (1-|b|)^r}{(1+|b|)^r + (1-|b|)^r} \dfrac{b}{|b|}$.

Case $|c| < \alpha$. In a similar fashion to the previous case, we can show that (a_n) converges to $\operatorname{sgn}(b)$.

Case $|c| = \alpha$, that is, $a = -\alpha \otimes_M b$. Consider $\lim_{n\to\infty} a_n$:

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} ((F_{n-1} \otimes_M a) \oplus_M (F_n \otimes_M b)) \\
&= \lim_{n\to\infty} ((F_{n-1} \otimes_M (-\alpha \otimes_M b)) \oplus_M (F_n \otimes_M b)) \\
&= \lim_{n\to\infty} (((-F_{n-1}\alpha) \otimes_M b) \oplus_M (F_n \otimes_M b)) \\
&= \lim_{n\to\infty} ((-F_{n-1}\alpha + F_n) \otimes_M b) \\
&= \left(\lim_{n\to\infty} (-F_{n-1}\alpha + F_n)\right) \otimes_M b.
\end{aligned}$$

We note that the fifth equality comes from the fact that $\otimes_M$ is continuous, and so the limit is distributive. Next, we show that $\lim_{n\to\infty}(-F_{n-1}\alpha + F_n) = 0$. Recall the closed form of the classical Fibonacci sequence, which is

$$F_n = \frac{1}{\sqrt{5}} \left(\phi_1^n - \phi_2^n\right),$$

where $\phi_1 = \dfrac{1+\sqrt{5}}{2}$ and $\phi_2 = \dfrac{1-\sqrt{5}}{2}$. Therefore,

$$\lim_{n\to\infty}(-F_{n-1}\alpha + F_n) = \lim_{n\to\infty} \left(-\frac{1}{\sqrt{5}}(\phi_1^{n-1} - \phi_2^{n-1})\alpha + \frac{1}{\sqrt{5}}(\phi_1^n - \phi_2^n)\right)$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{5}} \lim_{n\to\infty} \left(-(\phi_1^{n-1} - \phi_2^{n-1})\alpha + (\phi_1^n - \phi_2^n)\right) \\
&= \frac{1}{\sqrt{5}} \lim_{n\to\infty} ((\phi_2^{n-1} - \phi_1^{n-1})\phi_1 + (\phi_1^n - \phi_2^n)) \\
&= \frac{1}{\sqrt{5}} \lim_{n\to\infty} (\phi_1\phi_2^{n-1} - \phi_1^n + \phi_1^n - \phi_2^n) \\
&= \frac{1}{\sqrt{5}} \lim_{n\to\infty} (\phi_1\phi_2^{n-1} - \phi_2^n) \\
&= \frac{1}{\sqrt{5}} \lim_{n\to\infty} (\phi_2^{n-1}(\phi_1 - \phi_2)) \\
&= \frac{1}{\sqrt{5}} \lim_{n\to\infty} \left(\phi_2^{n-1}\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right)\right) \\
&= \frac{1}{\sqrt{5}} \lim_{n\to\infty} (\phi_2^{n-1}\sqrt{5}) \\
&= \lim_{n\to\infty} (\phi_2^{n-1}) \\
&= \lim_{n\to\infty} \left(\frac{1-\sqrt{5}}{2}\right)^{n-1} \\
&= 0.
\end{aligned}
$$

Hence, $\lim_{n\to\infty} a_n = (\lim_{n\to\infty}(-F_{n-1}\alpha + F_n)) \otimes_M b = 0 \otimes_M b = 0$. This completes the proof of part 2. $\square$

We are now in a position to state the main theorem of this section, as shown below.

Theorem 2. *Let a and b be on the same diameter $D \subseteq \mathbb{D}$, and let (a_n) be the right-Fibonacci sequence generated by a and b. We write $a = c \otimes_M b$ for some $c \in \mathbb{R}$. Then, (a_n) converges in the closure $\overline{\mathbb{D}}$ of $\mathbb{D}$ with the following conditions.*

(i) If $a = b = 0$, then $\lim_{n\to\infty} a_n = 0$.

(ii) If one of a and b is non-zero and they are on the same radius, then $\lim_{n\to\infty} (a_n) = a/|a| = b/|b|$.

(iii) If one of a and b is non-zero and they are on the opposite radii, then

$$
\lim_{n\to\infty} a_n = \begin{cases} a/|a|, & |c| > \alpha; \\ b/|b|, & |c| < \alpha; \\ 0, & |c| = \alpha. \end{cases}
$$

Proof. The theorem follows directly from Lemma 2, using the fact that any rotation is an isomorphism from $(-1, 1)$ to D. $\square$

We exhibit three concrete examples regarding our results.

