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On the Diophantine Equation $2 + (m^2 - 2)^x = y^2$

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Abstract. This paper establishes conditions under which the Diophantine equation $2+(m^2-2)^x = y^2$ has exactly one solution in non-negative integers. As applications, we show that $2 + 7^x = y^2$ and $2 + 23^x = y^2$ have exactly one solution, namely $(x, y) = (1, 3)$ and $(1, 5)$, respectively.

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1. Introduction

In 1844, Catalan conjectured that the Diophantine equation

$$a^x - b^y = 1,$$

where a, b, x, y are integers with $\min\{a, b, x, y\} > 1$, has the unique solution $(a, b, x, y) = (3, 2, 2, 3)$. This conjecture was proved by P. Mihăilescu [1] in 2004.

Exponential Diophantine equations involving perfect powers and quadratic forms have been investigated extensively. A recurring theme is to determine all solutions or to prove uniqueness for equations of the form

$$A^x + B^y = z^2, \qquad A^x + C = z^2, \qquad \text{or more generally} \qquad X^2 - dY^2 = N$$

under additional arithmetic restrictions. For example, Acu [2] proved that $2^x + 5^y = z^2$ has exactly two non-negative integer solutions, namely $(x, y, z) = (3, 0, 3)$ and $(2, 1, 3)$. Suvarnamani, Singta, and Chotchaisthit [3] showed that $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no solutions in non-negative integers, and Sroysang [4] proved that $7^x + 8^y = z^2$ has the unique solution $(x, y, z) = (0, 1, 3)$.

More recently, Borah and Dutta [5] investigated the equation $2^x+7^y = z^2$, determining all solutions for $x \neq 1$ and reducing the remaining case to the equation

$$2 + 7^y = z^2.$$

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This remaining case was subsequently solved by Zhang and Li [6], who proved by using elliptic curve techniques that $(y, z) = (1, 3)$ is its unique non-negative integer solution.

In a different direction, several authors have developed Störmer-type divisibility criteria for Pell equations. Such results typically assert that, under suitable hypotheses, divisibility conditions on one component of a solution force that solution to be fundamental (or to lie in a distinguished class). For example, Luo [7] extended Störmer-type results to Pell-type equations of the form

$$kX^2 - \ell Y^2 = 2 \quad \text{and} \quad kX^2 - \ell Y^2 = 4,$$

and applied them to certain rational exponential Diophantine equations, such as

$$\frac{ax^n \pm 2}{ax \pm 2} = y^2.$$

The present paper studies the exponential Diophantine family

$$2 + (m^2 - 2)^x = y^2, \tag{1}$$

where $m > 1$ is an odd integer and x, y are non-negative integers. A key observation is that when x is odd, writing $x = 2t + 1$ and setting

$$d = m^2 - 2, \qquad u = d^t,$$

equation (1) becomes the Pell-type equation

$$y^2 - du^2 = 2, \tag{2}$$

together with the additional *perfect-power constraint* $u = d^t$. Thus, our problem is not merely to describe all solutions of (2), but rather to determine whether a *restricted subsequence* of Pell solutions can satisfy a perfect-power condition, and to deduce uniqueness for (1) from this constrained recurrence structure. This "Pell subsequence" viewpoint is structurally different from the rational exponential families treated in [7] and from approaches based on elliptic curves, such as those in [6].

Our main result gives a general criterion implying that (1) has exactly one non-negative integer solution, namely $(x, y) = (1, m)$. As applications, we verify the criterion for $m = 3$ and $m = 5$, yielding that

$$2 + 7^x = y^2 \quad \text{and} \quad 2 + 23^x = y^2$$

each has a unique non-negative integer solution, namely $(x, y) = (1, 3)$ and $(1, 5)$, respectively. We also formulate a natural open problem of determining all odd integers $m > 1$ for which the criterion holds.

2. Preliminaries

A Pell equation is a Diophantine equation of the form $x^2 - dy^2 = N$, where d is a non-square integer and N is a non-zero integer. If $N = \pm 1$, the equation is known as the classical Pell equation. The fundamental solution of $x^2 - dy^2 = 1$ is the pair (a, b) with $a, b > 0$ minimal. Two solutions $x + y\sqrt{d}$ and $x' + y'\sqrt{d}$ of $x^2 - dy^2 = N$ are called *associated* if one can be obtained from the other by multiplying by a solution of $x^2 - dy^2 = 1$. Associated solutions form a class. Within each class, the fundamental solution is defined as the solution $x_0 + y_0\sqrt{d}$ with y_0 minimal among solutions in that class. More details can be found in [8].

