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A Numerical Approach Based on the Optimization of a One-Step Block Method for Solving First-Order Fuzzy Initial Value Problems

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Abstract. Fuzzy differential equations have attracted the attention of many researchers. Based on a hybrid block method, we present a numerical approach for solving first-order fuzzy initial value problems. We investigate the proposed method theoretically and establish convergence and stability results. Two numerical examples are presented to illustrate the efficiency of the proposed method and to compare the results with those obtained by other methods in the literature.

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Key Words and Phrases: Hybrid block method, hukuhara derivative, fuzzy initial value problems.

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1. Introduction

Fuzzy differential equations have many real-life applications in areas such as physics, control theory, economics, population dynamics, and ecology [1–3]. These applications have attracted researchers to investigate such problems. However, finding exact solutions of fuzzy differential equations is a difficult task, and it is sometimes not possible. The reason behind this difficulty is that, we need to solve two min-max problems generated by the $\alpha-$levels. For this reason, several researchers investigate fuzzy initial value problems using numerical methods. Ghamtbari [4] used the variational and Adomian decomposition method to solve such problems, while Abbasbandy [5] used the Taylor method, Allahviranloo [6] used the predictor-corrector method, Devi [7] used Simpson's rule and Runge-Kutta

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method, Radhy [8] used the Runge-Kutta method, and Midpoint and Trapezoidal rules were used to solve by Ivas [9].

The derivative that is used in this paper is the Hukuhara derivative which was presented in [10]. After that, several theoretical results for initial value problems were investigated by many researches, see [11, 12]. Several numerical methods can be developed in the literature to solve fuzzy differential equations [13–22].

In this article, we present a numerical method for solving the following fuzzy initial value problem based on hybrid block method

$$y' = f(x, y),\ y(x_0) = y_{0,}\ x_0 \in [a, b], \tag{1}$$

where $f(x, y)$ is continuous and satisfies the existence and uniqueness theorem. Recently many researchers implemented hybrid block method with a different number of steps to solve ordinary initial value problem, see [23–25]. In this paper, we optimization the local truncation errors to find the best choice of the step-point.

Based on a hybrid block method, we present a numerical approach for solving first-order fuzzy initial value problems. We investigate the proposed method theoretically and establish convergence and stability results. Two numerical examples are presented to illustrate the efficiency of the proposed method and to compare the results with those obtained by other methods in the literature.

This paper is organized as follows. In Section 2, we present some preliminaries about fuzzy concepts. In sections 3 and 4, we present the optimized one-step hybrid block method and some related theoretical results. In section 5, we generalize the proposed method to fuzzy initial value problems. In section 6, we present some numerical results to show the efficiency of the proposed method. Finally, in Section 7, we discuss the results and we draw some conclusions.

2. Preliminarily

In this section, we present some preliminaries which will be used in this paper.

Definition 1. **[26]** *A fuzzy number is a function* $\gamma : \mathbb{R} \to [0, 1]$ *with the following properties*

1. *There exist* $a \in \mathbb{R}$*, such that* $\gamma(a) = 1$.
2. $\gamma(sa + (1 - s)b) \geq \min\{\gamma(a), \gamma(b)\}$ *for all* $a, b \in \mathbb{R}$ *and* $s \in [0, 1]$.
3. *The set* $\{a \in \mathbb{R} : \gamma(a) \geq \alpha\}$ *is closed subset of* $\mathbb{R}$ *for all* $\alpha \in (0, 1]$.
4. *The closure of the set* $\{a \in \mathbb{R} : \gamma(a) > 0\}$ *is a compact set.*

The set of all fuzzy numbers is denoted by $F_{\mathbb{R}}$. *If* $\gamma \in F_{\mathbb{R}}$ *and* $\alpha \in (0, 1]$, *the* $\alpha-$*level set is given by*

$$\gamma_\alpha = \{a \in \mathbb{R} : \gamma(a) \geq \alpha\},$$

and the $0-$*level set is given by*

$$\gamma_0 = \overline{\{a \in \mathbb{R} : \gamma(a) > 0\}}.$$

It is easy to see, that we can write $\gamma_\alpha = [\underline{\gamma}_\alpha, \overline{\gamma}_\alpha]$. *One can see that*

1. $\gamma_\alpha \subseteq \gamma_c$, *if* $0 < c \leq \alpha \leq 1$.
2. *If the sequence* $\{\alpha_n\}$ *is an increasing sequence on* $(0,1]$ *and converges to* α, *then* $\lim\limits_{n\to\infty} \gamma_{\alpha_n} = \gamma_\alpha$.
3. *For any* $\alpha \in (0,1]$, $-\infty < \underline{\gamma}_\alpha \leq \overline{\gamma}_\alpha < \infty$.

Example 1. *Let* $\gamma = (t_1, t_2, t_3)$ *be a triangular fuzzy number. Then,*

$$\gamma(s) = \begin{cases} \frac{s-t_1}{t_2-t_1}, & t_1 \leq s \leq t_2 \\ \frac{s-t_3}{t_2-t_3}, & t_2 \leq s \leq t_3 \\ 0, & otherwise \end{cases}$$

and $\gamma_\alpha = [(1-\alpha)t_1 + \alpha t_2, (1-\alpha)t_3 + \alpha t_2]$. *The graph of the symmetric fuzzy triangle* $\gamma = (0,1,2)$ *is given in Figure 1.*

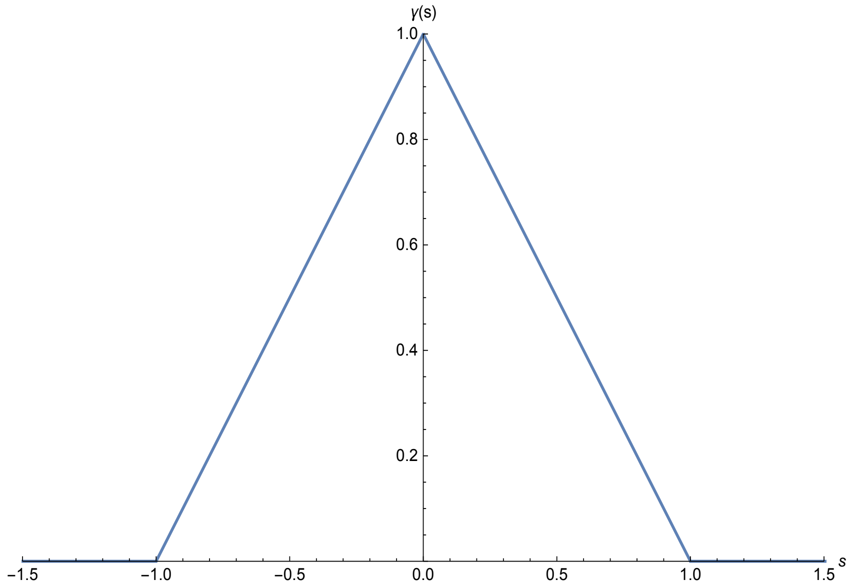

Figure 1: The Fuzzy triangle number with $\gamma = (-1, 0, 1)$

Definition 2. **[26]** *Let* $\gamma, \varphi \in F_{\mathbb{R}}$, $\alpha \in [0,1]$, *and* $c \in \mathbb{R}$ *where* $\gamma_\alpha = [\underline{\gamma}_\alpha, \overline{\gamma}_\alpha]$ *and* $\varphi_\alpha = [\underline{\varphi}_\alpha, \overline{\varphi}_\alpha]$. *Then,*

