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Qualitative Analysis of Integro-Differential Equation Involving a Generalized Caputo Fractional Operator

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Abstract. In this study, we investigated the existence and uniqueness of solutions for integro-fractional differential equations with a generalized Caputo operator involving boundary conditions. Utilizing the fixed-point technique, we examined the existence of a solution, and the Banach contraction technique was employed to determine the uniqueness of the solutions. Moreover, the stability conditions of the Ulam-Hyers are discussed. Subsequently, a suitable application is given to illustrate the validity of this work. The findings presented in this study serve to expand and generalize the existing body of knowledge about nonlocal and classical fractional differential equations.

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Key Words and Phrases: Integro-differential equation, fixed point technique, existence and uniqueness of solutions, stability result, generalized fractional derivative.

1. Introduction

A discussion of integer order derivatives and integrals to arbitrary order is explored in the field of fractional calculus to study mathematical concepts. Due to its significant

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applicability across a wide range of scientific domains, there has been an increased interest in fractional differential equations (FDEs). Compared to classical models, non-integer models have more reliability. The exploration of differential equations with a fractional order has become extensively researched. Models with non-integer order can grant an astonishing description of memory and hereditary properties of many physical phenomena. Moreover, using fractional derivatives in non-integer order models of real systems can be more accurate and adequate than the integer order models. Fractional differential equations can provide a comprehensive scheme to examine complex and regular systems in numerous fields, such as chemical physics, electrical circuits, physics, chemistry, biophysics, groundwater problems, viscoelasticity, polymers, environmental science, image processing, and capacitor theory. For more applications in this work, see, for example, [1–5].

Several authors have shown that integrals and derivatives of fractional order are more convenient for modelling some hereditary properties and disordered regions of several complex phenomena than the integer-order derivatives [6–8]. Researchers are now highly interested in looking into existence and uniqueness (EU) results for different FDEs. In recent times, a number of researchers used a variety of mathematical techniques to develop EU conclusions for various types of FDEs including fractional operators. Here, we highlight a few significant and recent papers concerning the solvability of different kinds of FDEs, see the articles [9–14]. Another subsequent aspect of fractal fractional calculus is investigating the stability analysis of initial value problems of FDEs. Stability analysis is a technique used to find the ability of the system to get back to its equilibrium position after introducing some changes. There are different kinds of stability analysis in the present literature of fractional calculus [15]. Ulam raised a question that under what conditions does there exist an additive mapping near an approximately additive mapping?. In reply to this question, Hyers answered Ulam's question for the "additive mapping in complete norm spaces" [16]. After it was considered a type of stability, it was called Hyers-Ulam (HU) stability.

Nowadays, the existence of solutions for FDEs of different orders is in the field of deep research. Now, we discuss some recent and important works of numerous authors on the existence of solutions for various classes of FDEs. The existence and uniqueness of solutions for local fractional differential equations with local fractional derivative operators were established by Jafari et al. [17]. The existence and uniqueness of solution for the fractional Langevin equation in terms of generalized Caputo derivatives with nonlocal boundary conditions involving generalized operators was investigated by Dhaniya et al. [18]. Aydogan et al. [19] explored the existence of solutions for the integro FDEs involving Caputo-Fabrizio derivative. Baleanu et al. [20] investigated the existence of hybrid-integro differential equation using the fixed-point theorem. Bohner et al. [21] studied the stability analysis for a fractional integro-differential equation involving nonlocal boundary conditions with constant delays. Boutiara et al.[22] discussed the existence and stability results for the coupled hybrid system of integro-differential equations using the fixed-point theorem. Ravichandran et al. [23] have presented sufficient conditions for the existence of solutions in the framework of Atangana-Baleanu derivative for fractional integro-differential equations

$$\begin{cases} {}_0^{ABC}\mathcal{D}^{\alpha}x(t) = f(t, x(t), I_1x(t), I_2x(t)) & t \in J : [0, 1], \\ x(a) = x_0, \end{cases}$$

where ${}_0^{ABC}\mathcal{D}^{\alpha}$ is the Atangana–Baleanu fractional derivative of order $0 < \alpha < 1$. $f : J \times \mathbb{R}^3 \to \mathbb{R}$, $x(t) : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Ahmad et al. [24] investigated the existence and uniqueness of solutions for the Langevin equation in terms of generalized fractional derivatives with nonlocal boundary conditions involving a generalized fractional integral

$$\begin{cases} {}_c^{\rho}\mathcal{D}_{a+}^{\alpha}({}_c^{\rho}\mathcal{D}_{a+}^{\beta} + \lambda)x(t) = f(t, x(t)) & t \in J : [a, T],\ \lambda \in \mathbb{R}, \\ x(a) = 0,\ x(\eta) = 0,\ x(T) = \mu {}_c^{\rho}I_{a+}^{\theta}x(\xi), & a < \eta < \xi < T, \end{cases}$$

