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Analytical Study of Multi Stages Breast Cancer Mathematical Model Using Fractional Calculus

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Abstract. In this manuscript, we investigate a mathematical model for breast cancer with chemotherapy treatment. Considered model is a six compartmental mathematical model for breast cancer having four stages of breast cancer, recovery and cardiotoxicity. Nonlocal fractional order derivative due to Caputo-Fabrizio is used for the considered study. The mentioned derivative has the ability to avoid singularity due to its nonlocal and nonsingular kernel. The considered model is investigated for qualitative analysis and also approximate solutions. Analysis is made on the basis of analytical method. Since breast reast cancer, is one of the most spreading cancer in the world which starts in the breast tissue. Type of breast cancer and overall health determine how it can be treated. Radiation, hormone therapy, chemotherapy, and surgery are among the available treatments. Usually, chemotherapy is used to treat it. Most chemotherapy drugs are quite effective at preventing cancer from spreading throughout the body. Chemotherapy, however, can result in cardiotoxicity due to its effects on blood arteries and heart muscle. A graphical presentation corresponding to some fundamental data is given via using MATLAB 2023.

2020 Mathematics Subject Classifications: 26A33, 34A08, 03C65.

Key Words and Phrases: Epidemiological model, multi stage breast cancer, cardiotoxicity, cFFD, laplace transform.

1. Introduction

Cancer is a threatful disease around the globe. Annually millions of people are dying throughout the world due to cancer disease. Any one of a wide range of illnesses known as cancer is defined by the growth of aberrant cells that divide uncontrolled and have the capacity to invade and kill healthy bodily tissue. It is frequently possible for cancer to spread throughout your body[1]. Changes in the body brought on by cancer are known

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as cancer symptoms. Although the condition might induce more general symptoms like fatigue or weight loss, they are typically brought on by the impact of a cancer on the area of the body where it is growing. Over 100 distinct forms of cancer exist, each with a variety of symptoms that might appear in various ways[2]. Breast cancer is a condition in which aberrant breast cells proliferate uncontrollably and develop into tumors. Fluid flowing from the nipple, skin dimpling, breast lumps, and changes in breast size or form are some of the symptoms. Treatment for breast cancer, which includes hormone therapy, radiation, chemotherapy, and surgery, is most effective when the cancer is discovered early, therefore early identification by screening and self-awareness is essential[3]. Mathematical models are the powerful tools to investigate various real world problems from different perspectives. With the help of mathematical models, prediction, future direction and control strategies can be analyzed. Researchers have produced plenty of research work in this regards to investigate various infectious and chronic diseases. For instance in recent times, the COVID-19 was very well explored by using mathematical models [4]. In the same way, tuberculosis a threatful infection has been studied very well in past by using mathematical models concepts [5]. Similarly, the Spanish flue [6] and influenza infections were studied by using mathematical models [7]. For the mentioned modeling purposes, researchers have used stochastic, ordinary, partial and fractional differential equations extensively[8]. Using of fractional order mathematical models for diseases investigation is an hot topic of research in recent times. Since, fractional derivatives are global operators which describe the dynamics more explicitly and comprehensively in many dynamical problems. Therefore, researchers have done valuable work by introducing different kinds of fractional differential operators. The concerned operators contain singular and nonsingular types kernels on the basis of which they have been classified. The concerned operators have various advantages and also some drawback been found there. As compared to drawback, they have large numbers of applications in real world problems. We refer to [9] and [10] for remarkable contribution. Recently, Khan [11] has studied uranium decay by using fractional calculus. Khan,et.al[12] studied co-infection model of COVID-19 and tuberculosis disease by using fractional calculus. In the same way, authors [13] investigated a mathematical model of epidemiology by using fractional calculus.

To solve problems of real world contain fractional order derivatives, researchers have established numerous techniques. They have deduced analytical as well as numerical procedure for finding the exact or approximate solutions. Mote of the used tools for finding the solution to fractional order differential problems, researchers have extended the traditional techniques. For instance, the famous Euler, Runge-Kutta and Adam Bashforth methods were increasingly extended to the mentioned area. We refer to [14] for some interesting numerical work. In the same way, researchers have also worked very well for investigating analytical work [15]. Tools like Laplace, Sumudu, and Natural transforms have also used to study various problems of fractional order equations. Researchers [16] have used the Laplace transform to study some problems. Also, researchers of [17] and [18] have applied other integral transforms for investigating solutions to fractional order problems.

Motivated from the mentioned use of mathematical models and analytical techniques,

we investigate the following breast cancer linear model [19] under fractional order Caputo-Fabrizio derivative (CFFD)

$$\begin{cases} \dfrac{d\mathcal{Y}_1}{dt} = C_{y_1} + \nu_1\mathbb{R} - (\alpha + n_1)\mathcal{Y}_1, \\ \dfrac{d\mathcal{Y}_2}{dt} = C_{y_2} + \nu_2\mathbb{R} + \alpha\mathcal{Y}_1 - (\beta + n_2)\mathcal{Y}_2, \\ \dfrac{d\mathcal{Y}_3}{dt} = C_{y_3} + \beta\mathcal{Y}_2 + \nu_3\mathbb{R} - (p_3 + \mu_3 + \gamma + n_3)\mathcal{Y}_3, \\ \dfrac{d\mathcal{Y}_4}{dt} = C_{y_4} + \gamma\mathcal{Y}_3 + \nu_4\mathbb{R} - (p_4 + \mu_4 + n_4)\mathcal{Y}_4, \\ \dfrac{d\mathbb{R}}{dt} = n_1\mathcal{Y}_1 + n_2\mathcal{Y}_2 + n_3\mathcal{Y}_3 + n_4\mathcal{Y}_4 - (p_1 + \nu_1 + \nu_2 + \nu_3 + \nu_4)\mathbb{R}, \\ \dfrac{d\mathcal{C}}{dt} = p_1\mathbb{R} + p_3\mathcal{Y}_3 + p_4\mathcal{Y}_4 - \mu_c\mathcal{C}. \end{cases} \tag{1}$$

Since, breast cancer is a big health issue throughout the world now a days. A detailed survey [20] about Karachi a populated big city of Pakistan has been published recently which shows rapid growth of mentioned disease. The concerned disease has been recently investigated by using various analysis, we refer to [21] and [22]. The classes are organized based on the epidemiology of breast malignancy in model (1) which are defined as inpatient cancer patients represented by the Y_i symbol, where $i = 1, 2, 3, 4$. Components and parameters used in the model (2) are given in .

