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Convergence Analysis of the HK-Iteration for Garcia–Falset Mappings in Banach Spaces with Application to Fractional Boundary Value Problems

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Abstract. In this paper, a novel three-step iterative scheme, referred to as the HK-iteration, is introduced and analyzed for the approximation of fixed points of mappings that obey condition (E) within Banach spaces. Theoretical outcomes ensuring both weak and strong convergence of the proposed scheme are established, accompanied by a detailed example of a mapping satisfying condition (E). In addition, numerical experiments reveal that the HK-iteration attains a faster convergence rate and enhanced stability when compared with several existing iterative methods. To further illustrate its practical significance, the HK-iteration is applied to a class of fractional boundary value problems characterized by Caputo-type derivatives. The overall analysis demonstrates that the HK-iteration not only generalizes and connects many previously known iterative processes but also provides improved convergence efficiency and computational robustness.

2020 Mathematics Subject Classifications: 47H09, 47H10, 34B15

Key Words and Phrases: Banach spaces, fixed point approximation, HK-iteration, Garcia-Falset mappings, convergence analysis, fractional boundary value problems

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1. Introduction

Functional analysis is a subfield of mathematics that studies the topic of distance on vector spaces. or size, with particular care as regards spaces which are complete in this measure. It began with the study of the linear operators and infinite-dimensional spaces, as an extension of the concepts of the calculus and linear algebra. Functional analysis takes the classical analysis a step further by addressing infinite dimensions, such as, optimization of mapping, differential equations and analysis of several different convergences. It is a highly related field to the other branches of mathematics, including probability, measure theory and topology. Banach spaces provide a natural context for examining continuous linear operators and their attributes in the framework of functional analysis. Banach spaces have a wide range of applications. They aid in the creation and resolution of issues pertaining to infinite-dimensional spaces in optimization. The completeness of Banach spaces helps differential equations by enabling stability analysis and well-defined solutions. Banach spaces are used in approximation theory to investigate and create function approximation techniques. Furthermore, Banach spaces are essential to probability theory because they offer a framework for examining random variables and stochastic processes. The interactions between Banach space methods and integral inequalities have been the subject of interest over the last few years, as most classical inequalities including the HermiteHadamard, Fejer, and Pachpatte type inequalities can be studied in the context of Banach spaces and used on issues of fractional operators, integral equations, and nonlinear mappings. In general, the study of Banach spaces offers crucial resources for applied and theoretical mathematics, providing understanding of both practical and abstract issues in a range of scientific fields. For the literature see the references [1–8].

The foundation of modern mathematics, fixed point theory, dates back to the early 1900s and the groundbreaking work of mathematicians such as Stefan Banach and Brouwer. One important theorem in this area is the Banach Contraction Principle. The Banach contraction principle gave rise to fixed point theory, which has long been the subject of research. Its use primarily depends on the existence of solutions to mathematical issues derived from engineering and economics. Considered spaces defined by the intuitive axioms are crucial for fixed points of function. Specifically, different metrics spaces are suggested, such as probabilistic, and partial metric spaces. Different kinds of fixed point theorems will arise in different spaces. Stated differently, the literature contains a wide variety of fixed point theorems. Fixed point theory continues to be a dynamic and active field that links abstract mathematics to practical issues through continued research and applications in fields like data science and machine learning.

Numerous researchers from a wide range of mathematical and applied science fields have been drawn to the fractional calculus. Thus far, several fractional or non-local operators have been presented. For the literature see the references [9–13].

This paper will be structured in the following way: In Section 2 and 3, give a few related theorems and concepts and comments, which are crucial to subsequent sections. In Section 4 we give various new convergence properties in Banach spaces through the use of HK-iteration. The summary of new concept is dealt with in Section 5. In Section, we explore the numerical example and the performance comparison of the HK-iterative scheme in Section 6. The application with respect to the recently developed HK-iteration method to the fractional boundary value problems is added in Section 7. We provide a conclusion in Section 8. Out of interest of the readers we include the future scope in section 9.

2. Review of Related Literature

Let $\mathcal{X}$ be a Banach space and let $\emptyset \neq \mathcal{C} \subseteq \mathcal{X}$, with a mapping $T : \mathcal{C} \to \mathcal{C}$. Suppose the point $p \in \mathcal{C}$ is said to be fixed point of T if $Tp = p$. $Fix(T)$ represent the set of all fixed points of T. Throughout this work, $\mathbb{N}$ denotes the set of all natural numbers. If T is a nonexpansive mapping, that is, for any $u, u' \in \mathcal{C}$,

$$||Tu - Tu'|| \leq ||u - u'||,$$

then it is well known that $Fix(T)$ is nonempty whenever $\mathcal{X}$ is uniformly convex and $\mathcal{C}$ is a closed, convex, and bounded subset of $\mathcal{X}$ (see [14–16] and the references therein).

In 2008, Suzuki introduced a wider category of nonlinear operators that generalizes the concept of nonexpansive mappings. A mapping $T : \mathcal{C} \to \mathcal{C}$ satisfies condition (C) (commonly called a Suzuki mapping) if, for every $u, u' \in \mathcal{C}$,

$$\frac{1}{2}||u - Tu'|| \leq ||u - u'|| \text{ implies that } ||Tu - Tu'|| \leq ||u - u'||.$$

Garcia-Falset et al. [17] extended the above condition (C), which is defined as follows:

A function $T : \mathcal{C} \to \mathcal{C}$ is called satisfy condition (E_μ) if there $\exists$ a constant $\mu \geq 1$ such that

$$||u - Tu'|| \leq \mu||u - Tu|| + ||u - u'|| \quad \text{for all } u, u' \in \mathcal{C}.$$

Furthermore, a mapping T is said to satisfy condition (E) (or equivalently, to be a Garcia-Falset mapping) whenever it fulfills condition (E_μ) for some $\mu \geq 1$.

It was shown by Garcia-Falset et al. [17] that each Suzuki mapping satisfies condition (E) with $\mu = 3$. Moreover, the class of Garcia-Falset mappings includes several well-known types of generalized nonexpansive mappings (see [18] for further details).

The study of iterative methods for approximating fixed points of nonlinear mappings has become an active and widely investigated research topic (see, e.g., [19–23] and the references therein). The well-known Banach contraction principle asserts that the Picard iteration provides a unique fixed point for a contraction operator. However, the Picard process may fail to converge in the case of nonexpansive mappings. To address this limitation and to improve the rate of convergence, various modified iterative schemes have been proposed in recent years.

Let $\mathcal{C}$ be a nonempty, convex subset of a Banach space $\mathcal{X}$, and let $T : \mathcal{C} \to \mathcal{C}$ be a mapping. Suppose that $a_n, b_n, c_n \in (0, 1)$ for all $n \in \mathbb{N}$. Then, several classical iteration schemes Picard [24], Mann [25], Ishikawa [26], Agarwal [27], Abbas [28], Thakur [29], Noor [30], Picard-Noor [31], and S_n [32] are respectively defined as follows:

$$\begin{cases} u_1 = u \in \mathcal{C}, \\ u_{n+1} = Tu_n. \end{cases} \tag{2.1}$$

$$\begin{cases} u_1 = u \in \mathcal{C}, \\ u_{n+1} = (1 - a_n)u_n + a_n Tu_n. \end{cases} \tag{2.2}$$

$$\begin{cases} u_1 \in \mathcal{C}, \\ w_n = (1 - b_n)u_n + b_n Tu_n, \\ u_{n+1} = (1 - a_n)u_n + a_n Tw_n. \end{cases} \tag{2.3}$$

$$\begin{cases} u_1 = u \in \mathcal{C}, \\ w_n = (1 - b_n)u_n + b_n Tu_n, \\ u_{n+1} = (1 - a_n)Tu_n + a_n Tw_n. \end{cases} \tag{2.4}$$

$$\begin{cases} u_1 = u \in \mathcal{C}, \\ v_n = (1 - c_n)u_n + c_n Tu_n, \\ w_n = (1 - b_n)Tu_n + b_n Tv_n, \\ u_{n+1} = (1 - a_n)Tw_n + a_n Tv_n. \end{cases} \tag{2.5}$$

$$\begin{cases} u_1 = u \in \mathcal{C}, \\ v_n = (1 - b_n)u_n + b_n Tu_n, \\ w_n = T\left((1 - a_n)u_n + a_n v_n\right), \\ u_{n+1} = Tw_n. \end{cases} \tag{2.6}$$