1. $(\gamma \oplus \varphi)_\alpha = [\underline{\gamma}_\alpha + \underline{\varphi}_\alpha, \overline{\gamma}_\alpha + \overline{\varphi}_\alpha]$.
2. $(c \odot \gamma)_\alpha = \begin{cases} [c\underline{\gamma}_\alpha, c\overline{\gamma}_\alpha] & c \geq 0 \\ [c\overline{\gamma}_\alpha, c\underline{\gamma}_\alpha], & c < 0 \end{cases}$.
3. *If there exists* $\sigma \in F_{\mathbb{R}}$ *such that* $\gamma = \varphi + \sigma$, *then* $\sigma = \gamma \ominus_H \varphi$ *is the* $H-$*difference of* γ *and* φ.
4. *The Hausdorff matrix* $\Gamma_H : F_{\mathbb{R}} \times F_{\mathbb{R}} \to [0, \infty)$ *is defined by*

$$\Gamma_H(\gamma, \varphi) = \sup_{0<\alpha\leq 1} \{\max\{\left|\underline{\gamma}_\alpha - \underline{\varphi}_\alpha\right|, |\overline{\gamma}_\alpha - \overline{\varphi}_\alpha|\}\}$$

and $(F_{\mathbb{R}}, \Gamma_H)$ *is a matrix space.*

Definition 3. **[27]** *A Fuzzy-valued function is a function* $\hat{h} : V \to F_{\mathbb{R}}$*, where* V *is a real vector space and* $F_{\mathbb{R}}$ *is the set of fuzzy numbers. The fuzzy function* $\hat{h}(x)$ *can be written as* $[f_\alpha(x), g_\alpha(x)]$ *for all* $x \in V$ *and any* $\alpha \in [0,1]$*. The functions* $f_\alpha(x)$ *and* $g_\alpha(x)$ *are called* α*-level functions of the fuzzy-valued function* $\hat{h}$*.*

Definition 4. **[27, 28]** *Let* $h : (a,b) \subset \mathbb{R} \to F_{\mathbb{R}}$ *be a fuzzy function and* $c \in (a,b)$*. If there exists* $h'(c) \in F_{\mathbb{R}}$ *such that*

$$h'(c) = \lim_{\Delta \to 0^+} \frac{h(c+\Delta) \ominus_H h(c)}{\Delta} = \lim_{\Delta \to 0^+} \frac{h(c) \ominus_H h(c-\Delta)}{\Delta},$$

then h *is called Hukuhara differentiable at* c*, and the Hukuhara derivative of* h *at* c *is* $h'(c)$*.*

Theorem 1. **[29]** *Let* $\gamma \in F_{\mathbb{R}}$ *and* $f : (a,b) \subset \mathbb{R} \to (0,\infty)$ *be differentiable at* $c \in (a,b)$*. Let* $h : (a,b) \to F_{\mathbb{R}}$ *be a fuzzy function defined by* $h(x) = \gamma \odot f(x)$*. If* $f'(c) > 0$*, then* h *is Hukuhara differentiable at* c *and the Hukuhara derivative of* h *at* c *is given by* $h'(c) = \gamma \odot f'(x)$*.*

Example 2. *Let* $\gamma \in F_{\mathbb{R}}$ *and* $h : \mathbb{R} \to F_{\mathbb{R}}$ *be a fuzzy function defined by* $h(x) = \gamma \odot x^2$*. Then,* $f(x) = x^2$*. If* $x > 0$*, then*

$$f(x) = x^2 > 0$$

and

$$f'(x) = 2x > 0.$$

Using Theorem 1, we have h *is Hukuhara differentiable on* $(0,\infty)$ *and the Hukuhara derivative of* h *is* $h'(c) = \gamma \odot 2x$*. If* $x < 0, \Delta > 0$ *but close to* 0*, and* $\alpha \in (0,1]$*, then* $2x\Delta + \Delta^2 < 0$*,*

$$\underline{h}_\alpha(x+\Delta) - \underline{h}_\alpha(x) = \overline{\gamma}_\alpha(2x\Delta + \Delta^2),$$

and

$$\overline{h}_\alpha(x+\Delta) - \overline{h}_\alpha(x) = \underline{\gamma}_\alpha(2x\Delta + \Delta^2),$$

where $h = [\underline{h}, \overline{h}]$*. Since*

$$\underline{h}_\alpha(x+\Delta) - \underline{h}_\alpha(x) \nleq \overline{h}_\alpha(x+\Delta) - \overline{h}_\alpha(x),$$

then $h(x+\Delta) \ominus_H h(x)$ *does not exist. Thus,* h *is not Hukuhara differentiable on* $(-\infty, 0)$*. If* $x = 0, \Delta > 0$ *but close to* 0*, and* $\alpha \in (0,1]$*, then*

$$h_\alpha = \begin{cases} [\underline{\gamma}_\alpha x^2, \overline{\gamma}_\alpha x^2], & x \geq 0 \\ [\overline{\gamma}_\alpha x^2, \underline{\gamma}_\alpha x^2], & x < 0 \end{cases}$$

which implies that h *is not Hukuhara differentiable at* $x = 0$*.*

3. One-step hybrid block method with one off-step point

In this section, we drive a numerical method based on the one-step hybrid block method with one off-step point (HBM1), t_{n+k} where $0 < k < 1$, to solve the following initial value problem (IVP)

$$y'(t) = f(t, y(t)), t \geq 0 \tag{2}$$

$$y(t_0) = y_0. \tag{3}$$

To derive HBM1, we assume that $t_n = nh$, where h is the stepsize. We approximate the solution of the IVP (2-3) by a polynomial of degree 3 as follows

$$y(t) \simeq \sum_{j=0}^{3} c_j\, t^j, \tag{4}$$

and its derivative by

$$y'(t) \simeq \sum_{j=1}^{2} j\, c_j\, t^{j-1}. \tag{5}$$

By interpolating Eq.(4) at the point t_n and collocating Eq.(5) at the points t_n, $t_{n+k} = t_n + kh$, and $t_{n+1} = t_n + h$, we end up with the following system

$$\begin{pmatrix} 1 & t_n & t_n^2 & t_n^3 \\ 0 & 1 & 2t_n & 3t_n^2 \\ 0 & 1 & 2t_{n+k} & 3t_{n+k}^2 \\ 0 & 1 & 2t_{n+1} & 3t_{n+1}^2 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} y_n \\ f_n \\ f_{n+k} \\ f_{n+1} \end{pmatrix}, \tag{6}$$

where $y_{n+j} \approx y(t_{n+j})$ and $f_{n+j} \approx y'(t_{n+j})$, $j = 0, k, 1$. We then, solve System (6) after substituting $t = t_n + wh$, to get

$$y(w) \approx y_n + h\,(\alpha_0 f_n + \alpha_k f_{n+k} + \alpha_1 f_{n+1}), \tag{7}$$

where α_0, α_k, and α_1 are functions of w. Evaluating the approximation of $y(w)$ at $w = k$ and 1 yields

$$y_{n+j} = y_n + h\,(\alpha_{0,j}\ f_n + \alpha_{k,j}\ f_{n+k} + \alpha_{1,j}\ f_{n+1}), \tag{8}$$

where $j = k, 1$ and

$$\begin{aligned} \alpha_{0,1} &= \frac{3k-1}{6k}, \alpha_{k,1} = \frac{1}{6k-6k^2}, \alpha_{1,1} = \frac{3k-2}{6(k-1)}, \\ \alpha_{0,k} &= \frac{k(3-k)}{6}, \alpha_{k,k} = \frac{k(2k-3)}{6(k-1)}, \alpha_{1,k} = \frac{k^3}{6(k-1)}. \end{aligned}$$