Parameters	Physical Meaning
$\mathcal{Y}_1$	stage 1 carcinoma compartment
$\mathcal{Y}_2$	stage 2 carcinoma compartment
$\mathcal{Y}_3$	stage 3 carcinoma compartment
$\mathcal{Y}_4$	stage 4 carcinoma compartment
$\mathbb{R}$	recovered after chemotherapy compartment
$\mathcal{C}$	carcinoma patients experience cardiotoxicity compartment
C_{y_1}	Rate of newly diagnosed stage 1
C_{y_2}	Rate of newly diagnosed stage 2
C_{y_3}	Rate of newly diagnosed stage 3
C_{y_4}	Rate of newly diagnosed stage 4
α	patients number from $\mathcal{Y}_1$ to $\mathcal{Y}_2$
n_1	patients number at $\mathcal{Y}_1$ who get a complete response
n_2	patients number at $\mathcal{Y}_2$ who get a complete response
n_3	patients number at $\mathcal{Y}_3$ who get a complete response
n_4	patients number at $\mathcal{Y}_4$ who get a complete response
β	patients number from $\mathcal{Y}_2$ to $\mathcal{Y}_3$
ν_1	patients number at disease-free state who return back to $\mathcal{Y}_1$
ν_2	patients number at disease-free state who return back to $\mathcal{Y}_2$
ν_3	patients number at disease-free state who return back to $\mathcal{Y}_3$
ν_4	patients number at disease-free state who return back to $\mathcal{Y}_4$
γ	patients number from $\mathcal{Y}_3$ to $\mathcal{Y}_4$
p_1	patients number at $\mathcal{Y}_3$ carcinoma chemotherapy patients who feel cardiotoxic
p_3	patients number at $\mathcal{Y}_3$ carcinoma chemotherapy patients who feel cardiotoxic
p_4	patients number at disease-free state chemotherapy patients who feel cardiotoxic
μ_3	death rate of carcinoma at $\mathcal{Y}_3$
μ_4	death rate of carcinoma at $\mathcal{Y}_4$
μ_c	death rate due to cardiotoxic

Table 1: Model parameters and description (1).

It is assumed that chemotherapy is given to all cancer patients. Depending on the severity of the patient's cancer, each group will undergo modifications throughout time. For some of these individuals, cardiotoxicity is a side effect of their drug; while some recover, others experience catastrophic complications. Individuals without disease are shown in compartment $\mathcal{R}$ following chemotherapy. Patients with cardiotoxicity who have cancer are in compartment $\mathcal{C}$.

1.1. Formulation of proposed model

Driven by the aforementioned research, the existence, uniqueness, and semi-analytical outcomes of the suggested model with the CFFD have not been examined. In order to bridge this gap, we address the model (1) under CFFD for existence theory and semi-

analytical conclusions as

$$\begin{cases} {}^{CF}D_t^{\sigma}\mathcal{Y}_1(t) = C_{y_1} + \nu_1\mathbb{R} - (\alpha + n_1)\mathcal{Y}_1, \\ {}^{CF}D_t^{\sigma}\mathcal{Y}_2(t) = C_{y_2} + \nu_2\mathbb{R} + \alpha\mathcal{Y}_1 - (\beta + n_2)\mathcal{Y}_2, \\ {}^{CF}D_t^{\sigma}\mathcal{Y}_3(t) = C_{y_3} + \beta\mathcal{Y}_2 + \nu_3\mathbb{R} - (p_3 + \mu_3 + \gamma + n_3)\mathcal{Y}_3, \\ {}^{CF}D_t^{\sigma}\mathcal{Y}_4(t) = C_{y_4} + \gamma\mathcal{Y}_3 + \nu_4\mathbb{R} - (p_4 + \mu_4 + n_4)\mathcal{Y}_4, \\ {}^{CF}D_t^{\sigma}\mathbb{R}(t) = n_1\mathcal{Y}_1 + n_2\mathcal{Y}_2 + n_3\mathcal{Y}_3 + n_4\mathcal{Y}_4 - (p_1 + \nu_1 + \nu_2 + \nu_3 + \nu_4)\mathbb{R}, \\ {}^{CF}D_t^{\sigma}\mathcal{C}(t) = p_1\mathbb{R} + p_3\mathcal{Y}_3 + p_4\mathcal{Y}_4 - \mu_c\mathcal{C}. \end{cases} \tag{2}$$

The operator ${}^{CF}D_t^{\sigma}$ used for modified type CFFD. The concerned derivative has been utilized very well for various dynamical problems [23]. Further the applications and utilizations of CFFD, we refer to [24]. Since it's essential to determine whether a solution exists for the dynamical system. Thus, some qualitative findings are established initially, such as the existence and uniqueness of the solution corresponding to the model under consideration. Many techniques have been employed for this, and one of the most effective ones is fixed point theory. Consequently, adequate conditions for the existence and uniqueness of solutions to the suggested model are developed using fixed point theorems of Banach and Krassnoselskii [25]. Additionally, several semi-analytical results are derived for the approximate solution to different model compartments. Next, some approximate analytical results are investigated using Laplace transforms approach. By the end Matlab is used for graphical representations of approximate results.

2. Preliminaries

For our required results, we need some basic fractional calculus definitions and results given as:

Definition 1. *[23] Let $w \in \mathcal{H}^1(0,\theta)$, $\theta > 0$, $\sigma \in (0,1)$, then, the CFFD is given as*

$${}^{CF}\mathcal{D}_t^{\sigma}(w(t)) = \frac{\mathscr{M}(\sigma)}{1-\sigma}\int_0^t w'(t)\exp\left[-\sigma\frac{t-\xi}{1-\sigma}\right]d\xi.$$

However, if w is not in $\mathcal{H}^1(0,\theta)$, then the derivative is given as

$${}^{CF}\mathcal{D}_t^{\sigma}(w(t)) = \frac{\mathscr{M}(\sigma)}{1-\sigma}\int_0^t (w(t)-w(\xi))\exp\left[-\sigma\frac{t-\xi}{1-\sigma}\right]d\xi.$$

Definition 2. *[23] For $w \in \mathcal{H}^1(0,\theta)$, $\theta > 0$, by using $\nabla_1 = \frac{(1-\sigma)}{\mathscr{M}(\sigma)}$, $\nabla_2 = \frac{\sigma}{\mathscr{M}(\sigma)}$, the integral of CF is given*

$${}^{CF}\mathcal{I}_t^{\sigma}[w(t)] = \nabla_1 w(t) + \nabla_2\int_0^t w(\xi)d\xi,\ \sigma \in (0,1].$$

Definition 3. *[24] The LT of ${}^{CF}\mathcal{D}_t^{\sigma}w(t)$ with $\mathscr{M}(\sigma) = 1$ is given as*

$$\mathscr{L}[{}^{CF}\mathcal{D}_t^{\sigma}w(t)] = \frac{s\mathscr{L}[w(t)] - w(0)}{s + \sigma(1-s)}, s \geq 0,\ \sigma \in (0,1].$$

Note: Corresponding to existence theory, let $\mathcal{J} = [0, \theta]$ and $0 \leq t \leq \theta < \infty$ and we define space as $\mathbf{Z} = C([0, \theta] \times \mathbb{R}^6, \mathbb{R})$, with norm given as $\|(x_1, x_2, x_3, x_4, x_5, x_6)\| = \sup_{t \in \mathcal{J}} \sum_{i=1}^{6} |x_i(t)|$.