Theorem 1. *[8] Let $u + v\sqrt{d}$ be the fundamental solution of $x^2 - dy^2 = 1$. Then for each fundamental solution $x^* + y^*\sqrt{d}$ of $x^2 - dy^2 = N$, the following inequalities hold:*

$$0 \leq y^* \leq \frac{v}{2(u+\varepsilon)}\sqrt{|N|},$$
$$|x^*| \leq \sqrt{\frac{1}{2}(u+\varepsilon)|N|},$$

where $\varepsilon = 1$ if $N > 0$ and $\varepsilon = -1$ if $N < 0$. In particular, there are only finitely many fundamental solutions (and hence only finitely many classes).

3. Main Results

Lemma 1. *Let $m > 1$ be an odd integer. Then the equation $2 + (m^2 - 2)^{2x} = y^2$ has no integer solution.*

Proof. Assume that m is odd. Then $m^2 - 2$ is odd, and hence y is odd. Thus

$$(m^2 - 2)^{2x} \equiv y^2 \equiv 1 \pmod 4,$$

which implies

$$y^2 - (m^2 - 2)^{2x} \equiv 0 \pmod 4.$$

Since $y^2 - (m^2 - 2)^{2x} = 2$, we have $2 \equiv 0 \pmod 4$, which is a contradiction. Therefore, $2 + (m^2 - 2)^{2x} = y^2$ has no integer solution.

Theorem 2. *Let $m > 1$ be an odd integer. Let M, N be defined by the identity*

$$M + N\sqrt{m^2 - 2} = (m^2 - 1 + m\sqrt{m^2 - 2})^{m^2-2}.$$

Let u_0^* be defined by

$$u_0^* = \sum_{k \text{ odd}} \binom{\frac{m^2-3}{2}}{k} (m^2 - 1)^{\frac{m^2-3}{2}-k} m^{k+1} (\sqrt{m^2 - 2})^{k-1}$$

$$+\sum_{k\ even}\binom{\frac{m^2-3}{2}}{k}(m^2-1)^{\frac{m^2-3}{2}-k}m^k(\sqrt{m^2-2})^k.$$

Suppose there exists a prime q relatively prime to m^2-2 such that $q \mid u_0^*$ and $q \mid N$. Then the equation

$$2+(m^2-2)^x = y^2$$

has exactly one non-negative integer solution, namely $(x,y)=(1,m)$.

Proof. By Lemma 1, there are no solutions when x is even and positive. If $x=0$, then $y^2=3$, which is impossible. Hence it suffices to consider x a positive odd integer. Write $x=2x_0+1$, where $x_0 \geq 0$.

Consider

$$2+(m^2-2)^{2x_0+1} = y^2.$$

Let $u=(m^2-2)^{x_0}$. Then

$$y^2-(m^2-2)u^2 = 2.$$

Since the continued fraction of $\sqrt{m^2-2}$ is $\left[m-1;\overline{1,m-2,1,2(m-1)}\right]$, the pair (m^2-1,m) is the fundamental solution of

$$y^2-(m^2-2)u^2 = 1.$$

By Theorem 1, any fundamental solution of $y^2-(m^2-2)u^2=2$ satisfies

$$0 \leq u \leq 1 \quad \text{and} \quad |y| \leq m.$$

If $u=0$, then $y^2=2$, which is impossible. If $u=1$, then $y^2=m^2$, so $(\pm m,1)$ is a fundamental solution of $y^2-(m^2-2)u^2=2$.

Hence all solutions of $y^2-(m^2-2)u^2=2$ are of the form

$$y_n+u_n\sqrt{m^2-2} = \pm(m+\sqrt{m^2-2})(m^2-1+m\sqrt{m^2-2})^n, \tag{3}$$

or

$$y_n+u_n\sqrt{m^2-2} = \pm(-m+\sqrt{m^2-2})(m^2-1+m\sqrt{m^2-2})^n. \tag{4}$$

The goal is to determine whether u_n can be of the form $(m^2-2)^\ell$ with $\ell>0$.

For (3), taking conjugates yields

$$y_n-u_n\sqrt{m^2-2} = \pm(m-\sqrt{m^2-2})(m^2-1-m\sqrt{m^2-2})^n.$$

Subtracting gives

$$\begin{aligned}2u_n\sqrt{m^2-2} = {} & \pm(m+\sqrt{m^2-2})\left(m^2-1+m\sqrt{m^2-2}\right)^n \\ & \mp(m-\sqrt{m^2-2})\left(m^2-1-m\sqrt{m^2-2}\right)^n.\end{aligned}$$

Expanding by the binomial theorem,

$$\pm 2u_n\sqrt{m^2-2} = 2m\sum_{k \text{ odd}}\binom{n}{k}(m^2-1)^{n-k}\left(m\sqrt{m^2-2}\right)^k + 2\sqrt{m^2-2}\sum_{k \text{ even}}\binom{n}{k}(m^2-1)^{n-k}\left(m\sqrt{m^2-2}\right)^k.$$

Solving for u_n yields

$$\pm u_n = \sum_{k \text{ odd}}\binom{n}{k}(m^2-1)^{n-k}m^{k+1}\left(\sqrt{m^2-2}\right)^{k-1} + \sum_{k \text{ even}}\binom{n}{k}(m^2-1)^{n-k}m^k\left(\sqrt{m^2-2}\right)^k.$$

Similarly, the conjugate of (4) yields an analogous expression for u_n.