In 2000, Noor [30] introduced a three-step iterative process defined by:

$$\left.\begin{array}{l} u_1 = u \in \mathcal{C}, \\ w_n = (1-c_n)u_n + c_n\mathcal{T}u_n, \\ v_n = (1-b_n)u_n + b_n\mathcal{T}w_n, \\ u_{n+1} = (1-a_n)u_n + a_n\mathcal{T}v_n. \end{array}\right\} \tag{2.7}$$

Subsequently, in 2019, Kanayo Stella and Hudson [31] proposed a four-step iterative process, referred to as the Picard-Noor iteration, which is formulated as follows:

$$\left.\begin{array}{l} u_1 = u \in \mathcal{C}, \\ w_n = (1-c_n)u_n + c_n\mathcal{T}u_n, \\ v_n = (1-b_n)u_n + b_n\mathcal{T}w_n, \\ u_n = (1-a_n)u_n + a_n\mathcal{T}v_n, \\ u_{n+1} = \mathcal{T}u_n. \end{array}\right\} \tag{2.8}$$

In 2016, W. Sintunavarat and A. Pitea [32] developed an iterative method known as the S_n iteration, which is defined by:

$$\left.\begin{array}{l} u_1 = u \in \mathcal{C}, \\ w_n = (1-c_n)u_n + c_n\mathcal{T}u_n, \\ v_n = (1-b_n)u_n + b_n w_n, \\ u_{n+1} = (1-a_n)\mathcal{T}v_n + a_n\mathcal{T}w_n. \end{array}\right\} \tag{2.9}$$

According to Agarwal et al. [27], the iterative process (2.4) demonstrates a higher rate of convergence than the methods (2.1), (2.2), and (2.3) when applied to contraction mappings. Similarly, Thakur et al. [29], through numerical analysis, verified that the scheme (2.6) converges more efficiently than all iterative methods listed from (2.1)–(2.5) in the framework of Suzuki mappings. Furthermore, Ullah and Arshad [19], using a numerical illustration, confirmed that their proposed iterative technique outperforms the established scheme (2.6) for Suzuki-type generalized mappings in Banach spaces.

An iterative process, known as the Picard-S_n iteration, is described as follows:

$$\left.\begin{array}{l} u_1 = u \in \mathcal{C}, \\ w_n = (1-c_n)u_n + c_nTu_n, \\ v_n = (1-b_n)u_n + b_n w_n, \\ u_n = (1-a_n)Tv_n + a_nTw_n, \\ u_{n+1} = Tu_n. \end{array}\right\} \tag{2.10}$$

Inspired by these advancements, we introduce a new three-step iterative algorithm, termed the **HK-Iteration**, defined as:

$$\left\{\begin{array}{l} \quad u_i = (1-\zeta_i)w_i + \zeta_i\mathcal{T}w_i, \\ \quad v_i = (1-\kappa_i)\mathcal{T}w_i + \kappa_i\mathcal{T}^2u_i, \\ w_{i+1} = \mathcal{T}v_i, \end{array}\right. \tag{2.11}$$

where the control sequences $\{\zeta_i\}$ and $\{\kappa_i\}$ are real-valued and satisfy $0 \le \zeta_i, \kappa_i \le 1$ for all $i \in \mathbb{N}$. This newly proposed process extends and unifies several existing iterations while exhibiting a superior rate of convergence for mappings that fulfill condition (E) in Banach spaces.

3. Preliminaries

Suppose $\mathcal{X}$ be a Banach space, $\mathcal{C} \subset \mathcal{X}$, and $\{u_n\}$ denote a bounded sequence. For any $u \in \mathcal{X}$, define

$$r(u, \{u_n\}) = \limsup_{n\to\infty} \|u - u_n\|.$$

The asymptotic radius of $\{u_n\}$ with respect to $\mathcal{C}$ is expressed as

$$r(\mathcal{C},\{u_n\}) = \inf\{r(u,\{u_n\}) : u \in \mathcal{C}\}.$$

The asymptotic center of $\{u_n\}$ relative to $\mathcal{C}$ is defined by

$$A(\mathcal{C},\{u_n\}) = \{u \in \mathcal{C} : r(u,\{u_n\}) = r(\mathcal{C},\{u_n\})\}.$$

If $\mathcal{X}$ is a uniformly convex space [33], then $A(\mathcal{C},\{u_n\})$ contains a unique element. Furthermore, the set $A(\mathcal{C},\{u_n\})$ is convex and nonempty whenever $\mathcal{C}$ is weakly compact and convex (see, for example, [34, 35]).

A Banach space $\mathcal{X}$ is said to possess the Opial property [36] if and only if, for every sequence $\{u_n\}$ in $\mathcal{C}$ that converges weakly to some $u \in \mathcal{X}$ and for each $w \in \mathcal{X}\setminus\{u\}$, the following inequality holds:

$$\limsup_{n\to\infty} \|u_n - u\| < \limsup_{n\to\infty} \|u_n - w\|.$$

The next lemma provides several examples of Garcia-Falset mappings.

Lemma 1. *[17] Suppose a mapping T is defined on $\mathcal{C} \subset \mathcal{X}$. With $\mu = 3$, T satisfies condition (E) if it satisfies condition (C).*

Lemma 2. *[17] Suppose a mapping T is defined on $\mathcal{C} \subset \mathcal{X}$. If T obeys condition (E), then for each $p \in Fix(T)$ and every $u \in \mathcal{C}$, we have $||Tu - p|| \le ||u - p||$.*

Lemma 3. *[17] Let T be a mapping on a subset $\mathcal{C}$ of a Banach space $\mathcal{X}$ having the Opial property. Suppose that T satisfies condition (E). If $\{u_n\}$ converges weakly to q and $\lim_{n\to\infty} ||u_n - Tu_n|| = 0$, then $q \in Fix(T)$.*

Schu [37] established the following key result in 1991.

Lemma 4. *Suppose $\mathcal{X}$ be a uniformly convex Banach space (UCBS), and assume that $0 < u \le \theta_n \le v < 1$ for all $n \in \mathbb{N}$. If $\{u_n\}$ and $\{w_n\}$ are two sequences in $\mathcal{X}$ satisfying $\limsup_{n\to\infty} ||u_n|| \le \lambda$, $\limsup_{n\to\infty} ||w_n|| \le \lambda$, and $\lim_{n\to\infty} ||\theta_n u_n + (1-\theta_n)w_n|| = \lambda$ for some $\lambda \ge 0$, then $\lim_{n\to\infty} ||u_n - w_n|| = 0$.*

The above notions serve as fundamental tools in the analysis of the convergence behavior of iterative schemes. Specifically, for the proposed **HK-iteration** (2.11), these lemmas will be employed repeatedly to establish both weak and strong convergence outcomes. Hence, our findings generalize the theoretical foundation of Picard-type and S_n-type iterative processes to a broader framework represented by the HK-iteration.

4. Main Results

This section is dedicated to establishing and proving the principal convergence results of this study. We start with the following essential lemma.