Theorem 1. *[25] Let $\mathbb{Y} \subset \mathbf{Z}$ be the convex closed set, such that $\mathfrak{Q}_1, \mathfrak{Q}_2$ be two operators satisfy the following conditions*

1) $\mathfrak{Q}_1 y + \mathfrak{Q}_2 y \in \mathbb{Y}$ for every $y \in \mathbb{Y}$;

2) $\mathfrak{Q}_1$ is a condensing map;

3) $\mathfrak{Q}_2$ is completely continuous,

then, at least one solution for the operator equation $Q_1 z + Q_2 z = z$ exists.

3. Main result for proposed fractional order model

Some results about the proposed model (2), existence and uniqueness of the solution, are derived in this portion. Applying integral operator ${}^{CF}\mathcal{I}_t^{\sigma}$ on both sides of (2) and using initial conditions, we have

$$\begin{cases} \mathcal{Y}_1(t) = \mathcal{Y}_1(0) + \nabla_1 \varphi_1(t, \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4, \mathbb{R}, \mathcal{C}) \\ + \frac{\sigma}{\mathscr{M}(\sigma)\Gamma(\sigma)} \int_0^t (t-\xi)^{\sigma-1} \varphi_1(\xi, \mathcal{Y}_1(\xi), \mathcal{Y}_2(\xi), \mathcal{Y}_3(\xi), \mathcal{Y}_4(\xi), \mathbb{R}(\xi), \mathcal{C}(\xi)) d\xi, \\ \mathcal{Y}_2(t) = \mathcal{Y}_2(0) + \nabla_1 \varphi_2(t, \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4, \mathbb{R}, \mathcal{C}) \\ + \frac{\sigma}{\mathscr{M}(\sigma)\Gamma(\sigma)} \int_0^t (t-\xi)^{\sigma-1} \varphi_2(\xi, \mathcal{Y}_1(\xi), \mathcal{Y}_2(\xi), \mathcal{Y}_3(\xi), \mathcal{Y}_4(\xi), \mathbb{R}(\xi), \mathcal{C}(\xi)) d\xi, \\ \mathcal{Y}_3(t) = \mathcal{Y}_3(0) + \nabla_1 \varphi_3(t, \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4, \mathbb{R}, \mathcal{C}) \\ + \frac{\sigma}{\mathscr{M}(\sigma)\Gamma(\sigma)} \int_0^t (t-\xi)^{\sigma-1} \varphi_3(\xi, \mathcal{Y}_1(\xi), \mathcal{Y}_2(\xi), \mathcal{Y}_3(\xi), \mathcal{Y}_4(\xi), \mathbb{R}(\xi), \mathcal{C}(\xi)) d\xi, \\ \mathcal{Y}_4(t) = \mathcal{Y}_4(0) + \nabla_1 \varphi_4(t, \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4, \mathbb{R}, \mathcal{C}) \\ + \frac{\sigma}{\mathscr{M}(\sigma)\Gamma(\sigma)} \int_0^t (t-\xi)^{\sigma-1} \varphi_4(\xi, \mathcal{Y}_1(\xi), \mathcal{Y}_2(\xi), \mathcal{Y}_3(\xi), \mathcal{Y}_4(\xi), \mathbb{R}(\xi), \mathcal{C}(\xi)) d\xi, \\ \mathbb{R}(t) = \mathbb{R}(0) + \nabla_1 \varphi_5(t, \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4, \mathbb{R}, \mathcal{C}) \\ + \frac{\sigma}{\mathscr{M}(\sigma)\Gamma(\sigma)} \int_0^t (t-\xi)^{\sigma-1} \varphi_5(\xi, \mathcal{Y}_1(\xi), \mathcal{Y}_2(\xi), \mathcal{Y}_3(\xi), \mathcal{Y}_4(\xi), \mathbb{R}(\xi), \mathcal{C}(\xi)) d\xi, \\ \mathcal{C}(t) = \mathcal{C}(0) + \nabla_1 \varphi_6(t, \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4, \mathbb{R}, \mathcal{C}) \\ + \frac{\sigma}{\mathscr{M}(\sigma)\Gamma(\sigma)} \int_0^t (t-\xi)^{\sigma-1} \varphi_6(\xi, \mathcal{Y}_1(\xi), \mathcal{Y}_2(\xi), \mathcal{Y}_3(\xi), \mathcal{Y}_4(\xi), \mathbb{R}(\xi), \mathcal{C}(\xi)) d\xi. \end{cases} \tag{3}$$

This further can be simplified as

$$\Omega(t) = \Omega_0 + \mho(t, \Omega(t))\nabla_1 + \nabla_2 \int_0^t \Psi(\xi, \Omega(\xi)) d\xi, \tag{4}$$

where

$$\Omega(t) = \begin{cases} \mathcal{Y}_1(t) \\ \mathcal{Y}_2(t) \\ \mathcal{Y}_3(t) \\ \mathcal{Y}_4(t) \\ \mathbb{R}(t) \\ \mathcal{C}(t) \end{cases}, \quad \Omega_0 = \begin{cases} \mathcal{Y}_{1_0} \\ \mathcal{Y}_{2_0} \\ \mathcal{Y}_{3_0} \\ \mathcal{Y}_{4_0} \\ \mathbb{R}_0 \\ \mathcal{C}_0 \end{cases}, \quad \mho(t, \Omega(t)) = \begin{cases} \varphi_1(t, \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4, \mathbb{R}, \mathcal{C}) \\ \varphi_2(t, \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4, \mathbb{R}, \mathcal{C}) \\ \varphi_3(t, \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4, \mathbb{R}, \mathcal{C}) \\ \varphi_4(t, \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4, \mathbb{R}, \mathcal{C}) \\ \varphi_5(t, \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4, \mathbb{R}, \mathcal{C}) \\ \varphi_6(t, \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4, \mathbb{R}, \mathcal{C}). \end{cases} \tag{5}$$

For further analysis we take following assumptions into account.

(G1) For constants $\mathbf{L}_{\wp} > 0$ with $\Omega, \bar{\Omega} \in \mathbf{Z}$, one has

$$|\wp(t, \Omega(t)) - \wp(t, \bar{\Omega}(t))| \leq \mathbf{L}_{\wp}[|\Omega - \bar{\Omega}|].$$

(G2) For fixed real values $C_{\wp}, C_{\wp} > 0$ and $\mathbf{M}_{\wp} > 0$, one has

$$|\wp(t, \Omega(t))| \leq C_{\wp}|\Omega| + M_{\wp}.$$

Using (4) and (5), the two operators are defined as:

$$\begin{aligned} \mathfrak{Q}_1(\Omega) &= \Omega_0(t) + \wp(t, \Omega(t))\nabla_1, \\ \mathfrak{Q}_2(\Omega) &= \nabla_2 \int_0^t \wp(\xi, \Omega(\xi))d\xi. \end{aligned} \tag{6}$$

Theorem 2. *Using assumptions* $(G1)$ *and* $(G2)$*, the integral system* (4) *has at least one solution with the restriction* $\nabla_1 \mathbf{L}_{\wp} < 1$.