where ${}_c^{\rho}\mathcal{D}_{a+}^{\alpha}$, ${}_c^{\rho}\mathcal{D}_{a+}^{\beta}$ denotes the Liouville–Caputo-type generalized fractional differential operators of order $1 < \alpha \leq 2$, $0 < \beta < 1$, $\rho > 0$ respectively, ${}_c^{\rho}I_{a+}^{\theta}$ is the generalized fractional integral operator of order $\theta > 0$ and $f : [a, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given continuous function. Jan et al. [25] studied the existence and uniqueness of solutions for a dynamic model via Schaefer's and Banach's fixed-point theorems. Nyamoradi et al. [26] discussed the EU of solutions for fractional integro-differential equation involving the Hadamard derivative. For more interesting results on FDEs, see [27–32]. Motivated by the aforementioned facts, in the present manuscript, we consider the following generalized Caputo derivatives fractional integro-differential equation involving the boundary condition:

$$\begin{cases} {}_c^{\rho}\mathcal{D}_{a+}^{\xi}y(t) = \nu(t, y(t), I_1y(t), I_2y(t)) & t \in J : [a, b], \\ y(a) = y_0, \end{cases} \tag{1}$$

where ${}_c^{\rho}\mathcal{D}_{a+}^{\xi}$ is the generalized Caputo fractional derivative of order $0 < \xi \leq 1$. $\nu : J \times \mathbb{R}^3 \to \mathbb{R}$, $y(t) : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $\rho > 0$, $y_0, T \in [a, b], \nu(a, y(a), a, \int_0^T \phi_2(a, s, y(s))ds = 0$. Suppose $I_1y(t) = \int_0^t \phi_1(t, s, y(s))ds$ and $I_2y(t) = \int_0^T \phi_2(t, s, y(s))ds$. The structure of the paper is organized as follows: In Section 2, we present the foundational definitions, properties, and lemmas that serve as the basis for this study. In Section 3, we demonstrate the existence results of the problem (1). In Section 4, we establish the uniqueness of the solution using the Banach fixed-point theorem. In Section 5, the stability of our problem is also studied based on Hyers-Ulam stability analysis. Finally, illustrative examples are provided in the concluding section to highlight the applicability of our results.

2. Preliminaries results

In this manuscript, we consider $Q = C([a, b], \mathbb{R})$ is the Banach space of all continuous functions on the interval $[a, b]$ equiped with the norm $\|y\| = \sup_{t\in[a,b]} |y(t)|$.

Definition 2.1 ([11]) Suppose $\nu \in \mathbb{X}_c^p(\epsilon,\theta)$ and $-\infty < \epsilon < t < \theta < \infty$, then the left-sided generalized fractional integral of order $\delta > 0$ is defined as:

$$({}^{\rho}I_{\epsilon+}^{\delta}\nu(t)) = \frac{\rho^{1-\delta}}{\Gamma(\delta)}\int_{\epsilon}^{t}\frac{\mathcal{s}^{\rho-1}}{(t^{\rho}-\mathcal{s}^{\rho})^{1-\delta}}\nu(\mathcal{s})d\mathcal{s}, \qquad \rho > 0, \tag{2}$$

where $\mathbb{X}_c^p(\epsilon,\theta)$ contains all the complex-valued Lebesgue measurable functions χ_1 on (ϵ,θ) and is defined with the norm as:

$$\|\chi_1\|_{\mathbb{X}_c^p} = \Big(\int_{\epsilon}^{\theta}|\mathcal{s}^c\chi_1(\mathcal{s})|^p\frac{d\mathcal{s}}{\mathcal{s}}\Big)^{\frac{1}{p}} < \infty, \quad c \in \mathbb{R},\ 1 \le p \le \infty,$$

and for $p = \infty,\ c \in \mathbb{R}$

$$\|\chi_1\| = \sup_{\epsilon \le \mathcal{s} \le \theta}\big[\mathcal{s}^c|\chi_1(\mathcal{s})|\big].$$

Similarly, the right-sided generalized integral operator of order $\delta > 0$ is define as:

$$({}^{\rho}I_{\theta-}^{\delta}\nu)(t) = \frac{\rho^{1-\delta}}{\Gamma(\delta)}\int_{e}^{\theta}\frac{\mathcal{s}^{\rho-1}}{(\mathcal{s}^{\rho}-e^{\rho})^{1-\delta}}\nu(\mathcal{s})d\mathcal{s}, \qquad \rho > 0. \tag{3}$$

Definition 2.2 ([12]) If $q = [\delta]+1$, where $\delta > 0$ and $0 \le \epsilon < x < \theta < \infty$, then generalized fractional derivative in the form of integral operators (2) and (3) are defined as:

$$\begin{aligned}({}^{\rho}\mathcal{D}_{\epsilon+}^{\delta}\nu)(t) &= \Big(t^{1-\rho}\frac{d}{dt}\Big)^q({}^{\rho}I_{\epsilon+}^{q-\delta}\nu)(t)\\ &= \Big(t^{1-\rho}\frac{d}{dt}\Big)^q\frac{\rho^{1-q+\delta}}{\Gamma(q-\delta)}\int_{\epsilon}^{t}\frac{\mathcal{s}^{\rho-1}}{(t^{\rho}-\mathcal{s}^{\rho})^{1-q+\delta}}\nu(\mathcal{s})d\mathcal{s}, \qquad \rho > 0.\end{aligned} \tag{4}$$