Lemma 5. *Suppose $\mathcal{C} \subset \mathcal{X}$ (closed and convex subset), and $T : \mathcal{C} \to \mathcal{C}$ be a mapping that satisfies condition (E) with $Fix(T) \neq \emptyset$. If the sequence $\{w_i\}$ is generated by the HK-iteration (2.11), then $\lim_{i\to\infty} \|w_i - p\|$ exists for every $p \in Fix(T)$.*

Proof. Consider $p \in Fix(T)$. From Lemma 2, we obtain

$$\begin{aligned} \|u_i - p\| &= \|(1-\zeta_i)w_i + \zeta_i T w_i - p\| \\ &= \|(1-\zeta_i)(w_i - p) + \zeta_i (T w_i - p)\| \\ &\le (1-\zeta_i)\|w_i - p\| + \zeta_i \|w_i - p\| \end{aligned}$$

$$= \quad \|w_i - p\|. \tag{4.1}$$

Similarly,

$$\begin{aligned}
\|v_i - p\| &= \|(1-\kappa_i)Tw_i + \kappa_i T^2 u_i - p\| \\
&= \|(1-\kappa_i)(Tw_i - p) + \kappa_i(T^2 u_i - p)\| \\
&\leq (1-\kappa_i)\|Tw_i - p\| + \kappa_i\|T^2 u_i - p\| \\
&\leq (1-\kappa_i)\|w_i - p\| + \kappa_i\|u_i - p\| \\
&\leq \|w_i - p\|.
\end{aligned} \tag{4.2}$$

Next, for the subsequent iteration:

$$\begin{aligned}
\|w_{i+1} - p\| &= \|Tv_i - p\| \\
&\leq \|v_i - p\| \\
&\leq \|w_i - p\|.
\end{aligned} \tag{4.3}$$

Combining (4.1), (4.2), and (4.3) gives

$$\|w_{i+1} - p\| \leq \|w_i - p\|.$$

Hence, the sequence $\{\|w_i - p\|\}$ is bounded and non-increasing. Therefore, $\lim_{i\to\infty} \|w_i - p\|$ exists for all $p \in Fix(T)$.

The next theorem establishes a basic result that will be instrumental for the subsequent analysis.

Theorem 1. *Let $\mathcal{C}$ is defined in Lemma 5 and a mapping $T : \mathcal{C} \to \mathcal{C}$ satisfying condition (E). If $\{w_i\}$ denotes the sequence produced by the HK-iteration, then $Fix(T) \neq \emptyset$ if and only if $\{w_i\}$ is bounded and $\lim_{i\to\infty} \|Tw_i - w_i\| = 0$.*

Proof. Suppose that $\{w_i\}$ is bounded and that $\lim_{i\to\infty} \|Tw_i - w_i\| = 0$. Let $p \in A(\mathcal{C}, \{w_i\})$. Our goal is to show that $Tp = p$. Since T obeys condition (E), we attain

$$\begin{aligned}
r(Tp, \{w_i\}) &= \limsup_{i\to\infty} \|w_i - Tp\| \\
&\leq \mu \limsup_{i\to\infty} \|Tw_i - w_i\| + \limsup_{i\to\infty} \|w_i - p\| \\
&= \limsup_{i\to\infty} \|w_i - p\| \\
&= r(p, \{w_i\}).
\end{aligned}$$

Thus, $Tp \in A(\mathcal{C}, \{w_i\})$. Since $A(\mathcal{C}, \{w_i\})$ is a singleton, it follows that $Tp = p$. Therefore, $Fix(T) \neq \emptyset$.

Conversely, suppose that $Fix(T) \neq \emptyset$ and let $p \in Fix(T)$. We will show that $\{w_i\}$ is bounded and that $\lim_{i\to\infty} \|w_i - Tw_i\| = 0$. By Lemma 5, $\lim_{i\to\infty} \|w_i - p\|$ exists, which implies that $\{w_i\}$ is bounded. Define

$$\lim_{i\to\infty} \|w_i - p\| = \lambda. \tag{4.4}$$

From the HK-iteration, we have

$$u_i = (1-\zeta_i)w_i + \zeta_i T w_i.$$

Applying Lemma 2, we get

$$\|u_i - p\| \leq \|w_i - p\| \quad \Rightarrow \quad \limsup_{i\to\infty} \|u_i - p\| \leq \lambda. \tag{4.5}$$

Also, using Lemma 2, we obtain

$$\|Tw_i - p\| \le \|w_i - p\| \quad \Rightarrow \quad \limsup_{i\to\infty} \|Tw_i - p\| \le \lambda. \tag{4.6}$$

From the HK-iteration, we further have

$$v_i = (1-\kappa_i)Tw_i + \kappa_i T^2 u_i, \qquad w_{i+1} = Tv_i.$$

Using these relations together with Lemma 2, we find

$$\|w_{i+1} - p\| = \|Tv_i - p\| \le \|v_i - p\| \le \|w_i - p\|. \tag{4.7}$$

Hence, $\{\|w_i - p\|\}$ is nonincreasing, and by (4.4) and (4.7) we have

$$\lambda \le \liminf_{i\to\infty} \|u_i - p\|. \tag{4.8}$$

Combining (4.5) and (4.8), we obtain

$$\lim_{i\to\infty} \|u_i - p\| = \lambda. \tag{4.9}$$

Note that $u_i = (1-\zeta_i)w_i + \zeta_i Tw_i$. Using (4.4) and (4.9), we get

$$\lambda = \lim_{i\to\infty} \|(1-\zeta_i)(w_i - p) + \zeta_i(Tw_i - p)\|. \tag{4.10}$$

By applying Lemma 4 (Schus lemma) to the sequences $\{w_i\}$ and $\{Tw_i\}$, along with the convex coefficients $\{\zeta_i\}$ (under the usual boundedness conditions on the control sequences stated in the preliminaries), and using (4.4), (4.6), and (4.10), we deduce that

$$\lim_{i\to\infty} \|Tw_i - w_i\| = 0.$$

This completes the proof of the converse part.

Theorem 2. *Let $\mathcal{C}$ is defined in Lemma 5 and T and $\{w_i\}$ be as described in Theorem 1 with $Fix(T) \neq \emptyset$. If $\mathcal{C}$ is compact, then the sequence $\{w_i\}$ converges strongly to a fixed point of T.*

Proof. From Theorem 1, we have $\lim_{i\to\infty} \|Tw_i - w_i\| = 0$. Since $\mathcal{C}$ is compact, there exists a subsequence $\{w_{i_l}\}$ of $\{w_i\}$ that converges strongly to some $q \in \mathcal{C}$. Because T satisfies condition (E), there exists a constant $\mu \ge 1$ such that

$$\|w_{i_l} - Tq\| \le \mu\|w_{i_l} - Tw_{i_l}\| + \|w_{i_l} - q\|.$$

Taking the limit as $l \to \infty$, we get $Tq = q$. By Lemma 5, $\lim_{i\to\infty} \|w_i - q\|$ exists. Therefore, q is the strong limit of $\{w_i\}$.

Theorem 3. *Let $\mathcal{C}$ is defined in Lemma 5 and let T and $\{w_i\}$ be as defined in Theorem 1. If $Fix(T) \neq \emptyset$ and $\liminf_{i\to\infty} d(w_i, Fix(T)) = 0$, then $\{w_i\}$ converges strongly to a fixed point of T.*

Proof. Since $Fix(T) \neq \emptyset$, Theorem 1 shows that $\{w_i\}$ is bounded. The result follows by a standard argument using the monotone decrease of $||w_i - p||$ for each $p \in Fix(T)$ together with the hypothesis $\liminf_{i\to\infty} dist(w_i,\ Fix(T)) = 0$.

The next result establishes the strong convergence property under condition (I).

Definition 1. *[38] Let $\mathcal{C}$ be a nonempty subset of a Banach space $\mathcal{X}$. A mapping $T : \mathcal{C} \to \mathcal{C}$ is said to satisfy condition (I) if there exists a function $\alpha : [0, \infty) \to [0, \infty)$ such that $\alpha(0) = 0$ and $\alpha(t) > 0$ for all $t \in (0, \infty)$, satisfying*

$$\|u - Tu\| \geq \alpha(dist(u, Fix(T))) \quad \text{for every } u \in \mathcal{C}.$$

Theorem 4. *Let $\mathcal{C}$ is defined in Lemma 5 and let T and $\{w_i\}$ be as defined in Theorem 1 with $Fix(T) \neq \emptyset$. If T fulfills condition (I), then the sequence $\{w_i\}$ converges strongly to a fixed point of T.*

Proof. By Theorem 1, we have

$$\liminf_{i\to\infty} \|Tw_i - w_i\| = 0. \tag{4.11}$$

Since the mapping T satisfies condition (I), we obtain

$$\|w_i - Tw_i\| \geq \alpha(d(w_i, Fix(T))). \tag{4.12}$$

Combining (4.11) and (4.12), it follows that

$$\liminf_{i\to\infty} \alpha(d(w_i, Fix(T))) = 0.$$

Because $\alpha : [0, \infty) \to [0, \infty)$ is a nondecreasing function with $\alpha(0) = 0$ and $\alpha(t) > 0$ for all $t \in (0, \infty)$, we deduce that

$$\liminf_{i\to\infty} d(w_i, Fix(T)) = 0.$$

The desired result then follows directly from Theorem 3.