Proof. Take $\mathbb{Y} = \{\Omega \in \mathbf{Z} : \ \|\Omega\| \leq \rho, \ \rho > 0\}$. It is required to prove that $P_1 : \mathbb{Y} \to \mathbb{Y}$ is a contraction. As given that $\Omega, \bar{\Omega} \in \mathbf{Z}$, we have

$$\begin{aligned} \|P_1\Omega - P_1\bar{\Omega}\| &= \sup_{t\in\mathcal{J}} \left| \Big(\wp(t, \Omega(t)) - (\wp(t, \bar{\Omega}(t)) \Big) \nabla_1 \right| \\ &\leq \nabla_1 \mathbf{L}_{\wp} \sup_{t\in\mathcal{J}} |\Omega(t) - \bar{\Omega}(t)| \\ &\leq \nabla_1 \mathbf{L}_{\wp} \|\Omega - \bar{\Omega}\|. \end{aligned}$$

Thus P_1 is a condensing map. To show that P_2 is completely continuous operator, take

$$\begin{aligned} \|P_2(\Omega)\| &= \sup_{t\in\mathcal{J}} \left| \int_0^t \wp(\xi, \Omega(\xi))d\xi \right| \\ &\leq \nabla_2 \theta[C_{\wp}\rho + \mathbf{M}_{\wp}] := \Delta. \end{aligned} \tag{7}$$

Due to continuity of $\wp$, P_2 is also continuous and bounded. Let take $t_1 < t_2 \in \mathcal{J}$, one has

$$\begin{aligned} \|P_2(\Omega)(t_2) - P_2(\Omega)(t_1)\| &= \nabla_2 \sup_{t\in\mathcal{J}} \left| \int_0^{t_2} \wp(\xi, \Omega(\xi))d\xi - \int_0^{t_1} \wp(\xi, \Omega(\xi))d\xi \right| \\ &\leq \nabla_2 \sup_{t\in\mathcal{J}} \int_{t_1}^{t_2} |\wp(\xi, \Omega(\xi))|d\xi \\ &\leq \nabla_2 \theta[C_{\wp}\rho + \mathbf{M}_{\wp}](t_2 - t_1) \to 0, \text{ as } t_2 \to t_1, \end{aligned} \tag{8}$$

hence it is equicontinues as $\|P_2(\Omega)(t_2) - P_2(\Omega)(t_1)\| \to 0$. Thus P_2 is completely continuous. Therefore, in view of the Theorem 1, the problem 2 has at least one solution.

Theorem 3. *By hypothesis* $(D1)$*, the system* (2) *has a unique solution under the condition* $[\nabla_1 + \nabla_2\theta]\mathbf{L}_\wp < 1$*.*

Proof. Let define $\mathbf{P} : \mathbf{Z} \to \mathbf{Z}$ by

$$\mathbf{P}(\Omega) = \Omega_0 + \nabla_1 \wp(t, \Omega(t)) + \int_0^t \nabla_2 \wp(\xi, \Omega(\xi))d\xi.$$

Let $\Omega, \bar{\Omega} \in \mathbf{Z}$, one has

$$\begin{aligned} \|\mathbf{P}(\Omega) - \mathbf{P}(\bar{\Omega})\| &\leq \sup_{t\in\mathcal{J}} \nabla_1 \left| \wp(t, \Omega(t)) - \wp(t, \bar{\Omega}(t)) \right| \\ &+ \nabla_2 \sup_{t\in\mathcal{J}} \int_0^t |\wp(\xi, \Omega(\xi)) - \wp(\xi, \bar{\Omega}(\xi))|d\xi \\ &\leq [\nabla_1 + \nabla_2\theta]\mathbf{L}_\wp \|\Omega - \bar{\Omega}\|. \end{aligned} \tag{9}$$

This shows that $\mathbf{P}$ is a condensing map. From which we conclude that the concerned problem (2) has a unique solution, hence the proposed model (2) has a unique solution.

4. Algorithm for approximate solution of fractional order model (2)

We use Laplace transform on both sides of the system (2) for computing the algorithm for the required approximate solution of the proposed model. Further for simplicity take $\mathscr{M}(\sigma) = 1$ and $\Xi(\sigma, s) = \frac{s+\sigma(1-s)}{s}$, we have

$$\begin{cases} \mathscr{L}[\mathcal{Y}_1(t)] = \dfrac{\mathcal{Y}_1(0)}{s} + \Xi(\sigma, s)\mathscr{L}[C_{y_1} + \nu_1\mathbb{R} - (\alpha + n_1)\mathcal{Y}_1], \\ \mathscr{L}[\mathcal{Y}_2(t)] = \dfrac{\mathcal{Y}_2(0)}{s} + \Xi(\sigma, s)\mathscr{L}[C_{y_2} + \nu_2\mathbb{R} + \alpha\mathcal{Y}_1 - (\beta + n_2)\mathcal{Y}_2], \\ \mathscr{L}[\mathcal{Y}_3(t)] = \dfrac{\mathcal{Y}_3(0)}{s} + \Xi(\sigma, s)\mathscr{L}[C_{y_3} + \beta\mathcal{Y}_2 + \nu_3\mathbb{R} - (p_3 + \mu_3 + \gamma + n_3)\mathcal{Y}_3], \\ \mathscr{L}[\mathcal{Y}_4(t)] = \dfrac{\mathcal{Y}_4(0)}{s} + \Xi(\sigma, s)\mathscr{L}[C_{y_4} + \gamma\mathcal{Y}_3 + \nu_4\mathbb{R} - (p_4 + \mu_4 + n_4)\mathcal{Y}_4], \\ \mathscr{L}[\mathbb{R}(t)] = \dfrac{\mathbb{R}(0)}{s} + \Xi(\sigma, s)\mathscr{L}[n_1\mathcal{Y}_1 + n_2\mathcal{Y}_2 + n_3\mathcal{Y}_3 + n_4\mathcal{Y}_4 - (p_1 + \nu_1 + \nu_2 + \nu_3 + \nu_4)\mathbb{R}], \\ \mathscr{L}[\mathcal{C}(t)] = \dfrac{\mathcal{C}(0)}{s} + \Xi(\sigma, s)\mathscr{L}[p_1\mathbb{R} + p_3\mathcal{Y}_3 + p_4\mathcal{Y}_4 - \mu_c\mathcal{C}]. \end{cases} \tag{10}$$