To maximize the order of the implicit block method (8), we optimize the local truncation errors in the formula for y_{n+1}. We have

$$\mathcal{L}\left(y\left(t_{n+1}\right); h\right) = \frac{2k-1}{72} h^4 y^{(4)}\left(t_n\right) + \frac{h^5\ \left(5k^2+5k-4\right)}{720} y^{(5)}\left(t_n\right) + \mathcal{O}\left(h^6\right). \tag{9}$$

To maximize the order of formula (8) where $j = 1$, we solve the following equation for k, where $0 < k < 1$,

$$\frac{2k-1}{72} = 0. \tag{10}$$

Hence,

$$k = \frac{1}{2}, \tag{11}$$

and the local truncation errors is given by

$$\mathcal{L}\left[y\left(t_{n+1}\right);h\right] = -\frac{h^5 y^{(5)}\left(t_n\right)}{2880} + \mathcal{O}\left(h^6\right) = -3.4722 * 10^{-4} h^5 y^{(5)}\left(t_n\right) + \mathcal{O}\left(h^6\right).$$

Therefore, the HBM1 is given by

$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{6}(f_n + 4f_{n+\frac{1}{2}} + f_{n+1}), \\ y_{n+\frac{1}{2}} &= y_n + \frac{h}{24}(5f_n + 8f_{n+\frac{1}{2}} - f_{n+1}). \end{aligned} \tag{12}$$

4. Analysis of the proposed method

In this section, we present some theoretical results on the one-step hybrid block method with one off-step point which was derived in the previous section. These results include consistency, stability, and convergence results. Let us write the system (12) as follows

$$\begin{aligned} y_{n+\frac{1}{2}} &= y_n + \frac{h}{24}(5f_n + 8f_{n+\frac{1}{2}} - f_{n+1}), \\ y_{n+1} &= y_n + \frac{h}{6}(f_n + 4f_{n+\frac{1}{2}} + f_{n+1}). \end{aligned} \tag{13}$$

We first rewrite the HBM1 in Eq.(13) in the form

$$A^1 Y_{n+1} = A^0 Y_n + h\left(B^0 F_n + B^1 F_{n+1}\right), \tag{14}$$

where A^0, A^1, B^0, B^1 are 2×2 matrices whose coefficients are given by

$$\begin{aligned} A^0 &= \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \; A^1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; B^0 = \begin{pmatrix} 0 & \frac{5}{24} \\ 0 & \frac{1}{6} \end{pmatrix}, B^1 = \begin{pmatrix} \frac{8}{24} & -\frac{1}{24} \\ \frac{4}{6} & \frac{1}{6} \end{pmatrix}, \\ Y_n &= \begin{pmatrix} y_{n-k} \\ y_n \end{pmatrix}, Y_{n+1} = \begin{pmatrix} y_{n+k} \\ y_{n+1} \end{pmatrix}, F_n = \begin{pmatrix} f_{n-k} \\ f_n \end{pmatrix}, F_{n+1} = \begin{pmatrix} f_{n+k} \\ f_{n+1} \end{pmatrix}. \end{aligned}$$

Thus, the characteristic equation of the HBM1 in Eq.(14) is given by

$$\rho(z) = \det\left(zA^1 - A^0\right) = z(z-1) = 0, \tag{15}$$

which implies that $z_1 = 0$ and $z_2 = 1$. Then, the multiplicity of the nonzero root of the characteristic equation is 1, which does not exceed the order of the differential equation.

Hence, the method is zero stable.
From Section 3, we see that the local truncation error of the system (12) is

$$\begin{aligned}\mathcal{L}\left[y\left(t_n\right);h\right] &= \left(\mathcal{L}\left[y\left(t_{n+\frac{1}{2}}\right);h\right],\mathcal{L}\left[y\left(t_{n+1}\right);h\right]\right)\\ &= \sum_{i=0}^{\infty}\alpha h^i y^{(i)}(t_n) = \alpha h^4 y^{(4)}(t_n)+\sum_{i=5}^{\infty}\alpha h^i y^{(i)}(t_n),\end{aligned} \tag{16}$$

where $\alpha_0=\alpha_1=\alpha_2=\alpha_3=0$ and $\alpha_4=(\frac{1}{384},0)^T$. Thus, System (12) has order $(3,3)^T$. For simplicity, we write the order as 3. Since the order of the System (12) is $3>1$, then it is consistent. The consistency and the zero stability of the System (16) imply that it is convergent [23, 25].
Applying the HBM1 to the test problem $y'=\lambda y$ with $\lambda<0$, we have

$$Y_{n+1}=M(z)Y_n,\ z=\lambda h, \tag{17}$$

where the matrix $M(z)$ is given by

$$M(z)=\left(A^1-zB^1\right)^{-1}\left(A^0+zB^0\right). \tag{18}$$

The matrix $M(z)$ has eigenvalues $\left\{0,-\frac{2(z-3)(z+6)}{3(z^2-6z+12)}\right\}$. Consider $R(z):\mathbb{C}\longrightarrow\mathbb{C}$ defined by $R(z)=-\frac{2(z-3)(z+6)}{3(z^2-6z+12)}$. The region of absolute stability S is defined as $S=\{z\in\mathbb{C}: |R(z)|<1\}$. The Region of absolute stability of the method is presented in Figure 2.

The stability region contains the entire left half complex plane and thus, the method is A-stable.

5. Fuzzy initial value problem

Consider the following fuzzy initial value problem

$$y'(t) = f(t,y), t\geq 0, \tag{19}$$
$$y(0) = y_0. \tag{20}$$

Denote the $\alpha-$level of the solution $y(t)$, y_0, and the function $f(t,y)$ by

$$\begin{aligned}y(t,\alpha) &= \left[y_1(t,\alpha),y_2(t,\alpha)\right],\\ y(0,\alpha) &= \left[y_{1,0},y_{2,0}\ \right],\\ f(t,y,\alpha) &= \left[g(t,y(t,\alpha)),h(t,y(t,\alpha))\right].\end{aligned}$$

Following the technique described in previous sections, the fuzzy HBM1 is given by

$$\begin{aligned}y_{1,n+1,\alpha} &= y_{1,n,\alpha}+\frac{h}{6}(f_{1,n,\alpha}+4f_{1,n+\frac{1}{2},\alpha}+f_{1,n+1,\alpha}),\\ y_{1,n+\frac{1}{2},\alpha} &= y_{1,n,\alpha}+\frac{h}{24}(5f_{1,n,\alpha}+8f_{1,n+\frac{1}{2},\alpha}-f_{1,n+1,\alpha}),\end{aligned}$$