We now establish the weak convergence of the sequence $\{w_i\}$ for mappings that satisfy condition (E).

Theorem 5. *Let $\mathcal{X}$ be a UCBS endowed with the Opial property. If $\mathcal{C}$ is defined in Lemma 5 and T and $\{w_i\}$ are as defined in Theorem 1 with $Fix(T) \neq \emptyset$, then the sequence $\{w_i\}$ converges weakly to a fixed point of T.*

Proof. By Theorem 1, $\{w_i\}$ is bounded and $\lim_{i\to\infty} ||Tw_i - w_i|| = 0$. By the Milman–Pettis theorem, X is reflexive, so by the Eberlein theorem there exists a subsequence $\{w_{i_k}\}$ converging weakly to $q_1 \in C$. Lemma 3 gives $q_1 \in Fix(T)$.

Suppose for contradiction that $\{w_i\}$ does not converge weakly to q_1. Then some subsequence $\{w_{i_\ell}\}$ converges weakly to $q_2 \neq q_1$, and again $q_2 \in Fix(T)$. By Lemma 3 and the Opial property,

$$\begin{aligned}
\lim_{i\to\infty} \|w_i - q_1\| &= \lim_{k\to\infty} \|w_{i_k} - q_1\| \\
&< \lim_{k\to\infty} \|w_{i_k} - q_2\| = \lim_{i\to\infty} \|w_i - q_2\| \\
&= \lim_{l\to\infty} \|w_{i_l} - q_2\| < \lim_{l\to\infty} \|w_{i_l} - q_1\| \\
&= \lim_{i\to\infty} \|w_i - q_1\|.
\end{aligned}$$

a contradiction. Hence $\{w_i\}$ converges weakly to q_1.

□

5. Summary of the HK-Iteration

In this work, we presented the **HK-iteration**, a new three-step iterative process developed for approximating fixed points of nonlinear mappings that satisfy condition (E) within uniformly convex Banach spaces. The established weak and strong convergence theorems demonstrate both the efficiency and stability of the proposed method compared with existing iterative approaches.

Not only HK-iteration generalizes a variety of classical schemes, including the ones produced by Picard, Mann, Ishikawa, Agarwal, Noor, Abbas, Thakur, Picard-Noor, and Sn, but it also has an even faster convergence rate. This capability of integrating various iteration processes and enhancement of computation performance shows the strength and applicability of the HK-iteration.

In general, the theory results demonstrate that the HK-iteration is an effective and versatile device used to estimate the fixed point of the Banach spaces. Furthermore, this framework offers the basis of subsequent studies on the extensions of the framework to further general classes of mappings and its use in other fields including optimization, variational inequalities, and nonlinear analysis.

Remark 1. *The HK-iteration scheme defined in* (2.11) *can be interpreted as a unifying structure that incorporates various well-known iterative methods through specific selections of the control sequences* $\{\zeta_i\}$ *and* $\{\kappa_i\}$*:*

- *For* $\zeta_i = 0$ *and* $\kappa_i = 0$*, we obtain* $u_i = w_i$*,* $v_i = Tw_i$*, and* $w_{i+1} = Tv_i = T^2 w_i$*, which corresponds to a modified form of the classical Picard iteration.*
- *When* $\zeta_i = \alpha_i$ *and* $\kappa_i = 0$*, the HK-iteration reduces to the Mann iteration.*
- *If* $\zeta_i = 0$ *and* $\kappa_i = \beta_i$*, the process becomes the Ishikawa iteration.*
- *Choosing* $\zeta_i = c_i$ *and* $\kappa_i = b_i$ *(with suitable renaming) recovers the Noor and Abbas-type three-step iterations.*
- *Appropriate choices of* $\zeta_i = a_i, b_i, c_i$ *following the structure of the Picard-Noor or* S_n *methods yield these iterations as particular cases.*

Thus, the HK-iteration is not only a general framework that incorporates a number of extinguished iterative schemes, but also a space in which new variants with improved convergence properties can be constructed.

6. Numerical Example

It is now time to explain our theoretical results by a simple but instructive example of a GarciaFalset mapping, which in general is not a Suzuki condition. The given example is an essential example of a numerical benchmark that is specified on the small interval, and the operator has only a unique natural fixed point. It provides a clear demonstration of the superior convergence rate achieved by the HK-iteration in contrast to several classical iterative methods such as those of Agarwal, Noor, and Abbas.

Example 1. *Let* $\mathcal{D} = [0, 3]$*. Define the operator* $T : \mathcal{D} \to \mathcal{D}$ *by*

$$Tt = \begin{cases} \dfrac{t^2+3}{4}, & \text{if } 0 \le t < 1, \\ \dfrac{t+3}{4}, & \text{if } 1 \le t \le 3. \end{cases}$$

Take $\mu = 3$. We aim to show that

$$\|t - Tt'\| \leq \mu\|t - Tt\| + \|t - t'\| \qquad \text{for all } t, t' \in \mathcal{D},$$

which verifies that T satisfies condition (E_μ).

Case (i). *For $t, t' \in [0, 1)$, we have $Tt = \frac{t^2+3}{4}$ and $Tt' = \frac{t'^2+3}{4}$. By applying the triangle inequality and noting that $|t^2 - t'^2| \leq |t - t'|(|t| + |t'|) \leq 2|t - t'|$ on $[0, 1]$, we obtain*

$$\begin{aligned}\|t - Tt'\| &= \left|t - \frac{t'^2+3}{4}\right| \leq \left|t - \frac{t^2+3}{4}\right| + \frac{1}{4}\left|t^2 - t'^2\right| \\ &\leq |t - Tt| + \frac{1}{4} \cdot 2|t - t'| = |t - Tt| + \tfrac{1}{2}|t - t'| \\ &\leq 3\|t - Tt\| + \|t - t'\|.\end{aligned}$$

Case (ii). *If $t, t' \in [1, 3]$, then $Tt = \frac{t+3}{4}$ and $Tt' = \frac{t'+3}{4}$. Thus,*

$$\begin{aligned}\|t - Tt'\| &= \left|t - \frac{t'+3}{4}\right| \leq \left|t - \frac{t+3}{4}\right| + \frac{1}{4}|t - t'| \\ &= |t - Tt| + \tfrac{1}{4}|t - t'| \leq 3\|t - Tt\| + \|t - t'\|.\end{aligned}$$

Case (iii). *When $t \in [0, 1)$ and $t' \in [1, 3]$, we have $Tt = \frac{t^2+3}{4}$ and $Tt' = \frac{t'+3}{4}$. Therefore,*

$$\begin{aligned}\|t - Tt'\| &= \left|t - \frac{t'+3}{4}\right| \leq |t - Tt| + \left|\frac{t^2+3}{4} - \frac{t'+3}{4}\right| \\ &= |t - Tt| + \frac{1}{4}\left|t^2 - t'\right| \\ &\leq |t - Tt| + \frac{1}{4}\left(|t^2 - t| + |t - t'|\right) \\ &\leq |t - Tt| + \frac{1}{4}|t - t'| \qquad (\text{since } t^2 \leq t \text{ on } [0, 1]) \\ &\leq 3\|t - Tt\| + \|t - t'\|.\end{aligned}$$

Case (iv). *Finally, if $t \in [1, 3]$ and $t' \in [0, 1)$, then $Tt = \frac{t+3}{4}$ and $Tt' = \frac{t'^2+3}{4}$. Hence,*

$$\begin{aligned}\|t - Tt'\| &= \left|t - \frac{t'^2+3}{4}\right| \leq |t - Tt| + \left|\frac{t+3}{4} - \frac{t'^2+3}{4}\right| \\ &= |t - Tt| + \frac{1}{4}\left|t - t'^2\right| \\ &\leq |t - Tt| + \frac{1}{4}\left(|t - t'| + |t' - t'^2|\right) \\ &\leq 3\|t - Tt\| + \|t - t'\|.\end{aligned}$$

From the above four cases, it follows that T satisfies (E_μ) with $\mu = 3$. Therefore, T is a Garcia–Falset operator defined on $\mathcal{D}$.