The required solution can be obtained in the form as

$$\begin{aligned}
\mathcal{Y}_1(t) &= \sum_{r=0}^{\infty} \mathcal{Y}_{1,q}(t),\ \mathcal{Y}_2(t) = \sum_{r=0}^{\infty} \mathcal{Y}_{2,q}(t),\\
\mathcal{Y}_3(t) &= \sum_{r=0}^{\infty} \mathcal{Y}_{3,q}(t),\ \mathcal{Y}_4(t) = \sum_{r=0}^{\infty} \mathcal{Y}_{4,q}(t),\\
\mathbb{R}(t) &= \sum_{r=0}^{\infty} \mathbb{R}_q(t),\ \mathcal{C}(t) = \sum_{r=0}^{\infty} \mathcal{C}_q(t),
\end{aligned} \tag{11}$$

Hence, using (11), the system (10) becomes

$$\begin{cases}
\mathscr{L}\left[\sum_{r=0}^{\infty} \mathcal{Y}_{1,q}(t)\right] = \frac{\mathcal{Y}_1(0)}{s} + \Xi(\sigma,s)\mathscr{L}\left[C_{y_1} + \nu_1 \sum_{r=0}^{\infty} \mathbb{R} - (\alpha + n_1) \sum_{r=0}^{\infty} \mathcal{Y}_{1,q}\right],\\
\mathscr{L}\left[\sum_{r=0}^{\infty} \mathcal{Y}_{2,q}(t)\right] = \frac{\mathcal{Y}_2(0)}{s} + \Xi(\sigma,s)\mathscr{L}\left[C_{y_2} + \nu_2 \sum_{r=0}^{\infty} \mathbb{R} + \alpha \sum_{r=0}^{\infty} \mathcal{Y}_{1,q} - (\beta + n_2) \sum_{r=0}^{\infty} \mathcal{Y}_{2,q}\right],\\
\mathscr{L}\left[\sum_{r=0}^{\infty} \mathcal{Y}_{3,q}(t)\right] = \frac{\mathcal{Y}_3(0)}{s} + \Xi(\sigma,s)\mathscr{L}\left[C_{y_3} + \beta \sum_{r=0}^{\infty} \mathcal{Y}_{2,q} + \nu_3 \sum_{r=0}^{\infty} \mathbb{R} - (p_3 + \mu_3 + \gamma + n_3) \sum_{r=0}^{\infty} \mathcal{Y}_{3,q}\right],\\
\mathscr{L}\left[\sum_{r=0}^{\infty} \mathcal{Y}_{4,q}(t)\right] = \frac{\mathcal{Y}_4(0)}{s} + \Xi(\sigma,s)\mathscr{L}\left[C_{y_4} + \gamma \sum_{r=0}^{\infty} \mathcal{Y}_{3,q} + \nu_4 \sum_{r=0}^{\infty} \mathbb{R} - (p_4 + \mu_4 + n_4) \sum_{r=0}^{\infty} \mathcal{Y}_{4,q}\right],\\
\mathscr{L}\left[\sum_{r=0}^{\infty} \mathbb{R}_q(t)\right] = \frac{\mathbb{R}_q(0)}{s} + \Xi(\sigma,s)\mathscr{L}\Big[n_1 \sum_{r=0}^{\infty} \mathcal{Y}_{1,q} + n_2 \sum_{r=0}^{\infty} \mathcal{Y}_{2,q} + n_3 \sum_{r=0}^{\infty} \mathcal{Y}_{3,q} + n_4 \sum_{r=0}^{\infty} \mathcal{Y}_{4,q}\\
- (p_1 + \nu_1 + \nu_2 + \nu_3 + \nu_4) \sum_{r=0}^{\infty} \mathbb{R}\Big],\\
\mathscr{L}\left[\sum_{r=0}^{\infty} \mathcal{C}_q(t)\right] = \frac{\mathcal{C}(0)}{s} + \Xi(\sigma,s)\mathscr{L}\left[p_1 \sum_{r=0}^{\infty} \mathbb{R} + p_3 \sum_{r=0}^{\infty} \mathcal{Y}_{3,q} + p_4 \sum_{r=0}^{\infty} \mathcal{Y}_{4,q} - \mu_c \sum_{r=0}^{\infty} \mathcal{C}_q\right].
\end{cases} \tag{12}$$

From (12), we equate terms as

$$\begin{cases}
\mathscr{L}[\mathcal{Y}_{1,0}(t)] = \frac{\mathcal{Y}_{1_0}}{s} + \Xi(\sigma,s)\mathscr{L}(C_{y_1}),\\
\mathscr{L}[\mathcal{Y}_{2,0}(t)] = \frac{\mathcal{Y}_{2_0}}{s} + \Xi(\sigma,s)\mathscr{L}(C_{y_2}),\\
\mathscr{L}[\mathcal{Y}_{3,0}(t)] = \frac{\mathcal{Y}_{3_0}}{s} + \Xi(\sigma,s)\mathscr{L}(C_{y_3}),\\
\mathscr{L}[\mathcal{Y}_{4,0}(t)] = \frac{\mathcal{Y}_{4_0}}{s} + \Xi(\sigma,s)\mathscr{L}(C_{y_4}),\\
\mathscr{L}[\mathbb{R}_0(t)] = \frac{\mathbb{R}_0}{s},\\
\mathscr{L}[\mathcal{C}_0(t)] = \frac{\mathcal{C}_0}{s},
\end{cases} \tag{13}$$

$$\begin{cases} \mathscr{L}[\mathcal{Y}_{1,1}(t)] = \Xi(\sigma,s)\mathscr{L}\Big[\nu_1\mathbb{R}_0 - (\alpha+n_1)\mathcal{Y}_{1,0}\Big], \\ \mathscr{L}[\mathcal{Y}_{2,1}(t)] = \Xi(\sigma,s)\mathscr{L}\Big[\nu_2\mathbb{R}_0 + \alpha\mathcal{Y}_{1,0} - (\beta+n_2)\mathcal{Y}_{2,0}\Big], \\ \mathscr{L}[\mathcal{Y}_{3,1}(t)] = \Xi(\sigma,s)\mathscr{L}\Big[\beta\mathcal{Y}_{2,0} + \nu_3\mathbb{R}_0 - (p_3+\mu_3+\gamma+n_3)\mathcal{Y}_{3,0}\Big], \\ \mathscr{L}[\mathcal{Y}_{4,1}(t)] = \Xi(\sigma,s)\mathscr{L}\Big[\gamma\mathcal{Y}_{3,0} + \nu_4\mathbb{R}_0 - (p_4+\mu_4+n_4)\mathcal{Y}_{4,0}\Big], \\ \mathscr{L}[\mathbb{R}_1(t)] = \Xi(\sigma,s)\mathscr{L}\Big[n_1\mathcal{Y}_{1,0} + n_2\mathcal{Y}_{2,0} + n_3\mathcal{Y}_{3,0} + n_4\mathcal{Y}_{4,0} - (p_1+\nu_1+\nu_2+\nu_3+\nu_4)\mathbb{R}_0\Big], \\ \mathscr{L}[\mathcal{C}_1(t)] = \Xi(\sigma,s)\mathscr{L}\Big[p_1\mathbb{R}_0 + p_3\mathcal{Y}_{3,0} + p_4\mathcal{Y}_{4,0} - \mu_c\mathcal{C}_0\Big] \end{cases} \tag{14}$$