Similarly,

$$\begin{aligned}({}^{\rho}\mathcal{D}_{\theta-}^{\delta}\nu)(t) &= \Big(-t^{1-\rho}\frac{d}{dt}\Big)^q({}^{\rho}I_{\theta-}^{q-\delta}\nu)(t)\\ &= \Big(-t^{1-\rho}\frac{d}{dt}\Big)^q\frac{\rho^{1-q+\delta}}{\Gamma(q-\delta)}\int_{t}^{\theta}\frac{\mathcal{s}^{\rho-1}}{(\mathcal{s}^{\rho}-t^{\rho})^{1-q+\delta}}\nu(\mathcal{s})d\mathcal{s}, \qquad \rho > 0,\end{aligned} \tag{5}$$

if the integrals in the above expressions exist.

Definition 2.3 ([33]) Suppose $q = [\delta]+1$, where $\delta > 0$ and $\nu \in \mathbb{AC}_{\lambda}^{q}[\epsilon,\theta]$, then the generalized Liouvilli-Caputo fractional derivative ${}_{c}^{\rho}\mathcal{D}_{\epsilon+}^{\delta}\nu$ and ${}_{c}^{\rho}\mathcal{D}_{\theta-}^{\delta}\nu$, respectively is defined as:

$${}_{c}^{\rho}\mathcal{D}_{\epsilon+}^{\delta}\nu(t) = {}^{\rho}\mathcal{D}_{\epsilon+}^{\delta}\Big[\nu(\mathcal{s}) - \sum_{k=1}^{q-1}\frac{\lambda^k\nu(\epsilon)}{k!}\Big(\frac{\mathcal{s}^{\rho}-\epsilon^{\rho}}{\rho}\Big)^k\Big](t) \qquad \lambda = t^{1-\rho}\frac{d}{dt}, \tag{6}$$

$${}_{c}^{\rho}\mathcal{D}_{\theta-}^{\delta}\nu(t) = {}^{\rho}\mathcal{D}_{\theta-}^{\delta}\Big[\nu(\mathcal{s}) - \sum_{k=1}^{q-1}\frac{(-1)^k\lambda^k\nu(\theta)}{k!}\Big(\frac{\theta^{\rho}-\mathcal{s}^{\rho}}{\rho}\Big)^k\Big](t), \quad \lambda = t^{1-\rho}\frac{d}{dt}, \tag{7}$$

where $q=[\delta]+1$, $\delta\in\mathbb{R}$ $(\delta>0)$ and $\mathbb{AC}^q_\lambda[\epsilon,\theta]$ is the class of all functions ν, which are absolutely-continuous and having λ^{q-1}-derivative $(\lambda^{q-1}\nu\in\mathbb{AC}([\epsilon,\theta],\mathbb{R}))$, with the norm $\|\nu\|_{\mathbb{AC}^q_\lambda}=\sum_{k=1}^{q-1}\|\lambda^k\nu\|_C$.

Lemma 2.4 ([33]) Suppose $\nu\in\mathbb{AC}^q_\lambda[\epsilon,\theta]$ and $\delta>0$, then

$$ {}^{\rho}_{c}\mathscr{D}^{\delta}_{\epsilon^+}\nu(t)=\frac{1}{\Gamma(q-\delta)}\int_\epsilon^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{q-\delta-1}\frac{(\lambda^q\nu)(s)ds}{t^{1-\rho}}, \tag{8}$$

$$ {}^{\rho}_{c}\mathscr{D}^{\delta}_{\theta^-}\nu(t)=\frac{1}{\Gamma(q-\delta)}\int_t^\theta\left(\frac{s^\rho-t^\rho}{\rho}\right)^{q-\delta-1}\frac{(-1^q)(\lambda^q\nu)(s)ds}{t^{1-\rho}}. \tag{9}$$

Lemma 2.5 ([33]) Suppose $\nu\in\mathbb{AC}^q_\lambda[\epsilon,\theta]$, or $C^q_\lambda[\epsilon,\theta]$ and $\delta\in\mathbb{R}$, then

$$ {}^{\rho}I^{\delta}_{\epsilon^+}\,{}^{\rho}_{c}\mathscr{D}^{\delta}_{\epsilon^+}\nu(t)=\nu(t)-\sum_{k=1}^{q-1}\frac{\lambda^k\nu(\epsilon)}{k!}\left(\frac{t^\rho-\epsilon^\rho}{\rho}\right)^k, \tag{10}$$

and

$$ {}^{\rho}I^{\delta}_{\theta^-}\,{}^{\rho}_{c}\mathscr{D}^{\delta}_{\theta^-}\nu(t)=\nu(t)-\sum_{k=1}^{q-1}\frac{(-1)^k\lambda^k\nu(\theta)}{k!}\left(\frac{\theta^\rho-t^\rho}{\rho}\right)^k. \tag{11}$$