To determine the fixed points, consider $Tt = t$. On the second branch,

$$t = \frac{t+3}{4} \quad \Longrightarrow \quad 4t = t + 3 \quad \Longrightarrow \quad t = 1.$$

On the first branch,

$$t = \frac{t^2+3}{4} \quad \Longrightarrow \quad 4t = t^2 + 3 \quad \Longrightarrow \quad t^2 - 4t + 3 = 0 \quad \Longrightarrow \quad (t-1)(t-3) = 0,$$

so the only root in $[0, 1)$ is $t = 1$ (the root $t = 3$ lies outside this interval). Hence, the operator admits a unique fixed point in $\mathcal{D}$, namely $p = 1$, which is a positive natural number.

However, T fails to satisfy the Suzuki condition for all pairs $(t, t') \in \mathcal{D} \times \mathcal{D}$. Thus, the Suzuki implication does not hold universally, and T is not a Suzuki mapping in general.

Table 1: Computation table obtained from **HK**, **Agarwal**,**Noor**,**Abbas** ,iteration process using Example 1.

n	HK	**Agarwal**	**Noor**	**Abbas**
1	2.5	2.5	2.5	2.5
2	1.01696875	1.22649999	1.262218750	1.295863281
3	1.00019195	1.03420150	1.045839115	1.058356721
4	1.00000217	1.00516442	1.008013250	1.011510407
5	1.00000002	1.00077982	1.001400816	1.002270338
6	1.00000000	1.00011775	1.000244880	1.000447806
7	1.00000000	1.00001778	1.000042808	1.000088326
8	1.00000000	1.00000268	1.000007483	1.000017421
9	1.00000000	1.00000040	1.000001308	1.000003436
10	1.00000000	1.00000006	1.000000229	1.000000677
11	1.00000000	1.00000000	1.000000039	1.000000133
12	1.00000000	1.00000000	1.000000007	1.000000026
13	1.00000000	1.00000000	1.000000001	1.000000005
14	1.00000000	1.00000000	1.000000000	1.000000001
15	1.00000000	1.00000000	1.000000000	1.000000000

Convergence Comparison

Table 1 reports the iterates generated by the HK, Agarwal, Noor, and Abbas schemes starting from the initial point $x_1 = 2.5$, with control parameters $a_n = b_n = c_n = \zeta_n = \kappa_n = 0.5$ for all n.

The Table 1 clearly shows that the HK-iteration reaches the fixed point $p = 1$ (to eight decimal places) by the *sixth* iterate, while Agarwal, Noor, and Abbas require at least 11, 13, and 14 iterates, respectively. This empirically confirms the superior convergence speed of the HK-iteration.

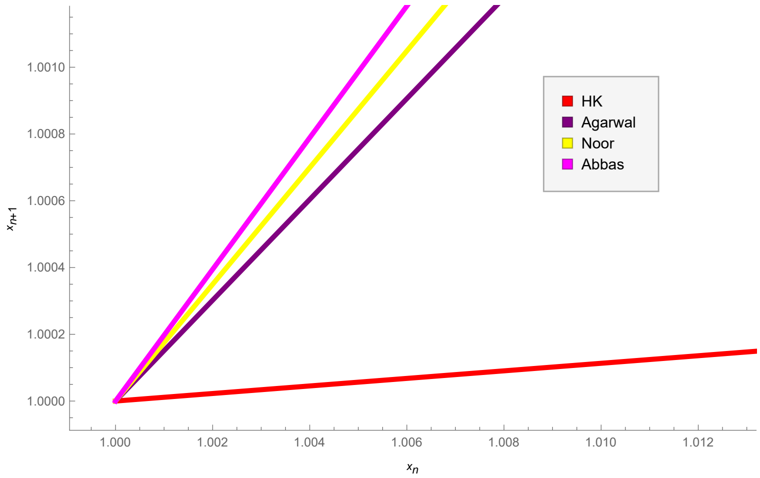

Figure 1: Cobweb diagram: x_{n+1} versus x_n for the four iteration schemes. A steeper slope toward the fixed point $p = 1$ indicates faster convergence.

Figure 1 displays the cobweb (staircase) diagram, which visualises the successive iterates $x_i \mapsto x_{i+1}$ for each scheme. The near-vertical trajectory of the HK-iterate (red line) against the diagonal $x_{n+1} = x_n$ confirms its rapid approach to the fixed point.

Example 2. *Let* $D = [0, 4]$*. Define the operator* $T : D \to D$ *by*

$$Tt = \begin{cases} \dfrac{t^2 + 8}{6}, & \text{if } 0 \leq t < 2, \\ \dfrac{t + 8}{6}, & \text{if } 2 \leq t \leq 4. \end{cases}$$

Take $\mu = 3$. We verify that T satisfies condition (E_μ), i.e.,

$$\|t - Tt'\| \ \leq \ \mu\|t - Tt\| + \|t - t'\| \quad \forall\, t, t' \in D.$$

Case (i). *For $t, t' \in [0, 2)$, we have $Tt = \frac{t^2+8}{6}$ and $Tt' = \frac{t'^2+8}{6}$. Since $|t^2 - t'^2| \leq 2|t - t'|$ on $[0, 2]$,*

$$\begin{aligned}
\|t - Tt'\| &= \left| t - \frac{t'^2 + 8}{6} \right| \\
&\leq \left| t - \frac{t^2 + 8}{6} \right| + \frac{1}{6}|t^2 - t'^2| \\
&\leq |t - Tt| + \frac{1}{3}|t - t'| \\
&\leq 3\|t - Tt\| + \|t - t'\|.
\end{aligned}$$

Case (ii). *For $t, t' \in [2, 4]$, we have $Tt = \frac{t+8}{6}$ and $Tt' = \frac{t'+8}{6}$. Thus,*

$$\begin{aligned}
\|t - Tt'\| &= \left| t - \frac{t' + 8}{6} \right| \\
&\leq \left| t - \frac{t + 8}{6} \right| + \frac{1}{6}|t - t'| \\
&= |t - Tt| + \frac{1}{6}|t - t'| \\
&\leq 3\|t - Tt\| + \|t - t'\|.
\end{aligned}$$

Case (iii). *For $t \in [0, 2)$ and $t' \in [2, 4]$:*

$$\begin{aligned}
\|t - Tt'\| &= \left| t - \frac{t' + 8}{6} \right| \\
&\leq |t - Tt| + \left| \frac{t^2 + 8}{6} - \frac{t' + 8}{6} \right| \\
&= |t - Tt| + \frac{1}{6}|t^2 - t'| \\
&\leq |t - Tt| + \frac{1}{6}\big(|t^2 - t| + |t - t'|\big) \\
&\leq 3\|t - Tt\| + \|t - t'\|,
\end{aligned}$$

where we used $t^2 \leq 2t$ on $[0, 2]$.
Case (iv). *For $t \in [2, 4]$ and $t' \in [0, 2)$:*

$$\begin{aligned}
\|t - Tt'\| &= \left| t - \frac{t'^2 + 8}{6} \right| \\
&\leq |t - Tt| + \left| \frac{t + 8}{6} - \frac{t'^2 + 8}{6} \right| \\
&= |t - Tt| + \frac{1}{6}|t - t'^2| \\
&\leq |t - Tt| + \frac{1}{6}\big(|t - t'| + |t' - t'^2|\big) \\
&\leq 3\|t - Tt\| + \|t - t'\|.
\end{aligned}$$

Hence T satisfies condition (E_μ) with $\mu = 3$, so T is a Garcia–Falset mapping on D.