and

$$\begin{cases} \mathscr{L}[\mathcal{Y}_{1,2}(t)] = \Xi(\sigma,s)\mathscr{L}\Big[\nu_1\mathbb{R}_1 - (\alpha+n_1)\mathcal{Y}_{1,1}\Big], \\ \mathscr{L}[\mathcal{Y}_{2,2}(t)] = \Xi(\sigma,s)\mathscr{L}\Big(\nu_2\mathbb{R}_1 + \alpha\mathcal{Y}_{1,1} - (\beta+n_2)\mathcal{Y}_{2_1}\Big], \\ \mathscr{L}[\mathcal{Y}_{3,2}(t)] = \Xi(\sigma,s)\mathscr{L}\Big[\beta\mathcal{Y}_{2,1} + \nu_3\mathbb{R}_1 - (p_3+\mu_3+\gamma+n_3)\mathcal{Y}_{3,1}\Big], \\ \mathscr{L}[\mathcal{Y}_{4,2}(t)] = \Xi(\sigma,s)\mathscr{L}\Big[\gamma\mathcal{Y}_{3,1} + \nu_4\mathbb{R}_1 - (p_4+\mu_4+n_4)\mathcal{Y}_{4,1}\Big], \\ \mathscr{L}[\mathbb{R}_2(t)] = \Xi(\sigma,s)\mathscr{L}\Big[n_1\mathcal{Y}_{1,1} + n_2\mathcal{Y}_{2,1} + n_3\mathcal{Y}_{3,1} + n_4\mathcal{Y}_{4,1} - (p_1+\nu_1+\nu_2+\nu_3+\nu_4)\mathbb{R}_1\Big], \\ \mathscr{L}[\mathcal{C}_2(t)] = \Xi(\sigma,s)\mathscr{L}\Big[p_1\mathbb{R}_1 + p_3\mathcal{Y}_{3,1} + p_4\mathcal{Y}_{4,1} - \mu_c\mathcal{C}_1\Big] \end{cases} \tag{15}$$

and so on. In general we can write a recursive result as follows:

$$\begin{cases} \mathscr{L}[\mathcal{Y}_{1,q+1}(t)] = \Xi(\sigma,s)\mathscr{L}\Big[\nu_1\mathbb{R}_q - (\alpha+n_1)\mathcal{Y}_{1,q}\Big], \\ \mathscr{L}[\mathcal{Y}_{2,q+1}(t)] = \Xi(\sigma,s)\mathscr{L}\Big(\nu_2\mathbb{R}_q + \alpha\mathcal{Y}_{1,q} - (\beta+n_2)\mathcal{Y}_{2,q}\Big], \\ \mathscr{L}[\mathcal{Y}_{3,q+1}(t)] = \Xi(\sigma,s)\mathscr{L}\Big[\beta\mathcal{Y}_{2,q} + \nu_3\mathbb{R}_q - (p_3+\mu_3+\gamma+n_3)\mathcal{Y}_{3,q}\Big], \\ \mathscr{L}[\mathcal{Y}_{4,q+1}(t)] = \Xi(\sigma,s)\mathscr{L}\Big[\gamma\mathcal{Y}_{3,q} + \nu_4\mathbb{R}_q - (p_4+\mu_4+n_4)\mathcal{Y}_{4,q}\Big], \\ \mathscr{L}[\mathbb{R}_{q+1}(t)] = \Xi(\sigma,s)\mathscr{L}\Big[n_1\mathcal{Y}_{1,q} + n_2\mathcal{Y}_{2,q} + n_3\mathcal{Y}_{3,q} + n_4\mathcal{Y}_{4,q} - (p_1+\nu_1+\nu_2+\nu_3+\nu_4)\mathbb{R}_q\Big], \\ \mathscr{L}[\mathcal{C}_{q+1}(t)] = \Xi(\sigma,s)\mathscr{L}\Big[p_1\mathbb{R}_q + p_3\mathcal{Y}_{3,q} + p_4\mathcal{Y}_{4,q} - \mu_c\mathcal{C}_q\Big]. \end{cases} \tag{16}$$

After simplifying (13) and (14), we get

$$\begin{cases} \mathcal{Y}_{1,0}(t) = \mathcal{Y}_1(0) + \Big(1+(t-1)\sigma\Big)C_{y_1}, \\ \mathcal{Y}_{2,0}(t) = \mathcal{Y}_2(0) + \Big(1+(t-1)\sigma\Big)C_{y_2}, \\ \mathcal{Y}_{3,0}(t) = \mathcal{Y}_3(0) + \Big(1+(t-1)\sigma\Big)C_{y_3}, \\ \mathcal{Y}_{4,0}(t) = \mathcal{Y}_4(0) + \Big(1+(t-1)\sigma\Big)C_{y_4}, \\ \mathbb{R}_0(t) = \mathbb{R}(0),\ \mathcal{C}_0(t) = \mathcal{C}(0). \end{cases} \tag{17}$$

Other terms can be calculated similarly. The required solutions will be given as follows:

$$\begin{cases} \mathcal{Y}_1(t) = \mathcal{Y}_{1,0} + \mathcal{Y}_{1,1}(t) + \mathcal{Y}_{1,2}(t) + \mathcal{Y}_{1,3}(t) + \ldots, \\ \mathcal{Y}_2(t) = \mathcal{Y}_{2,0} + \mathcal{Y}_{2,1}(t) + \mathcal{Y}_{2,2}(t) + \mathcal{Y}_{2,3}(t) + \ldots, \\ \mathcal{Y}_3(t) = \mathcal{Y}_{3,0} + \mathcal{Y}_{3,1}(t) + \mathcal{Y}_{3,2}(t) + \mathcal{Y}_{3,3}(t) + \ldots, \\ \mathcal{Y}_4(t) = \mathcal{Y}_{4,0} + \mathcal{Y}_{4,1}(t) + \mathcal{Y}_{4,2}(t) + \mathcal{Y}_{4,3}(t) + \ldots, \\ \mathcal{R}(t) = \mathcal{R}_0 + \mathcal{R}_1(t) + \mathcal{R}_2(t) + \mathcal{R}_3(t) + \ldots, \\ \mathcal{C}(t) = \mathcal{C}_0 + \mathcal{C}_1(t) + \mathcal{C}_2(t) + \mathcal{C}_3(t) + \ldots. \end{cases} \tag{18}$$

4.1. Results and discussion

Here, we use Matlab to plot the solutions of (18) for few terms. The numerical values for used parameters are given in Table 2. The solutions for various fractional orders are displayed in Figures 1–6.