In particular, for $0<\delta\le 1$, then we get

$$ {}^{\rho}I^{\delta}_{\epsilon^+}\,{}^{\rho}_{c}\mathscr{D}^{\delta}_{\epsilon^+}\nu(s)=\nu(s)-\nu(\epsilon),$$

and

$$ {}^{\rho}I^{\delta}_{\theta^-}\,{}^{\rho}_{c}\mathscr{D}^{\delta}_{\theta^-}\nu(s)=\nu(s)-\nu(\theta).$$

The following existence and uniqueness result for the problem (1) is based on fixed point techniques. We will need the following hypotheses:

• $(\mathscr{A}_1)$ Let $y\in C[a,\ b]$ and suppose that $\nu\in(J\times\mathbb{R}^3,\mathbb{R})$ is continuous function and there exist a positive constant w_1, w_2 and w such that

$\|\nu(t,x_1,y_1,z_1)-\nu(t,x_2,y_2,z_2)\|\le w_1(\|x_1-x_2\|+\|y_1-y_2\|+\|z_1-z_2\|)$
for all x_1,x_2,y_1,y_2,z_1,z_2 in Y, $w_2=\max_{t\in[a,b]}\|\nu(t,a,a,a)\|$ and $w=\max\{w_1,w_2\}$.

Let $Y=C[J,Q]$ be the set of continuous functions on J with values in the Banach spaces Q.

• $(\mathscr{A}_2)$ There exist a positive constants $\mathbf{E_1}$, $\mathbf{E_2}$ and $\mathbf{E}$ such that

$$\|I_1(t,s,x_1)-I_1(t,s,x_2)\|\le\mathbf{E_1}(\|x_1-x_2\|)$$

for all $x_1,x_2\in Y$, and $\mathbf{E_2}=\max_{(t,s)\in[a,b]}\|I_1(t,s,a)\|$ and $\mathbf{E}$=max$\{\mathbf{E_1},\mathbf{E_2}\}$.

• $(\mathscr{A}_3)$ There exist a positive constants ψ_1, ψ_2 and ψ, then

$$\|I_2(t,s,x_1)-I_2(t,s,x_2)\|\le\psi_1(\|x_1-x_2\|)$$

for all $x_1, x_2 \in Y$, and $\psi_2 = \max_{(t,s)\in[a,b]} \|I_2(t,s,a)\|$ and $\psi = \max\{\psi_1, \psi_2\}$.

• $(\mathscr{A}_4)$ For each positive r, suppose $\mathscr{B}_r \in \{y \in Y : \|y\| \leq r\} \subseteq Y$ where $r = ((1-\mu)^{-1}\|y_0\|)$ and consider $\mu = \omega r(1 + \mathbf{E}t + \psi T) \leq \omega r(1 + \mathbf{E}b + \psi T)$.

• $(\mathscr{A}_5)$ For each positive r_0, suppose $\mathscr{B}_{r_0} \in \{y \in C([a,b], J) : \|y\| \leq r_0\}$ then $\mathscr{B}_{r_0}$ is a nonempty closed, bounded convex subset in $C([a,b], J)$.

Lemma 2.6: If $\mathscr{A}_2$ and $\mathscr{A}_3$ are satisfied, then the estimate

$$\|I_1 x(t)\| \leq t(\mathbf{E}_1\|x\| + \mathbf{E}_2),\ \|I_1 x_1(t) - I_1 x_2(t)\| \leq \mathbf{E}t\|x_1 - x_2\|$$

and

$$\|I_2 x(t)\| \leq T(\psi_1\|x\| + \psi_2),\ \|I_2 x_1(t) - I_2 x_2(t)\| \leq \psi T\|x_1 - x_2\|$$

are satisfied for any $t \in [a,b]$ and $x_1, x_2 \in Y$.

Theorem 2.1 (Krasnoselskii Fixed Point Theorem) ([34]) Suppose $\mathbb{S}$ be a convex, bounded, closed, nonempty subset of a real Banach space $\mathbb{E}$. Suppose $\mathscr{H}_1, \mathscr{H}_2$ are operators on $\mathbb{S}$ satisfying the following conditions
(i) $\mathscr{H}_1 x_1 + \mathscr{H}_2 y_1 \in \mathbb{S}$, whenever $x_1, y_1 \in \mathbb{S}$;
(ii) $\mathscr{H}_1$ is a strict contraction on $\mathbb{S}$, i.e., there exists a $\eta \in [0,1)$ such that

$$\|\mathscr{H}_1 y - \mathscr{H}_1 z\| \leq \eta\|y - z\| \quad \forall\ y, z \in \mathbb{S}$$

(iii) $\mathscr{H}_1$ is compact and continuous.
Then, there exists a $\phi \in \mathbb{S}$ such that $\phi = \mathscr{H}_1\phi + \mathscr{H}_2\phi$.

Theorem 2.2 (Ascoli - Arzela Theorem) ([34]) Suppose Φ is a compact Housedroff metric space. Then $\mathcal{N} \subset C(\Phi)$ is relatively compact iff $\mathcal{N}$ is uniformly bounded and uniformly equicontinuous.