Fixed point. *On the second branch,* $Tt = t$ *gives* $\frac{t+8}{6} = t \Rightarrow t = \frac{8}{5} = 1.6$*; but* $1.6 < 2$*, so this root does not lie in* $[2, 4]$*. On the first branch,*

$$\frac{t^2+8}{6} = t \implies t^2 - 6t + 8 = 0 \implies (t-2)(t-4) = 0,$$

yielding $t = 2$ *(since* $t = 4$ *lies outside* $[0, 2)$*). Hence the unique fixed point of* T *on* D *is* $p = 2$*.*
Suzuki condition failure. *One can verify that* T *does not satisfy the Suzuki condition for all pairs* $(t, t') \in D \times D$*; hence this example lies strictly outside the Suzuki class while remaining within the Garcia–Falset class.*

Table 2: Iterates generated by the HK, Agarwal, Noor, and Abbas schemes for Example 2, starting from $w_1 = 3.5$ with $a_n = b_n = c_n = \zeta_n = \kappa_n = 0.5$.

n	HK	Agarwal	Noor	Abbas
1	3.500000000	3.500000000	3.500000000	3.500000000
2	2.039351852	2.401041667	2.458333333	2.502170139
3	2.000532201	2.055025685	2.073510531	2.087768940
4	2.000007583	2.007625312	2.012829627	2.017136045
5	2.000000108	2.001057736	2.002241356	2.003376812
6	2.000000002	2.000146594	2.000391621	2.000665817
7	2.000000000	2.000020335	2.000068455	2.000131283
8	2.000000000	2.000002820	2.000011963	2.000025889
9	2.000000000	2.000000391	2.000002090	2.000005108
10	2.000000000	2.000000054	2.000000365	2.000001007
11	2.000000000	2.000000008	2.000000064	2.000000199
12	2.000000000	2.000000001	2.000000011	2.000000039
13	2.000000000	2.000000000	2.000000002	2.000000008
14	2.000000000	2.000000000	2.000000000	2.000000002
15	2.000000000	2.000000000	2.000000000	2.000000000

Table 2 shows that the HK-iteration converges to the fixed point $p = 2$ *(to eight decimal places) by the* seventh *iterate, whereas Agarwal, Noor, and Abbas require at least* 12, 13, *and* 14 *iterates, respectively, confirming the faster convergence of the HK-scheme on this example.*

Example 3. *Let* $D = [0, 5]$*. Define the operator* $T : D \to D$ *by*

$$Tt = \begin{cases} \dfrac{\sqrt{t}+6}{5}, & \text{if } 0 \le t < 1, \\ \dfrac{t+6}{5}, & \text{if } 1 \le t \le 5. \end{cases}$$

Take $\mu = 3$*. We show that* T *satisfies condition* (E_μ)*.*
Case (i). *For* $t, t' \in [0, 1)$*, noting that* $|\sqrt{t} - \sqrt{t'}| \le |t - t'|^{1/2}$ *and that on* $[0, 1]$ *the inequality* $|\sqrt{t} - \sqrt{t'}| \le \frac{1}{2}|t - t'|/\min(\sqrt{t}, \sqrt{t'})$ *can be bounded as* $|\sqrt{t} - \sqrt{t'}| \le |t - t'|$*, we have*

$$\begin{aligned} \|t - Tt'\| = \left| t - \frac{\sqrt{t'}+6}{5} \right| &\le \left| t - \frac{\sqrt{t}+6}{5} \right| + \frac{1}{5}|\sqrt{t} - \sqrt{t'}| \\ &\le |t - Tt| + \frac{1}{5}|t - t'| \le 3\|t - Tt\| + \|t - t'\|. \end{aligned}$$

Case (ii). *For* $t, t' \in [1, 5]$*:*

$$\|t - Tt'\| = \left| t - \frac{t'+6}{5} \right| \le \left| t - \frac{t+6}{5} \right| + \frac{1}{5}|t - t'|$$

$$= |t - Tt| + \frac{1}{5}|t - t'| \leq 3\|t - Tt\| + \|t - t'\|.$$

Case (iii). *For $t \in [0,1)$ and $t' \in [1,5]$:*

$$\|t - Tt'\| = \left| t - \frac{t'+6}{5} \right| \leq |t - Tt| + \frac{1}{5}|\sqrt{t} - t'|$$
$$\leq |t - Tt| + \frac{1}{5}\big(|\sqrt{t} - t| + |t - t'|\big) \leq 3\|t - Tt\| + \|t - t'\|,$$

using $\sqrt{t} \leq t + (1-t) \leq 1$ on $[0,1)$.
Case (iv). *For $t \in [1,5]$ and $t' \in [0,1)$:*

$$\|t - Tt'\| = \left| t - \frac{\sqrt{t'}+6}{5} \right| \leq |t - Tt| + \frac{1}{5}|t - \sqrt{t'}|$$
$$\leq |t - Tt| + \frac{1}{5}\big(|t - t'| + |t' - \sqrt{t'}|\big) \leq 3\|t - Tt\| + \|t - t'\|.$$

Hence T satisfies (E_μ) with $\mu = 3$ and is a Garcia–Falset mapping.
Fixed point. *On the second branch, $\frac{t+6}{5} = t \Rightarrow t + 6 = 5t \Rightarrow t = \frac{3}{2} = 1.5$. Since $1.5 \in [1,5]$, we confirm $p = 1.5$ is the unique fixed point of T on D. On the first branch, $\frac{\sqrt{t}+6}{5} = t \Rightarrow 5t - \sqrt{t} - 6 = 0$; setting $s = \sqrt{t}$, we get $5s^2 - s - 6 = 0$, yielding $s = \frac{1 \pm \sqrt{121}}{10}$, so $s = \frac{6}{5}$ or $s = -1$. Thus $t = \frac{36}{25} = 1.44 \notin [0,1)$, so no fixed point arises from the first branch. The unique fixed point on D is therefore $p = 1.5$.*

Table 3: Iterates generated by the HK, Agarwal, Noor, and Abbas schemes for Example 3, starting from $w_1 = 4.0$ with $a_n = b_n = c_n = \zeta_n = \kappa_n = 0.5$.

n	HK	Agarwal	Noor	Abbas
1	4.000000000	4.000000000	4.000000000	4.000000000
2	1.530000000	2.075000000	2.143750000	2.200000000
3	1.500600000	1.628281250	1.703523438	1.762500000
4	1.500012000	1.533828369	1.569382324	1.595019531
5	1.500000240	1.518498730	1.541284180	1.558754761
6	1.500000005	1.511436219	1.525736702	1.537536621
7	1.500000000	1.506972635	1.515249677	1.522441864
8	1.500000000	1.504108605	1.508936286	1.513068199
9	1.500000000	1.502335673	1.505082875	1.507292709
10	1.500000000	1.501285819	1.502860756	1.504061638
11	1.500000000	1.500697561	1.501577297	1.502243866
12	1.500000000	1.500378527	1.000865040	1.501226234
13	1.500000000	1.500205213	1.500472968	1.500670116
14	1.500000000	1.500111277	1.500258778	1.500366427
15	1.500000000	1.500060336	1.500141601	1.500200376
16	1.500000000	1.500032682	1.500077488	1.500109577
17	1.500000000	1.500017703	1.500042406	1.500059934
18	1.500000000	1.500009589	1.500023201	1.500032783
19	1.500000000	1.500005195	1.500012695	1.500017933
20	1.500000000	1.500002815	1.500006947	1.500009809

The Tabel 3 show that the HK-iteration converges to $p = 1.5$ by the seventh *iterate, while Agarwal, Noor, and Abbas require substantially more iterations. This further corroborates the superior convergence speed of the HK-scheme established in Section 6.*

7. Application to a Class of Fractional Boundary Value Problems

Fractional differential equations (FDEs) have gained a lot of attention due to their ability to model memory and hereditary attributes of many physical, biological, and engineering systems. Unlike classical integer-order models, they offer a more accurate and flexible description of complicated dynamic effects by using fractional formulations. Nevertheless, it is usually very difficult to find analytical solutions to such equations and this is the reason why efficient iterative and numerical methods are developed. For the literature see the references [39–42].