Variable	numerical values	Variable	numerical
$\mathcal{Y}_1$	20	β	0.056
$\mathcal{Y}_2$	14	ν_1	0.024
$\mathcal{Y}_3$	30	ν_2	0.030
$\mathcal{Y}_4$	20	ν_3	0.036
$\mathbb{R}$	10	γ	0.042
$\mathcal{C}$	10	ν_4	0.062
C_{y_1}	15%	p_1	0.030
C_{y_2}	39%	p_3	0.030
C_{y_3}	43%	p_4	0.030
C_{y_4}	12%	μ_3	0.05
α	0.046	μ_4	0.08
n_1	0.060	μ_c	0.04
n_2	0.063	n_3	0.035
n_4	0.010		

Table 2: The model parameters are listed and given numerical values [19].

If we set values from Table 2 in model (2), we have

$$\begin{cases} {}^{CF}D_t^{\sigma}\mathcal{Y}_1(t) = 0.15 + 0.024\mathbb{R} - 0.106\mathcal{Y}_1, \\ {}^{CF}D_t^{\sigma}\mathcal{Y}_2(t) = 0.39 + 0.030\mathbb{R} + 0.046\mathcal{Y}_1 - 0.119\mathcal{Y}_2, \\ {}^{CF}D_t^{\sigma}\mathcal{Y}_3(t) = 0.43 + 0.056\mathcal{Y}_2 + 0.036\mathbb{R} - 0.107\mathcal{Y}_3, \\ {}^{CF}D_t^{\sigma}\mathcal{Y}_4(t) = 0.12 + 0.042\mathcal{Y}_3 + 0.062\mathbb{R} - 0.12\mathcal{Y}_4, \\ {}^{CF}D_t^{\sigma}\mathbb{R}(t) = 0.060\mathcal{Y}_1 + 0.063\mathcal{Y}_2 + 0.035\mathcal{Y}_3 + 0.010\mathcal{Y}_4 - 0.146\mathbb{R}, \\ {}^{CF}D_t^{\sigma}\mathcal{C}(t) = 0.030\mathbb{R} + 0.030\mathcal{Y}_3 + 0.030\mathcal{Y}_4 - 0.040\mathcal{C}, \\ \mathcal{Y}_1(0) = 20,\ \mathcal{Y}_2(0) = 14,\ \mathcal{Y}_3(0) = 30,\ \mathcal{Y}_4(0) = 20,\ \mathbb{R}(0) = 10,\ \mathcal{C}(0) = 10. \end{cases} \tag{19}$$

Therefore,(17) implies that

$$\begin{cases} \mathcal{Y}_{1,0}(t) = 20 + \Big(1 + (t-1)\sigma\Big)0.15, \\ \mathcal{Y}_{2,0}(t) = 14 + \Big(1 + (t-1)\sigma\Big)0.39, \\ \mathcal{Y}_{3,0}(t) = 30 + \Big(1 + (t-1)\sigma\Big)0.43, \\ \mathcal{Y}_{4,0}(t) = 20 + \Big(1 + (t-1)\sigma\Big)0.12, \\ \mathbb{R}_0(t) = 10,\ \mathcal{C}_0(t) = 10. \end{cases} \tag{20}$$

Repeating the same process, we can compute other terms. Now we, present some graphical illustrations for model (19) in subsequent figures 1-6 for different fractional orders. Here, we use first five terms solutions of the proposed model.

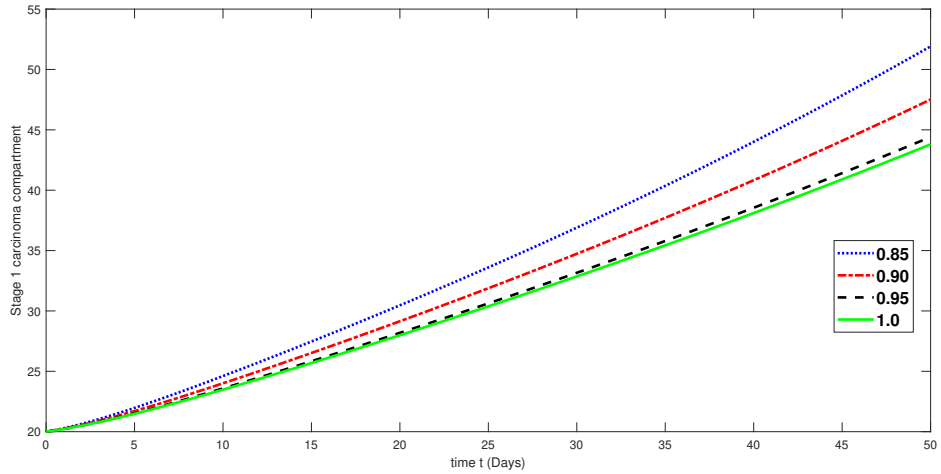

Figure 1: Graphical presentation of Stage 1 carcinoma compartment for different fractional order.

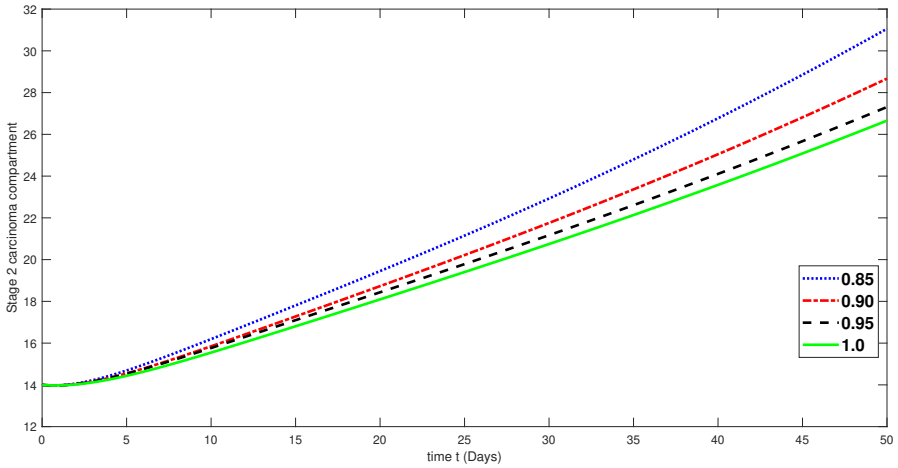

Figure 2: Graphical presentation of Stage 2 carcinoma compartment for different fractional order.