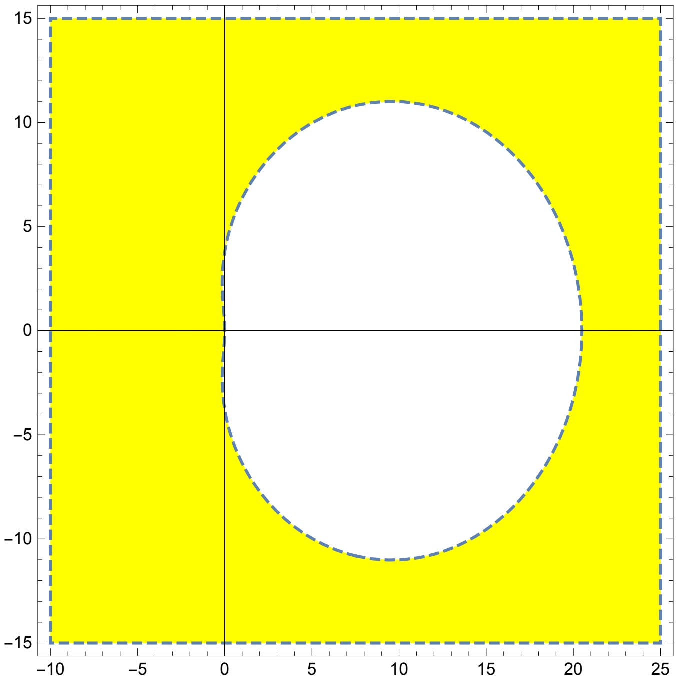

Figure 2: The region of absolute stability

$$
\begin{aligned}
y_{2,n+1,\alpha} &= y_{2,n,\alpha} + \frac{h}{6}(f_{2,n,\alpha} + 4f_{2,n+\frac{1}{2},\alpha} + f_{2,n+1,\alpha}),\\
y_{2,n+\frac{1}{2},\alpha} &= y_{2,n,\alpha} + \frac{h}{24}(5f_{2,n,\alpha} + 8f_{2,n+\frac{1}{2},\alpha} - f_{2,n+1,\alpha}),
\end{aligned}
$$

where

$$
\begin{aligned}
f_{1,n,\alpha} &= \min\{f(t_n, w) : w \in [y_{1,n,\alpha}, y_{2,n,\alpha}]\},\\
f_{2,n,\alpha} &= \max\{f(t_n, w) : w \in [y_{1,n,\alpha}, y_{2,n,\alpha}]\},\\
f_{1,n+\frac{1}{2},\alpha} &= \min\{f(t_{n+\frac{1}{2}}, w) : w \in \left[y_{1,n+\frac{1}{2},\alpha}, y_{2,n+\frac{1}{2},\alpha}\right]\},\\
f_{2,n+\frac{1}{2},\alpha} &= \max\{f(t_{n+\frac{1}{2}}, w) : w \in \left[y_{1,n+\frac{1}{2},\alpha}, y_{2,n+\frac{1}{2},\alpha}\right]\},\\
f_{1,n+1,\alpha} &= \min\{f(t_{n+1}, w) : w \in [y_{1,n+1,\alpha}, y_{2,n+1,\alpha}]\},\\
f_{2,n+1,\alpha} &= \min\{f(t_{n+1}, w) : w \in [y_{1,n+1,\alpha}, y_{2,n+1,\alpha}]\}.
\end{aligned}
\tag{21}
$$

It is worth mentioning that if $f(t, y)$ is increasing on y, then the fuzzy HBM1 becomes

$$
\begin{aligned}
y_{1,n+1,\alpha} &= y_{1,n,\alpha} + \frac{h}{6}(f(t_n, y_{1,n,\alpha}) + 4f(t_{n+\frac{1}{2}}, y_{1,n+\frac{1}{2},\alpha}) + f(t_{n+1}, y_{1,n+1,\alpha})),\\
y_{1,n+\frac{1}{2},\alpha} &= y_{1,n,\alpha} + \frac{h}{24}(5f(t_n, y_{1,n,\alpha}) + 8f(t_{n+\frac{1}{2}}, y_{1,n+\frac{1}{2},\alpha}) - f(t_{n+1}, y_{1,n+1,\alpha})),\\
y_{2,n+1,\alpha} &= y_{2,n,\alpha} + \frac{h}{6}(f(t_n, y_{2,n,\alpha}) + 4f(t_{n+\frac{1}{2}}, y_{2,n+\frac{1}{2},\alpha}) + f(t_{n+1}, y_{2,n+1,\alpha})),
\end{aligned}
$$

$$y_{2,n+\frac{1}{2},\alpha} = y_{2,n,\alpha} + \frac{h}{24}(5f(t_n, y_{2,n,\alpha}) + 8f(t_{n+\frac{1}{2}}, y_{2,n+\frac{1}{2},\alpha}) - f(t_{n+1}, y_{2,n+1,\alpha})),$$

while if $f(t,y)$ is decreasing on y, then the fuzzy HBM1 becomes

$$\begin{aligned}
y_{1,n+1,\alpha} &= y_{1,n,\alpha} + \frac{h}{6}(f(t_n, y_{2,n,\alpha}) + 4f(t_{n+\frac{1}{2}}, y_{2,n+\frac{1}{2},\alpha}) + f(t_{n+1}, y_{2,n+1,\alpha})),\\
y_{1,n+\frac{1}{2},\alpha} &= y_{1,n,\alpha} + \frac{h}{24}(5f(t_n, y_{2,n,\alpha}) + 8f(t_{n+\frac{1}{2}}, y_{2,n+\frac{1}{2},\alpha}) - f(t_{n+1}, y_{2,n+1,\alpha})),\\
y_{2,n+1,\alpha} &= y_{2,n,\alpha} + \frac{h}{6}(f(t_n, y_{1,n,\alpha}) + 4f(t_{n+\frac{1}{2}}, y_{1,n+\frac{1}{2},\alpha}) + f(t_{n+1}, y_{1,n+1,\alpha})),\\
y_{2,n+\frac{1}{2},\alpha} &= y_{2,n,\alpha} + \frac{h}{24}(5f(t_n, y_{1,n,\alpha}) + 8f(t_{n+\frac{1}{2}}, y_{1,n+\frac{1}{2},\alpha}) - f(t_{n+1}, y_{1,n+1,\alpha})).
\end{aligned}$$

Theorem 2. *Let* $b = [b_1, b_2]$ *be a fuzzy number,* a *be a real number and* $f(t,y) = a\ y + b$.