In this sub-topic, we apply the suggested **HK-Iteration** to a generalizations of nonlinear fractional boundary value problems (BVPs) with the view of proving its efficiency in computation and practicability of application. It is demonstrated here using the fixed point theory by using the HK-Iteration strategy, which can be effectively implemented in solving nonlinear fractional equations that are analytically non-resolvable. The example demonstrates the stability and flexibility of the HK-Iteration scheme and the increased convergence properties of the scheme to the solution of the fractional model of the Caputo-type derivative.

We consider the following fractional boundary value problem:

$$D^{\alpha}x(t) + D^{\beta}x(t) = f(t, x(t)), \quad t \in [0, b], \quad b > 0, \tag{7.1}$$

subject to the boundary conditions

$$x(0) = x(b) = 0, \tag{7.2}$$

where f is a continuous function and $\alpha, \beta \in (0, 1)$ denote the fractional derivatives in the Caputo sense.

The associated Greens function corresponding to the problem (7.1) (7.2) is given by

$$G(t) = t^{\alpha-1} M_{\alpha-\beta,\alpha}(-t^{\alpha-\beta}), \tag{7.3}$$

where $M_{\alpha-\beta,\alpha}(\cdot)$ represents the Mittag-Leffler function [43].

The solution of the boundary value problem (7.1) and (7.2) can be formulated as a fixed point of the operator $\mathcal{T} : C[0, b] \to C[0, b]$ defined by

$$\mathcal{T}x(t) = \int_0^b G(t - s) f(s, x(s))\, ds. \tag{7.4}$$

Building on recent developments in fixed-point iterative methods, we apply the three-step process called the **HK-Iteration**, which is defined as

$$\begin{cases} u_i = (1 - \zeta_i)w_i + \zeta_i \mathcal{T} w_i, \\ v_i = (1 - \kappa_i)\mathcal{T} w_i + \kappa_i \mathcal{T}^2 u_i, \\ w_{i+1} = \mathcal{T} v_i, \quad i = 0, 1, 2, \ldots, \end{cases} \tag{7.5}$$

where $\{\zeta_i\}$ and $\{\kappa_i\}$ are control sequences of real numbers satisfying $0 \le \zeta_i, \kappa_i \le 1$ for all $i \in \mathbb{N}$.

Theorem 6. *Let the operator $\mathcal{T}$ be defined by* (7.4)*. Suppose there exists a constant $\lambda < \alpha$ such that*

$$|f(t, a) - f(t, b)| \le \lambda |a - b|, \quad \text{for all } a, b \in \mathbb{R},\ t \in [0, b],$$

and $\frac{\lambda}{\alpha} < 1$. If the control sequences $\{\zeta_i\}$ and $\{\kappa_i\}$ satisfy $0 \le \zeta_i, \kappa_i \le 1$ and $\sum_{i=0}^{\infty} \kappa_i = \infty$, then the sequence $\{w_i\}$ generated by the HK-Iteration process (2.11) *converges strongly to the unique fixed point of $\mathcal{T}$, which corresponds to the unique solution of the fractional problem* (7.1) *and* (7.2)*.*

Proof. For any $x, z \in C[0, b]$, we have

$$\begin{aligned}
|\mathcal{T}x(t) - \mathcal{T}z(t)| &= \left| \int_0^b G(t-s)\big[f(s,x(s)) - f(s,z(s))\big]ds \right| \\
&\leq \int_0^b G(t-s)|f(s,x(s)) - f(s,z(s))|ds \\
&\leq \lambda \int_0^b G(t-s)|x(s) - z(s)|ds \\
&\leq \lambda \sup_{t\in[0,b]} \int_0^b G(t-s)ds\,|x(t) - z(t)| \\
&\leq \frac{\lambda}{\alpha}|x(t) - z(t)|.
\end{aligned}$$

Thus, $\|\mathcal{T}x - \mathcal{T}z\| \leq \frac{\lambda}{\alpha}\|x - z\|$ for all $x, z \in C[0, b]$, indicating that $\mathcal{T}$ is a contraction. Therefore, it possesses a unique fixed point x^*, which is the unique solution of (7.1) and (7.2).

Since the HK-Iteration (2.11) consists of convex combinations of iterates of $\mathcal{T}$, the contraction property yields

$$\|w_{i+1} - x^*\| = \|\mathcal{T}v_i - \mathcal{T}x^*\| \leq \frac{\lambda}{\alpha}\|v_i - x^*\| \leq \frac{\lambda}{\alpha}\|w_i - x^*\|.$$

By induction, we obtain

$$\|w_{i+1} - x^*\| \leq \left(\frac{\lambda}{\alpha}\right)^{i+1}\|w_0 - x^*\|,$$

which confirms that $\{w_i\}$ converges strongly to x^*. The uniqueness of x^* further ensures that this convergence is weakly w_2 stable in the sense of Berinde.

Remark 2. *The HK-Iteration technique illustrates a sound and dependable configuration of approximating the fixed-points of the nonlinear fractional operation. The approach does not only lengthen the classical techniques but also achieves improved and faster convergence. As a result of this reason, the technique is an invaluable and a portable method of solving complex fractional equation that occur in applied mathematics and engineering.*

8. Conclusions

This study proposed the HK-iteration (2.11), a novel three-step iterative scheme for approximating fixed points of Garcia–Falset mappings (satisfying condition (E)) in uniformly convex Banach spaces. The main contributions are:

(i) Existence of limits. Lemma 5 guarantees that $\lim_{i\to\infty} ||w_i - p||$ exists for every fixed point p.

(ii) **Characterization.** Theorem 1 characterises the nonemptiness of $fixT$ via the asymptotic behaviour of $\{w_i\}$.

(iii) **Strong convergence.** Theorems 2, 3, and 4 establish strong convergence under compactness, approximate fixed point, and condition (I), respectively.

(iv) **Weak convergence.** Theorem 5 establishes weak convergence in UCBS with the Opial property.

(v) **Numerical superiority.** The experiments in Section 6 confirm that the HK-iteration outperforms the Agarwal, Noor, and Abbas schemes.

(vi) **Fractional BVP application.** Theorem 6 shows that the HK-iteration converges to the unique solution of a Caputo-type fractional boundary value problem.

9. Future Work

The results presented here suggest several avenues for further research:

- **Explicit convergence rates.** Deriving quantitative bounds on $||w_i - p||$ in terms of the control sequences ζ_i, κ_i, and the contraction constant.
- **Stability analysis.** Providing a rigorous w^2-stability proof for the HK-iteration in the spirit of Berinde [44].
- **Broader mapping classes.** Extending the analysis to total asymptotically nonexpansive mappings, multi-valued mappings, and $(B_{\gamma,\mu})$-mappings.
- **Higher-dimensional fractional problems.** Applying the HK-iteration to fractional partial differential equations and multi-dimensional boundary value problems.
- **Optimisation and variational inequalities.** Investigating the performance of the HK-iteration for nonsmooth and constrained optimisation problems, and for variational inequalities arising in mechanics, physics, and transportation networks.
- **Comprehensive numerical benchmarking.** Comparing runtime, memory usage, stability, and accuracy against a wider set of iterative methods on large-scale problems.

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References

[1] H. Ullah, K. Ullah, A. Asghar, M. Tariq, H. Ahmad, and M. F. Abouelenein. Some new convergence approaches in banach spaces involving the hk-iteration method with application. *Contemporary Mathematics*, 7(2):1–22, 2026.

[2] H. Ullah, K. Ullah, A. Asghar, M. Tariq, H. Ahmad, and N. Roushdy. New approach of generalized α-nonexpansive mappings via hk-iteration process with applications to fractional differential equation and boundary value problems. *Contemporary Mathematics*, 7(2):1662–1682, 2026.

[3] Waqar Afzal, Mujahid Abbas, and Omar Mutab Alsalami. Bounds of different integral operators in tensorial hilbert and variable exponent function spaces. *Mathematics*, 12(16):2464, 2024.

[4] M. Tariq, S. K. Sahoo, J. Nasir, H. Aydi, and H. Alsamir. Some ostrowski type inequalities via n-polynomial exponentially s-convex functions and their applications. *AIMS Mathematics*, 6(12):13272–13290, Jan 2021.