Example 1. *Let* (a_n) *be the right-Fibonacci sequence generated by* $a = 0.2$ *and* $b = 0.4$. *Observe that* a *and* b *are on the same radius. Then, by Theorem 2,* $\lim_{n\to\infty} a_n = a/|a| = 1$; *see Fig. 1 (Left).*

Example 2. *Let* (a_n) *be the right-Fibonacci sequence generated by*

$$a = (-1.3) \otimes_M (0.2 + 0.2i)$$

and $b = 0.2 + 0.2i$. *Observe that* a *and* b *are on the opposite radii and* $|-1.3| < \alpha$, *where* α *is the golden ratio. Then, by Theorem 2,* $\lim_{n\to\infty} a_n = (0.2 + 0.2i)/|0.2 + 0.2i| = b/|b|$; *see Fig. 1 (Right).*

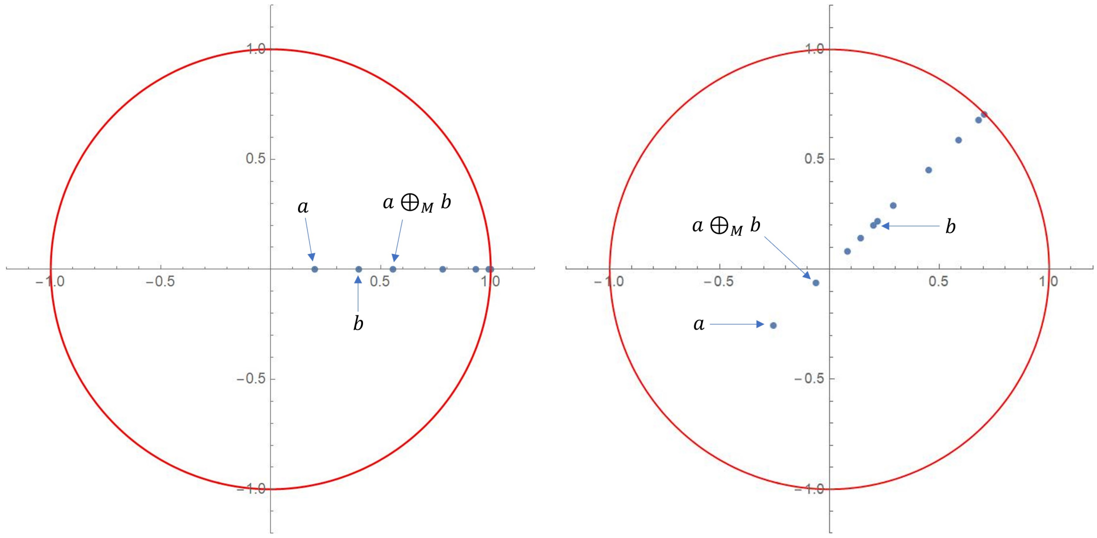

Figure 1: (Left) Part of the Fibonacci sequence in the Möbius gyrovector space generated by $a = 0.2$ and $b = 0.4$. (Right) Part of the Fibonacci sequence in the Möbius gyrovector space generated by $a = (-1.3) \otimes_M (0.2+0.2i)$ and $b = 0.2 + 0.2i$.

Example 3. *Let* (a_n) *be the right-Fibonacci sequence generated by* $a = (-\alpha) \otimes_M (0.5+0.5i)$ *and* $b = 0.5 + 0.5i$. *Observe that* a *and* b *are on the opposite radii and* $|-\alpha| = \alpha$, *where* α *is the golden ratio. Then, by Theorem 2,* $\lim_{n\to\infty} a_n = 0$; *see Fig. 2 (Left).*

4. Conclusions

We show that every Fibonacci sequence in the Möbius gyrovector space with the two generators on the same diameter converges in $\overline{\mathbb{D}}$ to one of the endpoints of that diameter or to 0. An interesting direction for further investigation is the extension of our result to the case where the two generators are not aligned on the same diameter. Numerical experiments suggest that, when the generators do not lie on the same diameter, the full sequence may fail to converge, while its even and odd subsequences may converge separately. See, for instance, Fig. 2 (Right).