1. If $a \geq 0$, *then*

$$y_{1,n+1,\alpha} = \varphi^{n+1} y_{1,0,\alpha} + \theta \sum_{k=0}^{n} \varphi^k,$$

and

$$y_{2,n+1,\alpha} = \varphi^{n+1} y_{2,0,\alpha} + \theta \sum_{k=0}^{n} \varphi^k.$$

2. If $a < 0$, *then*

$$y_{1,n+1,\alpha} = \theta_1 y_{1,n,\alpha} + \gamma_1 b_1 + \theta_2 y_{2,n,\alpha} + \gamma_2 b_2,$$

and

$$y_{2,n+1,\alpha} = \theta_2 y_{1,n,\alpha} + \gamma_2 b_1 + \theta_1 y_{2,n,\alpha} + \gamma_1 b_2.$$

Proof. 1) Let $a \geq 0$. Then, $f(t_{n+j}, y_{i,n+j,\alpha}) = b_i + a\ y_{i,n+j,\alpha}$, where $i = 1,2$ and $j = 0, \frac{1}{2}, 1$. Thus,

$$\begin{aligned}
y_{1,n+1,\alpha} &= y_{1,n,\alpha} + \frac{h}{6}(b_1 + a\ y_{1,n,\alpha} + 4(b_1 + a\ y_{1,n+\frac{1}{2},\alpha}) + b_1 + a\ y_{1,n+1,\alpha}),\\
y_{1,n+\frac{1}{2},\alpha} &= y_{1,n,\alpha} + \frac{h}{24}(5(b_1 + a\ y_{1,n,\alpha}) + 8(b_1 + a\ y_{1,n+\frac{1}{2},\alpha}) - (b_1 + a\ y_{1,n+1,\alpha}),\\
y_{2,n+1,\alpha} &= y_{2,n,\alpha} + \frac{h}{6}(b_2 + a\ y_{2,n,\alpha} + 4(b_2 + a\ y_{2,n+\frac{1}{2},\alpha}) + b_2 + a\ y_{2,n+1,\alpha}),\\
y_{2,n+\frac{1}{2},\alpha} &= y_{2,n,\alpha} + \frac{h}{24}(5(b_2 + a\ y_{2,n,\alpha}) + 8(b_2 + a\ y_{2,n+\frac{1}{2},\alpha}) - (b_2 + a\ y_{2,n+1,\alpha}).
\end{aligned}$$

We can rewrite the last system in the matrix form as

$$A_1 Y_1 = R_1,\ A_2 Y_2 = R_2, \tag{22}$$

where

$$\begin{aligned} A_1 &= \begin{pmatrix} 1-\frac{ah}{3} & \frac{ah}{24} \\ -\frac{2ah}{3} & 1-\frac{ah}{6} \end{pmatrix}, Y_1 = \begin{pmatrix} y_{1,n+\frac{1}{2},\alpha} \\ y_{1,n+1,\alpha} \end{pmatrix}, R_1 = \begin{pmatrix} \frac{h}{2} \\ h \end{pmatrix} b_1 + \begin{pmatrix} 1+\frac{5ah}{24} \\ 1+\frac{ah}{24} \end{pmatrix} y_{1,n,\alpha}, \\ A_2 &= \begin{pmatrix} 1-\frac{ah}{3} & \frac{ah}{24} \\ -\frac{2ah}{3} & 1-\frac{ah}{6} \end{pmatrix}, Y_2 = \begin{pmatrix} y_{2,n+\frac{1}{2},\alpha} \\ y_{2,n+1,\alpha} \end{pmatrix}, R_2 = \begin{pmatrix} \frac{h}{2} \\ h \end{pmatrix} b_2 + \begin{pmatrix} 1+\frac{5ah}{24} \\ 1+\frac{ah}{24} \end{pmatrix} y_{2,n,\alpha}. \end{aligned}$$

Since $\det(A_1) = \frac{1}{12}a^2h^2 - \frac{1}{2}ah + 1 = \frac{(ah-3)^2+3}{12} \neq 0$, we have

$$A_1^{-1} = \begin{pmatrix} \frac{12-2ah}{a^2h^2-6ah+12} & \frac{-ha}{2a^2h^2-12ah+24} \\ \frac{8ha}{a^2h^2-6ah+12} & \frac{12-4ah}{a^2h^2-6ah+12} \end{pmatrix}.$$

Thus,

$$\begin{pmatrix} y_{1,n+\frac{1}{2},\alpha} \\ y_{1,n+1,\alpha} \end{pmatrix} = b_1\ C_1 + y_{1,n,\alpha}\ C_2,$$

where

$$C_1 = \begin{pmatrix} -\frac{3}{2}h\frac{ah-4}{a^2h^2-6ah+12} \\ 12\frac{h}{a^2h^2-6ah+12} \end{pmatrix} and\ C_2 = \begin{pmatrix} -\frac{1}{16}\frac{7a^2h^2-192}{a^2h^2-6ah+12} \\ \frac{3}{2}\frac{a^2h^2+3ah+8}{a^2h^2-6ah+12} \end{pmatrix}.$$

Hence,

$$y_{1,n+1,\alpha} = \frac{12hb_1}{a^2h^2-6ah+12} + \frac{3}{2}\frac{a^2h^2+3ah+8}{a^2h^2-6ah+12}\ y_{1,n,\alpha}.$$

Let $\theta = \frac{12hb_1}{a^2h^2-6ah+12}$ and $\varphi = \frac{3}{2}\frac{a^2h^2+3ah+8}{a^2h^2-6ah+12}$. Then,

$$\begin{aligned} y_{1,n+1,\alpha} &= \theta + \varphi\ y_{1,n,\alpha} \\ &= \theta + \varphi(\theta + \varphi\ y_{1,n-1,\alpha}) \\ &= \theta + \varphi\theta + \varphi^2(\theta + \varphi\ y_{1,n-2,\alpha}) \\ &\ \ \vdots \\ &= \varphi^{n+1}y_{1,0,\alpha} + \theta\sum_{k=0}^{n}\varphi^k. \end{aligned}$$

Therefore,

$$y_{1,n+1,\alpha} = \varphi^{n+1}y_{1,0,\alpha} + \theta\sum_{k=0}^{n}\varphi^k$$

and similarly, we can see that

$$y_{2,n+1,\alpha} = \varphi^{n+1}y_{2,0,\alpha} + \theta\sum_{k=0}^{n}\varphi^k.$$

2) Let $a<0$. Then, the fuzzy HBM1 becomes

$$\begin{aligned}
y_{1,n+1,\alpha} &= y_{1,n,\alpha}+\frac{h}{6}\left(b_1+a\ y_{2,n,\alpha}+4(b_1+a\ y_{2,n+\frac{1}{2},\alpha})+b_1+a\ y_{2,n+1,\alpha}\right),\\
y_{1,n+\frac{1}{2},\alpha} &= y_{1,n,\alpha}+\frac{h}{24}\left(5(b_1+a\ y_{2,n,\alpha})+8(b_1+a\ y_{2,n+\frac{1}{2},\alpha})-(b_1+ay_{2,n+1,\alpha})\right),\\
y_{2,n+1,\alpha} &= y_{2,n,\alpha}+\frac{h}{6}\left(b_2+a\ y_{1,n,\alpha}+4(b_2+a\ y_{1,n+\frac{1}{2},\alpha})+b_2+a\ y_{1,n+1,\alpha}\right),\\
y_{2,n+\frac{1}{2},\alpha} &= y_{2,n,\alpha}+\frac{h}{24}\left(5(b_2+ay_{1,n,\alpha})+8(b_2+a\ y_{1,n+\frac{1}{2},\alpha})-(b_2+ay_{1,n+1,\alpha})\right).
\end{aligned}$$