[5] Waqar Afzal, Mujahid Abbas, Jorge E Macías-Díaz, Armando Gallegos, and Yahya Almalki. Boundedness and sobolev-type estimates for the exponentially damped riesz potential with applications to the regularity theory of elliptic pdes. *Fractal and fractional*, 9(7):458, 2025.

[6] S. K. Sahoo, M. Tariq, H. Ahmad, A. A. Aly, B. F. Felemban, and P. Thounthong. Some hermite–hadamard-type fractional integral inequalities involving twice-differentiable mappings. *Symmetry*, 13(11):2209, 2021.

[7] M. Tariq, S. I. Butt, and S. I. Butt. Some ostrowski type integral inequalities via generalized harmonic convex functions. *Open Journal of Mathematical Sciences*, 5(1):200–208, 2021.

[8] Mujahid Abbas, Aloraini, Waqar Afzal, Najla M, and Jongsuk Ro. Resolution of open problems via orlicz-zygmund spaces and new geometric properties of morrey spaces in the besov sense with non-standard growth. *AIMS mathematics*, 10(6):13908–13940, 2025.

[9] E. Kikianty. *Hermite–Hadamard Inequality in the Geometry of Banach Spaces*. PhD thesis, Victoria University, 2010.

[10] K.-H. Neeb. A cartan–hadamard theorem for banach–finsler manifolds. *Geometriae Dedicata*, 95(1):115–156, 2002.

[11] M. Tariq, S. K. Ntouyas, and A. A. Shaikh. New variant of hermite–hadamard, fejér and pachpatte-type inequality and its refinements pertaining to fractional integral operator. *Fractal and Fractional*, 7(5):405, 2023.

[12] M. Tariq, H. Ahmad, C. Cesarano, H. Abu-Zinadah, A. E. Abouelregal, and S. Askar. Novel analysis of hermite–hadamard type integral inequalities via generalized exponential type m-convex functions. *Mathematics*, 10(1):31, 2021.

[13] M. Tariq, S. K. Sahoo, H. Ahmad, A. A. Shaikh, B. Kodamasingh, and D. Khan. Some integral inequalities via new family of preinvex functions. *Mathematical Modelling and Numerical Simulation with Applications*, 2(2):117–126, 2022.

[14] F. E. Browder. Nonexpansive nonlinear operators in a banach space. *Proceedings of the National Academy of Sciences of the United States of America*, 54:1041–1044, 1965.

[15] D. Gohde. Zum prinzip der kontraktiven abbildung. *Mathematische Nachrichten*, 30:251–258, 1965.

[16] W. A. Kirk. A fixed point theorem for mappings which do not increase distance. *American Mathematical Monthly*, 72:1004–1006, 1965.

[17] J. Garcia-Falset, E. Llorens-Fuster, and T. Suzuki. Fixed point theory for a class of generalized nonexpansive mappings. *Journal of Mathematical Analysis and Applications*, 375(1):185–195, 2011.

[18] R. Pandey, R. Pant, V. Rakocevic, and R. Shukla. Approximating fixed points of a general class of nonexpansive mappings in banach spaces with applications. *Results in Mathematics*, 74(1):1–24, 2018.

[19] K. Ullah and M. Arshad. New iteration process and numerical reckoning fixed point in banach spaces. *UPB Scientific Bulletin, Series A*, 79(4):113–122, 2017.

[20] T. Abdeljawad, K. Ullah, and J. Ahmad. Iterative algorithm for mappings satisfying $(b_{\gamma,\mu})$ condition. *Journal of Function Spaces*, 2020:3492549, 2020.

[21] T. Abdeljawad, K. Ullah, J. Ahmad, and N. Mlaiki. Iterative approximation of endpoints for multivalued mappings in banach spaces. *Journal of Function Spaces*, 2020:2179059, 2020.

[22] G. A. Okeke. Iterative approximation of fixed points of contraction mappings in complex valued banach spaces. *Arab Journal of Mathematical Sciences*, 25(1):83–105, 2018.

[23] K. Ullah, F. Ayaz, and J. Ahmad. Some convergence results of m iterative process in banach spaces. *Asian-European Journal of Mathematics*, 12(5):1950017, 2019.

[24] E. Picard. Memoire sur la theorie des equations aux derivees partielles et la methode des approximations successives. *Journal de Mathematiques Pures et Appliquees*, 6:145–210, 1880.

[25] W. R. Mann. Mean value methods in iteration. *Proceedings of the American Mathematical Society*, 4:506–510, 1953.

[26] S. Ishikawa. Fixed points by a new iteration method. *Proceedings of the American Mathematical Society*, 44:147–150, 1974.

[27] R. P. Agarwal, D. O'Regan, and D. R. Sahu. Iterative construction of fixed points of nearly asymptotically nonexpansive mappings. *Journal of Nonlinear and Convex Analysis*, 8(1):61–79, 2007.

[28] M. Abbas and T. Nazir. A new faster iteration process applied to constrained minimization and feasibility problems. *Mathematica Vesnik*, 66:223–234, 2014.

[29] B. S. Thakur, D. Thakur, and M. Postolache. A new iterative scheme for numerical reckoning fixed points of suzuki's generalized nonexpansive mappings. *Applied Mathematics and Computation*, 275:147–155, 2016.

[30] M. A. Noor. New approximation schemes for general variational inequalities. *Journal of Mathematical Analysis and Applications*, 251(1):217–229, 2000.

[31] K. Eke and H. Akewe. Equivalence of picard-type hybrid iterative algorithms for contractive mappings. *Asian Journal of Scientific Research*, 12(3):298–307, 2019.

[32] W. Sintunavarat and A. Pitea. On a new iteration scheme for numerical reckoning fixed points of berinde mappings with convergence analysis. *Journal of Nonlinear Sciences and Applications*, 9:2553–2562, 2016.

[33] J. A. Clarkson. Uniformly convex spaces. *Transactions of the American Mathematical Society*, 40:396–414, 1936.

[34] W. Takahashi. *Nonlinear Functional Analysis*. Yokohama Publishers, Yokohama, 2000.

[35] R. P. Agarwal, D. O'Regan, and D. R. Sahu. *Fixed Point Theory for Lipschitzian-Type Mappings with Applications*. Springer, New York, 2009.

[36] Z. Opial. Weak and strong convergence of the sequence of successive approximations for nonexpansive mappings. *Bulletin of the American Mathematical Society*, 73:591–597, 1967.

[37] J. Schu. Weak and strong convergence to fixed points of asymptotically nonexpansive mappings. *Bulletin of the Australian Mathematical Society*, 43:153–159, 1991.

[38] H. F. Senter and W. G. Dotson. Approximating fixed points of nonexpansive mappings. *Proceedings of the American Mathematical Society*, 44:375–380, 1974.

[39] Najla M Aloraini, Mujahid Abbas, Waqar Afzal, Jong-Suk Ro, and Abdullah A Zaagan. Some novel kulisch-miranker type inclusions for a generalized class of godunova-levin stochastic processes. *AIMS Mathematics*, 9(2):5122–5146, 2024.

[40] Mujahid Abbas, Mutum Zico Meetei, Waqar Afzal, and Saïd Bourazza. Tensorial maclaurin approximation bounds and structural properties for mixed-norm orlicz–zygmund spaces. *Mathematics*, 13(6):917, 2025.

[41] Waqar Afzal and Luminita-Ioana Cotîrlă. New numerical quadrature functional inequalities on hilbert spaces in the framework of different forms of generalized convex mappings. *Symmetry*, 17(1):146, 2025.

[42] Waqar Afzal. Boundedness and regularity of the navier–stokes system in generalized herz spaces via a novel fractional potential framework. *Chaos, Solitons & Fractals*, 201:117086, 2025.

[43] A. M. Mathai and H. J. Haubold. *Special Functions for Applied Scientists*. Springer, New York, 2008.

[44] V. Berinde. On the stability of some fixed point procedures. *Buletinul Ştiinţific al Universităţii Baia Mare, Seria B*, 18(1):7–14, 2002.