3. Main Results

In this section, we demonstrate the existence results for $Eq.(1)$ using the fixed-point technique.

Lemma 3.1 If $y(t) \in C[a,b]$ is said to be a solution of $Eq.(1)$ and there exists $\nu \in (C[a,b] \times \mathbb{R}^3, \mathbb{R})$ such that $t \in [a,b]$ and the integral equation

$$y(t) = y_0 + {}^{\rho}I^{\xi}_{a^+}\nu(t, y(t), I_1 y(t), I_2 y(t)). \tag{12}$$

is satisfied.
Proof. Now, we prove that $y(t)$ satisfies the problem (1) iff $y(t)$ satisfies the $Eq.(12)$. Using Lemma 2.5 and the integral operator ${}^{\rho}I^{\xi}_{a^+}$ on $Eq.(1)$, the stated $Eq.(1)$ becomes

$$y(t) = {}^{\rho}I^{\xi}_{a^+}\nu(t, y(t), I_1 y(t), I_2 y(t)) + C_0. \tag{13}$$

Since, $y(a) = y_0$, then $Eq.(13)$ become

$$y(t) = y_0 + {}^{\rho}I^{\xi}_{a+}\nu(t, y(t), I_1y(t), I_2y(t)). \tag{14}$$

Now, we define the operator $\mathscr{H}$ such that

$$\mathscr{H}y(t) = y_0 + {}^{\rho}I^{\xi}_{a+}\nu(t, y(t), I_1y(t), I_2y(t)). \tag{15}$$

It can be observed that $y(t)$ is the result of the given $Eq.(1)$ if operator $\mathscr{H}$ has a fixed point.

3.2 Results on existence

Theorem 3.2 If the assumptions $(\mathscr{A}_1)-(\mathscr{A}_5)$ are satisfied and $\mathscr{P}(y_2 - y_1)$=$\{(\omega(\|y(t_2) - y(t_1)\| + \mathbf{E}t\|y(t_2) - y(t_1)\| + \psi T\|y(t_2) - y(t_1)\|)\}$, then the given $Eq.(1)$ has a solution.
Proof. For any positive constant r_0, and $y \in \mathscr{B}_{r_0}$ defined two operators $\mathscr{H}_1$ and $\mathscr{H}_2$ on $\mathscr{B}_{r_0}$ as follows

$$(\mathscr{H}_1y)(t) = y_0, \quad t \in [a, \mathscr{b}], \tag{16}$$

$$(\mathscr{H}_2y)(t) = {}^{\rho}I^{\xi}_{a+}\nu(t, y(t), I_1y(t), I_2y(t)). \tag{17}$$

Obviously, y is a solution of $Eq.(1)$ iff the operators $\mathscr{H}_1y+\mathscr{H}_2y = y$ has a solution $y \in \mathscr{B}_{r_0}$ on $[a, \mathscr{b}]$.
Here, we execute the following steps to demonstrate the required solutions.
Step 1. To show $\mathscr{H}y = \mathscr{H}_1y + \mathscr{H}_2y \in \mathscr{B}_{r_0}$.

$$\begin{aligned}
\|\mathscr{H}_1y(t) + \mathscr{H}_2y(t)\| &\leq \|y_0 + {}^{\rho}I^{\xi}_{a+}\nu(t, y(t), I_1y(t), I_2y(t))\| \\
&\leq \|y_0\| + \|{}^{\rho}I^{\xi}_{a+}\nu(t, y(t), I_1y(t), I_2y(t))\| \\
&\leq \|y_0\| + |\ {}^{\rho}I^{\xi}_{a+}\nu(t, y(t), I_1y(t), I_2y(t))\ | \\
&\leq \|y_0\| + \frac{(\mathscr{b}^{\rho} - a^{\rho})^{\xi}}{\rho^{\xi}\Gamma(\xi + 1)}(\omega(\|y\| + \mathbf{E}t\|y\| + \psi T\|y\|)) \\
&\leq \|y_0\| + \frac{(\mathscr{b}^{\rho} - a^{\rho})^{\xi}}{\rho^{\xi}\Gamma(\xi + 1)}(\omega(1 + \mathbf{E}t + \psi T))\|y\| \\
&\leq \|y_0\| + \frac{(\mathscr{b}^{\rho} - a^{\rho})^{\xi}}{\rho^{\xi}\Gamma(\xi + 1)}\mu\|y\| \\
&\leq \|y_0\| + \mu\|y\| \\
&\leq r(1 - \mu) + \mu r \\
&\leq r,
\end{aligned}$$

Hence, $\|\mathscr{H}_1y + \mathscr{H}_2y\| \leq r$.

Step 2. The operator $\mathscr{H}_1$ is contraction on $\mathscr{B}_{r_0}$ for any $y, \mathfrak{z} \in \mathscr{B}_{r_0}$, according to assumption $\mathscr{A}_5$, we have

$$\|\mathscr{H}_1y(t) - \mathscr{H}_1\mathfrak{z}(t)\| = 0 \leq \eta\|y - \mathfrak{z}\|, \quad 0 < \eta < 1.$$

Hence, $\mathcal{H}_1$ is Lipschitz continuous with constant 0 (< 1), and therefore it is a contraction mapping on $\mathcal{B}_{r_0}$.