We can rewrite the last system in the form

$$AY=R, \tag{23}$$

where

$$A=\begin{pmatrix} 1 & 0 & \frac{-ah}{3} & \frac{ah}{24}\\ 0 & 1 & -\frac{2ah}{3} & \frac{-ah}{6}\\ \frac{-ah}{3} & \frac{ah}{24} & 1 & 0\\ -\frac{2ah}{3} & \frac{-ah}{6} & 0 & 1\end{pmatrix}, Y=\begin{pmatrix} y_{1,n+\frac{1}{2},\alpha}\\ y_{1,n+1,\alpha}\\ y_{2,n+\frac{1}{2},\alpha}\\ y_{2,n+1,\alpha}\end{pmatrix},$$

$$R=\begin{pmatrix}\frac{h}{2} & 0\\ h & 0\\ 0 & \frac{h}{2}\\ 0 & h\end{pmatrix}\begin{pmatrix} b_1\\ b_2\end{pmatrix}+\begin{pmatrix} 1 & \frac{5ah}{24}\\ 1 & \frac{ah}{6}\\ \frac{5ah}{24} & 1\\ \frac{ah}{6} & 1\end{pmatrix}\begin{pmatrix} y_{1,n,\alpha}\\ y_{2,n,\alpha}\end{pmatrix}.$$

Since

$$\det(A)=\frac{144a^4h^4-1728a^2h^2+20736}{20736}\neq 0,$$

we have

$$A^{-1}=\begin{pmatrix}
\frac{144}{a^4h^4-12a^2h^2+144} & -\frac{3a^2h^2}{a^4h^4-12a^2h^2+144} & \frac{48ah-2a^3h^3}{a^4h^4-12a^2h^2+144} & -\frac{ah\left(a^2h^2+12\right)}{2a^4h^4-24a^2h^2+288}\\
\frac{48a^2h^2}{a^4h^4-12a^2h^2+144} & \frac{144-12a^2h^2}{a^4h^4-12a^2h^2+144} & \frac{8ah\left(a^2h^2+12\right)}{a^4h^4-12a^2h^2+144} & -\frac{4ah\left(a^2h^2-6\right)}{a^4h^4-12a^2h^2+144}\\
\frac{48ah-2a^3h^3}{a^4h^4-12a^2h^2+144} & -\frac{ah\left(a^2h^2+12\right)}{2a^4h^4-24a^2h^2+288} & \frac{144}{a^4h^4-12a^2h^2+144} & -\frac{3a^2h^2}{a^4h^4-12a^2h^2+144}\\
\frac{8ah\left(a^2h^2+12\right)}{a^4h^4-12a^2h^2+144} & -\frac{4ah\left(a^2h^2-6\right)}{a^4h^4-12a^2h^2+144} & \frac{48a^2h^2}{a^4h^4-12a^2h^2+144} & \frac{144-12a^2h^2}{a^4h^4-12a^2h^2+144}
\end{pmatrix}.$$

Thus,

$$\begin{pmatrix} y_{1,n+\frac{1}{2},\alpha}\\ y_{1,n+1,\alpha}\\ y_{2,n+\frac{1}{2},\alpha}\\ y_{2,n+1,\alpha}\end{pmatrix} = C_1\begin{pmatrix} b_1\\ b_2\end{pmatrix}+C_2\begin{pmatrix} y_{1,n,\alpha}\\ y_{2,n,\alpha}\end{pmatrix},$$

where

$$C_1 = \begin{pmatrix} \frac{72h-3a^2h^3}{a^4h^4-12a^2h^2+144} & \frac{36ah^2-3a^3h^4}{2a^4h^4-24a^2h^2+288} \\ \frac{12h\left(a^2h^2+12\right)}{a^4h^4-12a^2h^2+144} & \frac{72ah^2}{a^4h^4-12a^2h^2+144} \\ \frac{36ah^2-3a^3h^4}{2a^4h^4-24a^2h^2+288} & \frac{72h-3a^2h^3}{a^4h^4-12a^2h^2+144} \\ \frac{72ah^2}{a^4h^4-12a^2h^2+144} & \frac{12h\left(a^2h^2+12\right)}{a^4h^4-12a^2h^2+144} \end{pmatrix}$$

and

$$C_2 = \begin{pmatrix} \frac{-a^4h^4+12a^2h^2+288}{2a^4h^4-24a^2h^2+288} & \frac{72ah-3a^3h^3}{a^4h^4-12a^2h^2+144} \\ \frac{a^4h^4+60a^2h^2+144}{a^4h^4-12a^2h^2+144} & \frac{12ah\left(a^2h^2+12\right)}{a^4h^4-12a^2h^2+144} \\ \frac{72ah-3a^3h^3}{a^4h^4-12a^2h^2+144} & \frac{-a^4h^4+12a^2h^2+288}{2a^4h^4-24a^2h^2+288} \\ \frac{12ah\left(a^2h^2+12\right)}{a^4h^4-12a^2h^2+144} & \frac{a^4h^4+60a^2h^2+144}{a^4h^4-12a^2h^2+144} \end{pmatrix}.$$

Then,

$$\begin{aligned} y_{1,n+1,\alpha} &= \theta_1 y_{1,n,\alpha} + \gamma_1 b_1 + \theta_2 y_{2,n,\alpha} + \gamma_2 b_2, \\ y_{2,n+1,\alpha} &= \theta_2 y_{1,n,\alpha} + \gamma_2 b_1 + \theta_1 y_{2,n,\alpha} + \gamma_1 b_2, \end{aligned}$$

where

$$\theta_1 = \frac{a^4h^4+60a^2h^2+144}{a^4h^4-12a^2h^2+144},\ \theta_2 = \frac{12ah\left(a^2h^2+12\right)}{a^4h^4-12a^2h^2+144},$$
$$\gamma_1 = \frac{12h\left(a^2h^2+12\right)}{a^4h^4-12a^2h^2+144},\ \gamma_2 = \frac{72ah^2}{a^4h^4-12a^2h^2+144}.$$

6. Numerical examples

In this section, we present two numerical examples to show the efficiency of the proposed method. We start with a linear fuzzy initial value problem, and then study a nonlinear problem in the second example.

Example 3. *Consider the following linear first order fuzzy initial value problem*

$$y'(x) = (x^2 \odot y(x)) \oplus (\hat{\gamma} \odot x^2), \quad y(0) = \hat{\gamma}$$

where α*-level set for* $\hat{\gamma} = (1 \quad 2 \quad 3)$ *is* $\gamma_\alpha = [1+\alpha, 3-\alpha]$ *and,* $h = 0.1$*. Let* $y(x) = [y_1(x,\alpha), y_2(x,\alpha)]$ *be the fuzzy solution. By implementing* α*-level sets, the problem becomes*

$$[y_1'(x,\alpha), y_2'(x,\alpha)] = (x^2 \odot [y_1(x,\alpha), y_2(x,\alpha)]) \oplus [x^2 \odot (1+\alpha), x^2 \odot (3-\alpha)],$$

$$[y_1(0,\alpha), y_2(0,\alpha)] = [1+\alpha, 3-\alpha].$$

Using HBM1 for the lower bound, we get

$$y_{1,n+1} = y_{1,n} + \frac{h}{6}\left(x_n^2 y_{1,n} + x_n^2(1+\alpha) + 4\left(x_{n+\frac{1}{2}}^2 y_{1,n+\frac{1}{2}} + x_{n+\frac{1}{2}}^2(1+\alpha)\right) + x_{n+1}^2 y_{1,n+1} + x_{n+1}^2(1+\alpha)\right),$$

$$y_{1,n+\frac{1}{2}} = y_{1,n} + \frac{h}{24}\left(5\left(x_n^2 y_{1,n} + x_n^2(1+\alpha)\right) + 8\left(x_{n+\frac{1}{2}}^2 y_{1,n+\frac{1}{2}} + x_{n+\frac{1}{2}}^2(1+\alpha)\right) - \left(x_{n+1}^2 y_{1,n+1} + x_{n+1}^2(1+\alpha)\right)\right).$$