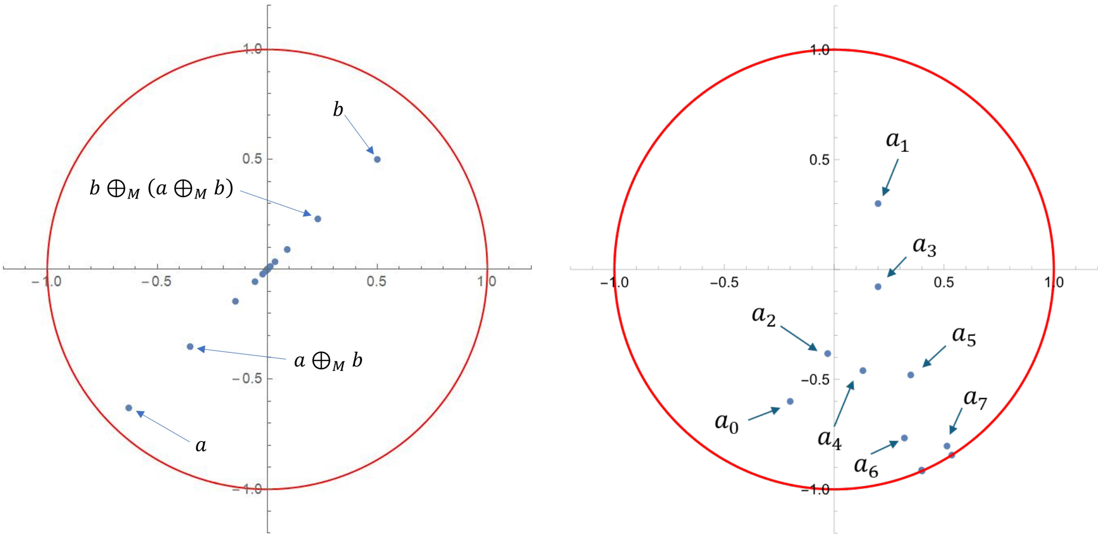

Figure 2: (Left) Part of the Fibonacci sequence in the Möbius gyrovector space generated by $a = (-(1+\sqrt{5})/2) \otimes_M (0.5+0.5i)$ and $b = 0.5+0.5i$. (Right) Part of the Fibonacci sequence in the Möbius gyrovector space generated by a_0 and a_1, where the generators are not on the same diameter.

Acknowledgements

The author would like to thank Watchareepan Atiponrat and Jaturon Wattanapan for their collaboration and their fruitful discussions. The author would like to thank the reviewers and the editor for their careful reading of the manuscript and for their valuable comments and suggestions, which helped improve the clarity and quality of the article. This work was supported by Fundamental Fund 2025, Chiang Mai University.

Conflict of interest

The authors declare no conflict of interest.

Data availability

Data sharing is not applicable to this article, as no datasets were generated or analyzed during the current study.

Use of Artificial Intelligence (AI) tools declaration

The authors declare that Artificial Intelligence (AI) tools were used in the preparation of this manuscript only for language editing and grammar correction. The author takes full responsibility for the content, interpretation, and conclusions of this work.

Author contributions

R. Maungchang: conceptualization, methodology, writing—original draft preparation, validation, investigation, visualization, supervision; T. Suwansri: validation, investigation, visualization; T. Suksumran: validation, investigation, writing—review and editing, project administration. All authors have read and agreed to the published version of the manuscript.

References

[1] T. Abe and K. Watanabe. Finitely generated gyrovector subspaces and orthogonal gyrodecomposition in the Möbius gyrovector space. *J. Math. Anal. Appl.*, 449(1):77–90, 2017.

[2] M. Ferreira. Harmonic analysis on the Möbius gyrogroup. *J. Fourier Anal. Appl.*, 21(2):281–317, 2015.

[3] A. A. Ungar. Hyperbolic trigonometry and its application in the Poincaré ball model of hyperbolic geometry. *Comput. Math. Appl.*, 41(1–2):135–147, 2001.

[4] K. Watanabe. On Lipschitz continuity with respect to the Poincaré metric of linear contractions between Möbius gyrovector spaces. *J. Inequal. Appl.*, 2021(1):Article 166, 2021.

[5] A. A. Ungar. *Analytic hyperbolic geometry and Albert Einstein's Special Theory of Relativity.* World Scientific, Hackensack, NJ, 2008.

[6] P. Ferreira, Z. Németh, and J. Zhu. Convexity of non-homogeneous quadratic functions on the hyperbolic space. *J. Optim. Theory Appl.*, 199(3):1085–1105, 2023.

[7] K. T. Atanassov, V. K. Atanassova, A. G. Shannon, and J. C. Turner. *New visual perspectives on Fibonacci numbers.* World Scientific, Hackensack, NJ, 2002.