Step 3. Now, we show that the operator $\mathcal{H}_2$ is continuous and compact.
First, we have to show that $\mathcal{H}_2$ is continuous on $\mathcal{B}_{r_0}$, with $\lim_{n\to\infty} \|y_n - y\| = 0$ for any $y_n, y \in \mathcal{B}_{r_0},\ n = 1, 2, ...,$ we get $\lim_{n\to\infty} y_n(t) = y(t)$, for $t \in [a, b]$ Thus, by $\mathcal{A}_1$, we have

$$\lim_{n\to\infty} \nu(t, y_n(t), I_1 y_n(t), I_2 y_n(t)) = \nu(t, y(t), I_1 y(t), I_2 y(t))$$

for $t \in [a, b]$, let us obtain

$$\begin{aligned}\|\mathcal{H}_2 y_n(t) - \mathcal{H}_2 y(t)\| &= \sup_{t\in[a,b]} \Big\{ \|{}^\rho I^\xi_{a+} \nu(t, y_n(t), I_1 y_n(t), I_2 y_n(t)) - {}^\rho I^\xi_{a+} \nu(t, y(t), I_1 y(t), I_2 y(t))\| \Big\} \\ &\leq \frac{(b^\rho - a^\rho)^\xi}{\rho^\xi \Gamma(\xi+1)} \times \sup_{t\in[a,b]} \|\nu(t, y_n(t), I_1 y_n(t), I_2 y_n(t)) - \nu(t, y(t), I_1 y(t), I_2 y(t))\|.\end{aligned}$$

Consequently, $\|\mathcal{H}_2 y_n(t) - \mathcal{H}_2 y(t)\| \longrightarrow 0,$ as $n \longrightarrow \infty$. It indicates that the operator $\mathcal{H}_2$ is a continuous mapping.
Now, we show that $\mathcal{H}_2$ is compact.

$$\begin{aligned}\|\mathcal{H}_2 y(t)\| &= \sup_{t\in[a,b]} \Big\{ \|{}^\rho I^\xi_{a+} \nu(t, y_n(t), I_1 y_n(t), I_2 y_n(t))\| \Big\} \\ &\leq \frac{(b^\rho - a^\rho)^\xi}{\rho^\xi \Gamma(\xi+1)} \times (\omega(\|y\| + \mathbf{E}t\|y\| + \psi T\|y\|)) < \infty,\end{aligned}$$

Subsequently, this demonstrates that operator $\mathcal{H}_2$ is uniformly bounded.
In order to study the compactness of $\mathcal{H}_2$, we must establish the equicontinuous nature of the operator $\mathcal{H}_2$.
For any $t \in \mathcal{B}_{r_0}$ and $t_1, t_2 \in [a, b]$ and $t_2 > t_1$, then we have

$$\begin{aligned}\|\mathcal{H}_2 y(t_2) - \mathcal{H}_2 y(t_1)\| &= \Big\| \frac{\rho^{1-\xi}}{\Gamma(\xi)} \int_{a+}^{t_2} \frac{s^{\rho-1}}{({t_2}^\rho - s^\rho)^{1-\xi}} \nu(s, y(s), I_1 y(s), I_2 y(s)) ds \\ &\quad - \frac{\rho^{1-\xi}}{\Gamma(\xi)} \int_{a+}^{t_1} \frac{s^{\rho-1}}{({t_1}^\rho - s^\rho)^{1-\xi}} \nu(s, y(s), I_1 y(s), I_2 y(s)) ds \Big\| \\ &\leq \frac{1}{\Gamma(\xi)} \Big[\int_{a+}^{t_1} \mid ({t_2}^\rho - s^\rho)^{\xi-1} - ({t_1}^\rho - s^\rho)^{\xi-1} \mid \ \|\nu(s, y(s), I_1 y(s), I_2 y(s))\| ds \\ &\quad + \int_{t_1}^{t_2} \mid ({t_2}^\rho - s^\rho)^{\xi-1} \mid \ \|\nu(s, y(s), I_1 y(s), I_2 y(s))\| ds \Big]. \\ &\|\mathcal{H}_2 y(t_2) - \mathcal{H}_2 y(t_1)\| \longrightarrow 0 \ \text{ as } \ t_2 \longrightarrow t_1.\end{aligned}$$

As a result, the operator $\mathcal{H}_2$ is relatively compact and equicontinuous on $[a, b]$. Hence, $\mathcal{H}_2$ is a compact operator in accordance with the Arzela Ascoli theorem. Consequently, using Theorem 2.1, there exists a point $y \in \mathcal{B}_{r_0}$ such that $y = \mathcal{H}_1 y + \mathcal{H}_2 y$. Hence, the given $Eq.(1)$ possesses at least a solution on $[a, b]$.

4. Results on uniqueness

In this section, the uniqueness results of solutions for $Eq.(1)$ is examined.