Let $x_n = nh,\ \ x_{n+\frac{1}{2}} = \frac{h}{2} + nh$ *and* $x_{n+1} = h + nh$. *Then, we can rewrite the above system in a matrix form as*

$$Y_m = A^{-1}B\, y_{1,n} + A^{-1}C_1,$$

where

$$Y_m = \begin{pmatrix} y_{1,n+1} \\ y_{1,n+\frac{1}{2}} \end{pmatrix}, \quad A = \begin{pmatrix} 1 - \frac{h}{6}(h+nh)^2 & \frac{4h}{6}(\frac{h}{2}+nh)^2 \\ \frac{h}{24}(h+nh)^2 & 1 - \frac{8h}{24}(\frac{h}{2}+nh)^2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 + \frac{h}{6}(nh)^2 \\ 1 + \frac{5h}{24}(nh)^2 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} \frac{h}{6}(1+\alpha)\left((nh)^2 + 4(\frac{h}{2}+nh)^2 + (h+nh)^2\right) \\ \frac{h}{24}(1+\alpha)\left(5(nh)^2 + 8(\frac{h}{2}+nh)^2 - (h+nh)^2\right) \end{pmatrix}.$$

Similarly, we can apply HBM1 for the upper bound. Then, we get

$$Y_m = A^{-1}B\, y_{2,n} + A^{-1}C_2$$

where

$$Y_m = \begin{pmatrix} y_{2,n+1} \\ y_{2,n+\frac{1}{2}} \end{pmatrix}, \quad A = \begin{pmatrix} 1 - \frac{h}{6}(h+nh)^2 & \frac{4h}{6}(\frac{h}{2}+nh)^2 \\ \frac{h}{24}(h+nh)^2 & 1 - \frac{8h}{24}(\frac{h}{2}+nh)^2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 + \frac{h}{6}(nh)^2 \\ 1 + \frac{5h}{24}(nh)^2 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} \frac{h}{6}(3-\alpha)\left((nh)^2 + 4(\frac{h}{2}+nh)^2 + (h+nh)^2\right) \\ \frac{h}{24}(3-\alpha)\left(5(nh)^2 + 8(\frac{h}{2}+nh)^2 - (h+nh)^2\right) \end{pmatrix}.$$

The errors of approximation of $y_{1,n}$ *and* $y_{2,n}$ *for* $\alpha = 0,\ 0.25,\ 0.5,\ 0.75,\ 1$, *are given in Tables 1 and 2, respectively, where the exact solution is given by*

$$[(2e^{\frac{x^3}{3}} - 1)(1+\alpha), (2e^{\frac{x^3}{3}} - 1)(3-\alpha)].$$

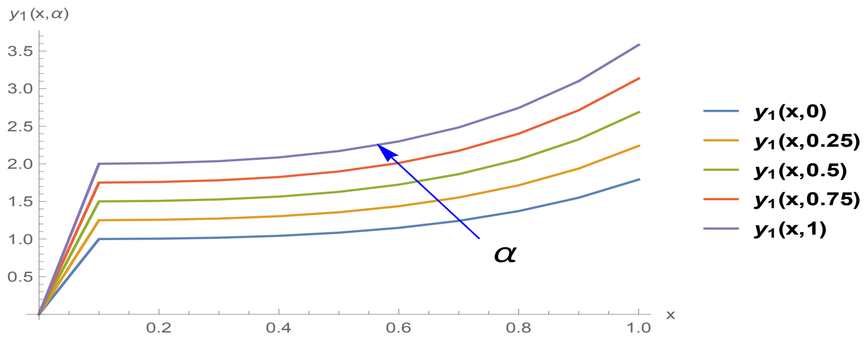

Figure 3: The approximate solution y_1 for $\alpha = 0, 0.25, 0.5, 0.75, 1$.

$y_{1,n}$					
$x/\ \alpha$	0	0.25	0.5	0.75	1
0	0	0	0	0	0
0.1	1.39×10^{-12}	1.74×10^{-12}	2.08×10^{-12}	2.43×10^{-12}	2.78×10^{-12}
0.2	5.56×10^{-12}	6.95×10^{-12}	8.34×10^{-12}	9.7×10^{-12}	1.11×10^{-11}
0.3	1.25×10^{-11}	1.57×10^{-11}	1.88×10^{-11}	2.20×10^{-11}	2.51×10^{-11}
0.4	2.24×10^{-11}	2.80×10^{-11}	3.36×10^{-11}	3.92×10^{-11}	4.48×10^{-11}
0.5	3.52×10^{-11}	4.39×10^{-11}	5.27×10^{-11}	6.15×10^{-11}	7.03×10^{-11}
0.6	5.078×10^{-11}	6.35×10^{-11}	7.62×10^{-11}	8.89×10^{-11}	1.02×10^{-10}
0.7	6.88×10^{-11}	8.60×10^{-11}	1.03×10^{-10}	1.20×10^{-10}	1.38×10^{-10}
0.8	8.75×10^{-11}	1.09×10^{-10}	1.31×10^{-10}	1.53×10^{-10}	1.75×10^{-10}
0.9	1.02×10^{-10}	1.28×10^{-10}	1.54×10^{-10}	1.79×10^{-10}	2.05×10^{-10}
1	1.03×10^{-10}	1.29×10^{-10}	1.55×10^{-10}	1.80×10^{-10}	2.06×10^{-10}

Table 1: The absolute error in approximating $y_{1,n}$ for $h = 0.01$.

Example 4. *Consider the nonlinear first order fuzzy initial value problem*

$$y'(x) = y(x)^2 + x^2, \quad y(0) = \hat{\gamma}, \quad x \geq 0,$$

where α*-level set for* $\hat{\gamma} = (-0.1 \ \ 0 \ \ 0.1)$ *is* $\hat{\gamma}_\alpha = [0.1(\alpha - 1), 0.1(1 - \alpha)]$*, and* $h = 0.1$*. Let* $y(x) = [y_1(x, \alpha), y_2(x, \alpha)]$ *be a fuzzy solution. By implementing* α*-level sets, the problem becomes*

$$[y_1'(x,\alpha), y_2'(x,\alpha)] = [y_1(x,\alpha), y_2(x,\alpha)]^2 + x^2, \ [y_1(0,\alpha), y_2(0,\alpha)] = [0.1(\alpha - 1), 0.1(1 - \alpha)],$$

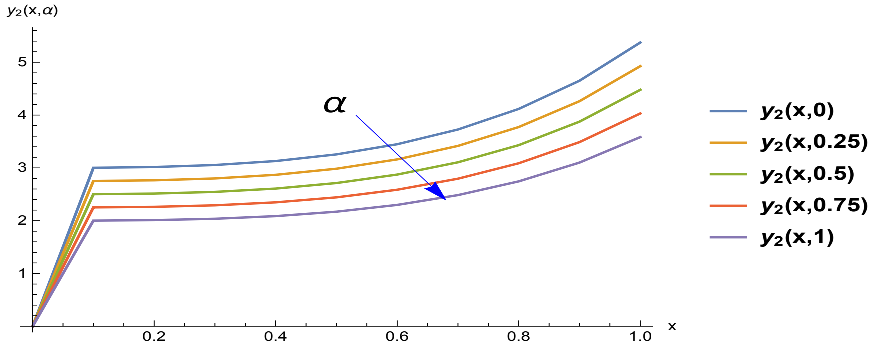

Figure 4: The approximate solution y_2 for $\alpha = 0, 0.25, 0.5, 0.75, 1$.