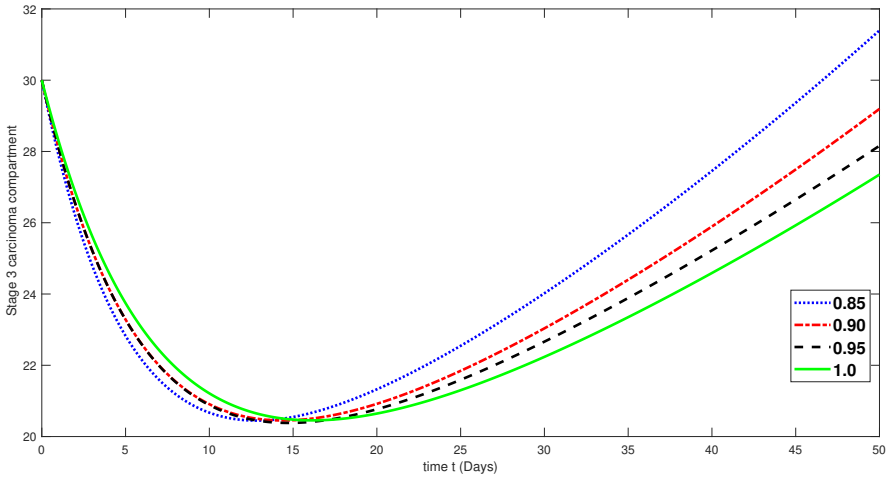

Figure 3: Graphical presentation of Stage 3 carcinoma compartment for different fractional order.

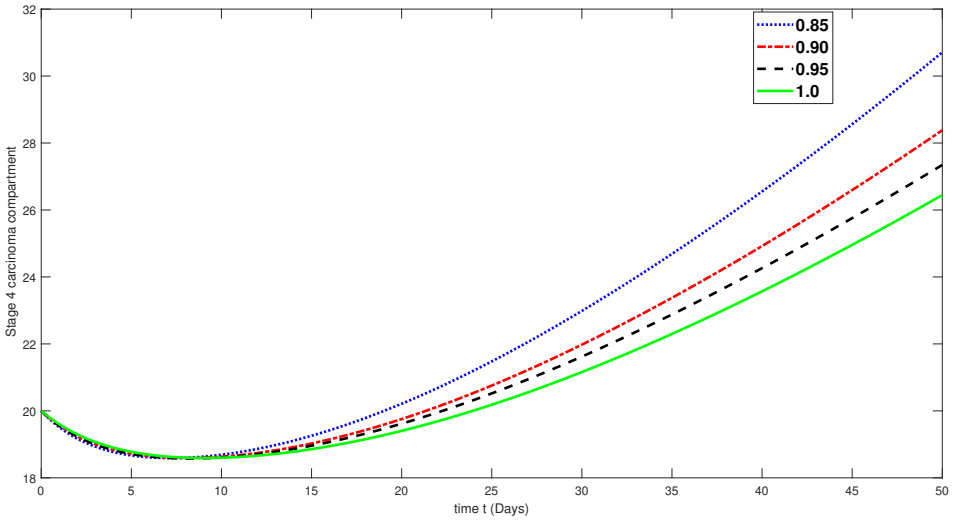

Figure 4: Graphical presentation of Stage 4 carcinoma compartment for different fractional order.

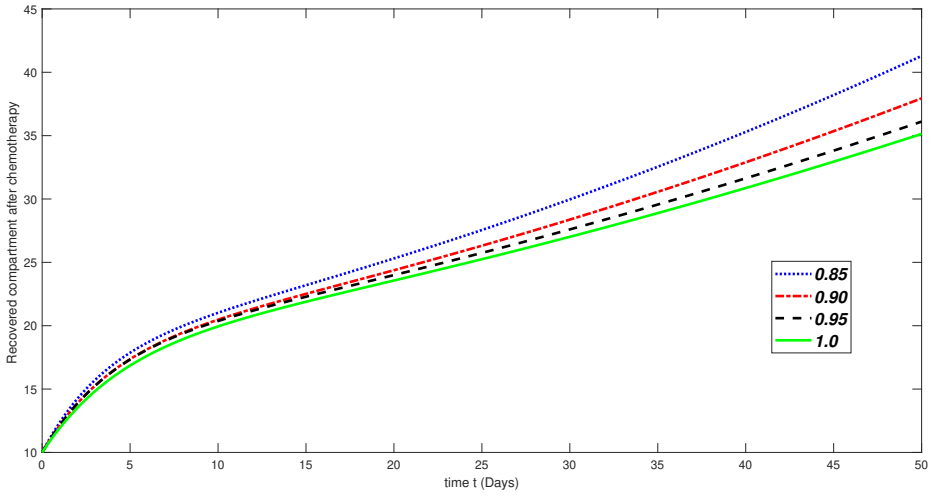

Figure 5: Graphical presentation of recovered compartment after chemotherapy for different fractional order.

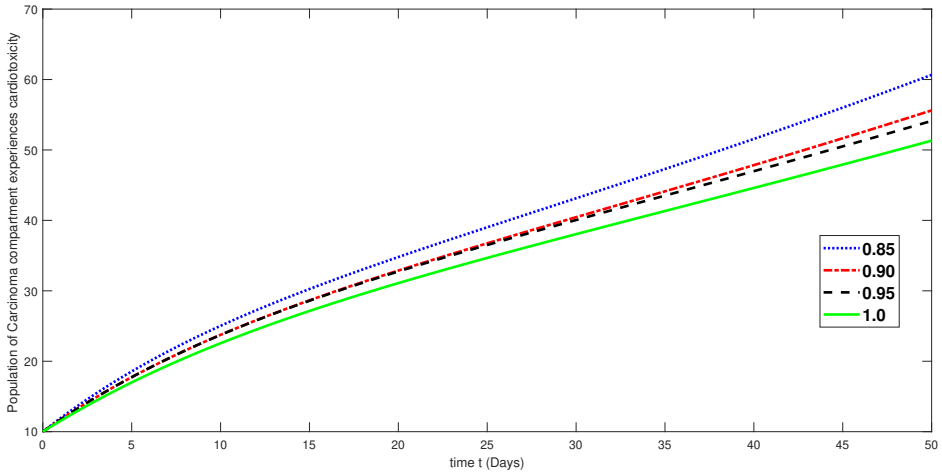

Figure 6: Graphical presentation of population of Carcinoma compartment experiences cardiotoxicity for different fractional order.

We have presented the analytical results for different fractional orders with taking first few terms solutions for the proposed model. We see that as the fractional order $\sigma \to 1$, the approximate solution tends to close to the solution at order 1.

5. Conclusions

Some analytical and theoretical results were investigated for breast cancer mathematical model with multi-stages. As breast cancer is a world wide serious health issue in women mostly. Also, in rare cases in men can be occurred. Therefore, we have considered a linear multi-stage cancer model under fractional order derivative. We have showed the existence theory of solution by using fixed point approaches. Also, with the help of Laplace transform a methodology was created to generate series type solution. Based on the used methodology, series type solutions corresponding to various compartment were produced. We have presented graphical illustrations of the considered model with respect to various fractional orders. Here, we have used the CFDO which is a nonlocal derivative of fractional order. In the future, the proposed method can be extended to other dynamical systems.

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