Theorem 4.1 Consider the assumption $\mathcal{A}_1 - \mathcal{A}_5$ are satisfied. Then the given problem (1) has a unique solution on $[a, b]$, if

$$\frac{\mu(b^{\rho} - a^{\rho})^{\xi}}{\rho^{\xi}\Gamma(\xi + 1)} < 1. \tag{18}$$

Proof. Let $\mathcal{B}_{\mathfrak{r}} \in \{y \in Q : \|y\| \leq \mathfrak{r}\}$ and by using Lemma 2.6, we get
Now, we show that $\mathcal{H}\mathcal{B}_{\mathfrak{r}}$ is bounded, i.e., $\mathcal{H}\mathcal{B}_{\mathfrak{r}} \subset \mathcal{B}_{\mathfrak{r}}$.

$$\begin{aligned}
\|\mathcal{H}y(t)\| &= \|y_0 + {}^{\rho}I^{\xi}_{a+}\nu(t, y(t), I_1y(t), I_2y(t))\| \\
&\leq \|y_0\| + \|{}^{\rho}I^{\xi}_{a+}\nu(t, y(t), I_1y(t), I_2y(t))\| \\
&\leq \|y_0\| + |\ {}^{\rho}I^{\xi}_{a+}\nu(t, y(t), I_1y(t), I_2y(t))\ | \\
&\leq \|y_0\| + \frac{(b^{\rho} - a^{\rho})^{\xi}}{\rho^{\xi}\Gamma(\xi + 1)}(\omega(\|y\| + \mathbf{E}t\|y\| + \psi T\|y\|)) \\
&\leq \|y_0\| + \frac{(b^{\rho} - a^{\rho})^{\xi}}{\rho^{\xi}\Gamma(\xi + 1)}\mu\|y\| \\
&\leq \|y_0\| + \mu\|y\| \\
&\leq \mathfrak{r}(1 - \mu) + \mu\mathfrak{r} \\
&\leq \mathfrak{r}.
\end{aligned}$$

This proves that $\mathcal{H}$ is bounded by $\mathcal{B}_{\mathfrak{r}}$.

We now show this uniqueness. Suppose that for each $y_1, y_2 \in Q$ we have

$$\begin{aligned}
\|\mathcal{H}y_1(t) - \mathcal{H}y_2(t)\| &= \|{}^{\rho}I^{\xi}_{a+}\nu(t, y_1(t), I_1y_1(t), I_2y_1(t)) - {}^{\rho}I^{\xi}_{a+}\nu(t, y_2(t), I_1y_2(t), I_2y_2(t))\| \\
&\leq \frac{(b^{\rho} - a^{\rho})^{\xi}}{\rho^{\xi}\Gamma(\xi + 1)}(\omega(\|y_1 - y_2\| + \mathbf{E}t\|y_1 - y_2\| + \psi T\|y_1 - y_2\|)) \\
&\leq \frac{(b^{\rho} - a^{\rho})^{\xi}}{\rho^{\xi}\Gamma(\xi + 1)}\mu\|y_1 - y_2\|.
\end{aligned}$$

nevertheless, it was assumed in $Eq.(18)$ that $\frac{(b^{\rho}-a^{\rho})^{\xi}}{\rho^{\xi}\Gamma(\xi+1)}\mu < 1$. Thus, by $Eq.(18)$, the operator $\mathcal{H}$ is a contraction mapping. As a result, $\mathcal{H}$ possesses a unique fixed point, that is a unique solution for $Eq.(1)$ according to the Banach contraction mapping principle.

5. Stability results

The notion of HU stability for $Eq.(1)$ is presented in this section.

Definition 5.1: The Integral form of $Eq.(14)$ exhibits HU stability if there exists $\nabla \geq 0$,

and given that the conditions below are satisfied.
If

$$| y(t) - y_0 + {}^{\rho}I^{\xi}_{a+}\nu(t, y(t), I_1 y(t), I_2 y(t)) | \leq \Omega, \quad \forall\ \Omega > 0, \tag{19}$$

then the continuous function $\mathfrak{z}(t)$ exists and it satisfies the integral form shown below:

$$\mathfrak{z}(t) = \mathfrak{z}_0 + {}^{\rho}I^{\xi}_{a+}\nu(t, \mathfrak{z}(t), I_1\mathfrak{z}(t), I_2\mathfrak{z}(t)), \tag{20}$$

with

$$| y(t) - \mathfrak{z}(t) | \leq \nabla\Omega. \tag{21}$$

Theorem 5.1: The given integro *FDEs* (1) are HU-stable under the assumptions $\mathscr{A}_1 - \mathscr{A}_5$.