$y_{2,n}$					
x / α	0	0.25	0.5	0.75	1
0	0	0	0	0	0
0.1	4.17×10^{-12}	3.82×10^{-12}	3.47×10^{-12}	3.13×10^{-12}	2.78×10^{-12}
0.2	1.67×10^{-11}	1.53×10^{-11}	1.39×10^{-11}	1.25×10^{-11}	1.11×10^{-11}
0.3	3.76×10^{-11}	3.45×10^{-11}	3.14×10^{-11}	2.8×10^{-11}	2.51×10^{-11}
0.4	6.72×10^{-11}	6.16×10^{-11}	5.60×10^{-11}	5.04×10^{-11}	4.48×10^{-11}
0.5	1.05×10^{-10}	9.67×10^{-11}	8.79×10^{-11}	7.91×10^{-11}	7.0×10^{-11}
0.6	1.52×10^{-10}	1.40×10^{-10}	1.27×10^{-10}	1.14×10^{-10}	1.02×10^{-10}
0.7	2.06×10^{-10}	1.89×10^{-10}	1.72×10^{-10}	1.55×10^{-10}	1.38×10^{-10}
0.8	2.62×10^{-10}	2.41×10^{-10}	2.19×10^{-10}	1.97×10^{-10}	1.75×10^{-10}
0.9	3.07×10^{-10}	2.82×10^{-10}	2.56×10^{-10}	2.31×10^{-10}	2.05×10^{-10}
1	3.09×10^{-10}	2.84×10^{-10}	2.58×10^{-10}	2.32×10^{-10}	2.06×10^{-10}

Table 2: Absolute error in approximating $y_{2,n}$ for $h = 0.01$.

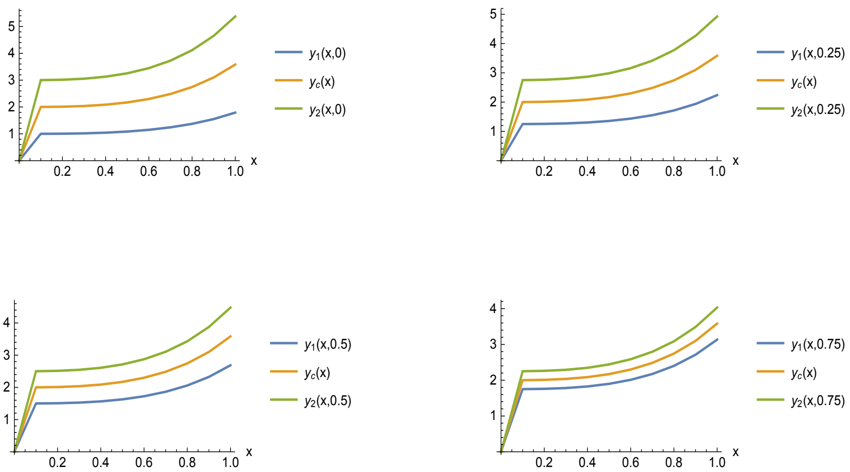

Figure 5: The approximate solutions of y_1, y_2, and crisp solution for $\alpha = 0, 0.25, 0.5, 0.75, 1$.

where the exact solution is given by

$$y_k(x,\alpha) = \frac{(1-\alpha)J_{\frac{-3}{4}}(\frac{x^2}{2})\,\Gamma(\frac{1}{4}) + (-1)^k 20x J_{\frac{3}{4}}(\frac{x^2}{2})\,\Gamma(\frac{3}{4})}{(\alpha-1)J_{\frac{1}{4}}(\frac{x^2}{2})\,\Gamma(\frac{1}{4}) + (-1)^k 20 J_{\frac{-1}{4}}(\frac{x^2}{2})\,\Gamma(\frac{3}{4})},$$

for $k = 1,\ 2, J_n(z)$ is the Bessel function of the first kind, and $\Gamma(z)$ is the Euler Gamma function. In Table 3, we present the absolute error of the results obtained by the current

(HBM1) method and the ones obtained in [30].

α/Error	$y_{1,0.5}$	$y_{2,0.5}$	$y_{1,0.5}$ (HPM) [30]	$y_{2,0.5}$ (HPM) [30]
0	5.57124×10^{-8}	5.43066×10^{-8}	1.63068×10^{-6}	1.99421×10^{-6}
0.2	5.65270×10^{-8}	5.58044×10^{-8}	1.03431×10^{-6}	1.27971×10^{-6}
0.4	5.72014×10^{-8}	5.68912×10^{-8}	6.33486×10^{-7}	8.09126×10^{-7}
0.6	5.77142×10^{-8}	5.76168×10^{-8}	3.51316×10^{-7}	4.87165×10^{-7}
0.8	5.80413×10^{-8}	5.80253×10^{-8}	1.34909×10^{-7}	2.50421×10^{-7}
1	5.81552×10^{-8}	5.81552×10^{-8}	5.46386×10^{-8}	5.46386×10^{-8}

Table 3: Absolute errors for $y_{1,0.5}$ and $y_{2,0.5}$.

7. Conclusion

In this paper, an optimized one-step hybrid block method has been proposed for solving fuzzy first order initial value problems of ordinary differential equations. The method is a self-starting method since it depends on the initial condition only. The proposed method is zero stable, has order 3, consistent, and thus it is convergent. Also, the method is A-stable as illustrated by the region of absolute stability in Figure 1. We solved two examples, linear and nonlinear using the current method. The numerical results show the efficiency of the current method where high precision is achieved even when we use only one-step point. From Tables 1 and 2, we note that the results are highly accurate with a small perturbation of errors. In Table 3, we compared our results with other results obtained in [30]. We noticed that our method is better and more accurate. Besides, Figure 3 shows the behavior of the lower bound solution y_1 is increasing as α increases, and Figure 4, shows the behavior of upper bound solution y_2 is decreasing as α increases. Consequently, from this behavior, we conclude that the solution is fuzzy. At the end, Figure 5, shows that the crisp solution is bounded by y_1 and y_2 and they become close to the crisp solution as α approaches to one. It is worth mention that this type of problem can devolped to solve several real applications in Physics and engineering [30–32].

Author Contributions:

All authors have the same contribution.

Data Availability Statement:

Not applicable.

Conflicts of Interest:

The authors declare no conflict of interest.

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