Now put $d = m^2-2$. If $u_n \equiv 0 \pmod{d}$, then $2n+1 \equiv 0 \pmod{d}$. Thus

$$n \equiv \frac{d-1}{2} \pmod{d},$$

so $n = \frac{d-1}{2} + td$ for some integer $t \geq 0$. Therefore, all solutions of $y^2 - du^2 = 2$ with $d \mid u$ are of the form

$$y_t^* + u_t^*\sqrt{d} = \pm(m \pm \sqrt{d})(m^2 - 1 + m\sqrt{d})^{\frac{d-1}{2}+td}.$$

It remains to show that $q \mid u_t^*$ for all $t \geq 0$. Assume $q \mid u_0^*$ and $q \mid u_t^*$ for some $t \geq 0$. Using

$$\begin{aligned} y_{t+1}^* + u_{t+1}^*\sqrt{d} &= \pm(m \pm \sqrt{d})(m^2-1+m\sqrt{d})^{\frac{d-1}{2}+(t+1)d} \\ &= \pm(m \pm \sqrt{d})(m^2-1+m\sqrt{d})^{\frac{d-1}{2}+td}(m^2-1+m\sqrt{d})^d \\ &= (y_t^* + u_t^*\sqrt{d})(M + N\sqrt{d}) \\ &= y_t^*M + u_t^*Nd + (u_t^*M + y_t^*N)\sqrt{d}, \end{aligned}$$

it follows that

$$u_{t+1}^* = u_t^*M + y_t^*N.$$

Since $q \mid u_t^*$ and $q \mid N$, we have $q \mid u_{t+1}^*$. Hence $q \mid u_t^*$ for all $t \geq 0$.

Because $\gcd(q, d) = 1$, no term u_t^* can be of the form d^ℓ with $\ell > 0$. Therefore, $(m^2-2)^{x_0} = u = d^{x_0}$ forces $x_0 = 0$. Hence the only non-negative integer solution is $x = 1$ and then $y = m$.

The condition on the existence of a prime q dividing both u_0^* and N is not guaranteed for arbitrary m. It is verified explicitly in Corollaries 1 and 2. A natural open problem is to characterize all odd integers $m > 1$ for which such a common prime divisor exists.

Applications

In this section we illustrate applications of Theorem 2.

Let m be an odd integer and put $d = m^2 - 2$. To determine whether the equation

$$2 + d^x = y^2$$

has a unique solution, it suffices to check whether there exists a prime q with $\gcd(q, d) = 1$ such that $q \mid u_0^*$ and $q \mid N$. If such a prime exists, then Theorem 2 implies that $x = 1$ is the only possible exponent.

We now apply Theorem 2 to the equations $2 + 7^y = z^2$ and $2 + 23^y = z^2$.

Corollary 1. *The equation* $2+7^y = z^2$ *has a unique non-negative integer solution, namely* $(y, z) = (1, 3)$.

Proof. Using the notation of Theorem 2, we set $m = 3$, which gives $m^2 - 2 = 7$. It can be verified that the prime 617 divides both u_0^* and N. Indeed, $u_0^* = 4319$, which is a multiple of 617. Furthermore, from the relation $M + N\sqrt{7} = (8 + 3\sqrt{7})^7$, we obtain $N = 49353213$, which is also divisible by 617.

Therefore, by Theorem 2, $(y, z) = (1, 3)$ is the unique non-negative integer solution.

Corollary 2. *The equation* $2 + 23^y = z^2$ *has a unique non-negative integer solution, namely* $(y, z) = (1, 5)$.

Proof. Applying Theorem 2 with $m = 5$, we have $m^2 - 2 = 23$. We observe that the prime 137 divides both u_0^* and N. Indeed, $u_0^* = 3167569837654840799$, which is divisible by 137. Moreover, from the relation $M + N\sqrt{23} = (24 + 5\sqrt{23})^{23}$, we obtain

$$N = 481189692414949104935838125684448691 15,$$

which is also divisible by 137.

Thus, Theorem 2 implies that $(y, z) = (1, 5)$ is the unique non-negative integer solution.

Corollaries 1 and 2 indicate that, as u_0^* and N grow rapidly, it becomes more difficult in practice to identify a common prime divisor q satisfying the hypothesis of Theorem 2.

Future Directions

Several natural questions arise from this work. First, it would be interesting to determine whether the condition $\gcd(u_0^*, N) > 1$ holds for infinitely many odd integers m. Second, one may attempt to extend the method to equations of the form

$$2 + (m^2 - k)^x = y^2,$$

for fixed integers k. Finally, further study of related exponential Diophantine families via recurrence methods may lead to additional classification results.

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