Proof. For integro *FDEs* (1), Let $\mathfrak{z}(t)$ be the approximate solution, under the suppositions that $\mathscr{A}_1 - \mathscr{A}_5$ satisfy *Eq.*(20). Following up, we get

$$\begin{aligned}
| y(t) - \mathfrak{z}(t) | &= \left| y_0 + {}^{\rho}I^{\xi}_{a+}\nu(t, y(t), I_1 y(t), I_2 y(t)) - \mathfrak{z}_0 - {}^{\rho}I^{\xi}_{a+}\nu(t, \mathfrak{z}(t), I_1\mathfrak{z}(t), I_2\mathfrak{z}(t)) \right| \\
&\leq | y_0 - \mathfrak{z}_0 | + \left| {}^{\rho}I^{\xi}_{a+}\nu(t, y(t), I_1 y(t), I_2 y(t)) - {}^{\rho}I^{\xi}_{a+}\nu(t, \mathfrak{z}(t), I_1\mathfrak{z}(t), I_2\mathfrak{z}(t)) \right| \\
&\leq | y_0 - \mathfrak{z}_0 | + {}^{\rho}I^{\xi}_{a+}\left| \nu(t, y(t), I_1 y(t), I_2 y(t)) - \nu(t, \mathfrak{z}(t), I_1\mathfrak{z}(t), I_2\mathfrak{z}(t)) \right| \\
&\leq | y_0 - \mathfrak{z}_0 | + \frac{(\mathscr{b}^{\rho} - a^{\rho})^{\xi}}{\rho^{\xi}\Gamma(\xi + 1)}(\mathscr{w}(\|y - \mathfrak{z}\| + \mathbf{E}t\|y - \mathfrak{z}\| + \psi T\|y - \mathfrak{z}\|)) \\
&\leq | y_0 - \mathfrak{z}_0 | + \frac{\mu(\mathscr{b}^{\rho} - a^{\rho})^{\xi}}{\rho^{\xi}\Gamma(\xi + 1)}\|y - \mathfrak{z}\| \\
&\leq | y_0 - \mathfrak{z}_0 | + \Omega\|y - \mathfrak{z}\| \\
&\leq \frac{| y_0 - \mathfrak{z}_0 |}{1 - \Omega} \quad \forall\ t \in [a, \mathscr{b}],
\end{aligned} \tag{22}$$

where $\Omega = \frac{\mu(\mathscr{b}^{\rho} - a^{\rho})^{\xi}}{\rho^{\xi}\Gamma(\xi+1)}$. Making use of *Eq.*(22), the stability analysis of HU types of the integral *Eq.*(14) is proved. Therefore, the given integro *FDE* (1) is HU-stable.

6. Applications

Application 6.1 Consider the following integro *FDEs*:

$$\begin{cases} {}^{\rho}_{c}\mathscr{D}^{\xi}_{a+}y(t) = \nu(t, y(t), I_1 y(t), I_2 y(t)) \quad & t \in [0, 1], \\ y(0) = y_0, \end{cases} \tag{23}$$

where

$$\rho = \frac{1}{3},\ \xi = \frac{1}{2},\ T = \frac{\pi}{3},\ y_0 = \frac{1}{4},\ \mathfrak{z}_0 = \frac{1}{5},\ a = 0,\ \mathscr{b} = 1.$$

$$\begin{aligned}
\nu\left((t, y(t), \int_0^t \phi_1(t, \mathscr{s}, y(\mathscr{s}))d\mathscr{s}, \int_0^T \phi_2(t, \mathscr{s}, y(\mathscr{s}))d\mathscr{s}\right) = \frac{t^2}{t^3 + 15}\Bigg(&\frac{1}{35}\int_0^t (2 + t)y(\mathscr{s})d(\mathscr{s}) \\
&+ \frac{1}{20}\int_0^T e^t y(\mathscr{s})d(\mathscr{s})\Bigg),
\end{aligned}$$

$$\phi_1(t, s, y(s)) = \frac{1}{35}(2+t)y(s), \ \phi_2(t, s, y(s)) = \frac{1}{20}e^t y(s).$$

Now,

$$| \phi_1(t, s, y(s))ds - \phi_1(t, s, y_1(s))ds | \leq \frac{3}{35}(\|y - y_1\|),$$

$$| \phi_2(t, s, y(s))ds - \phi_2(t, s, y_1(s))ds | \leq \frac{1}{20}(\|y - y_1\|),$$

$$\left| \nu(t, y(t), I_1 y(t), I_2 y(t)) - \nu(t, y_1(t), I_1 y_1(t), I_2 y_1(t)) \right| \leq 0.12091\|y - y_1\|.$$

Therefore, assumptions $(\mathscr{A}_1)$, $(\mathscr{A}_2)$, and $(\mathscr{A}_3)$ are satisfied. Thus, according to Theorem 3.2, there exists at least one solution of $Eq.(23)$.
Also, $\Omega = \frac{\mu(b^\rho - a^\rho)^\xi}{\rho^\xi \Gamma(\xi+1)} \approx 0.48314 < 1$. Consequently, from Theorem 4.1, the given problem (23) has a unique solution.
Additionally, Theorem 5.1 requirements are all satisfied. As a result, $Eq.(23)$ exhibits HU stability.

Application 6.2 We examine the following integro *FDEs*:

$$\begin{cases} {}^{\rho}_{c}\mathscr{D}^{\xi}_{a^+} y(t) = \nu(t, y(t), I_1 y(t), I_2 y(t)) & t \in [0, 1], \\ y(0) = y_0, \end{cases} \tag{24}$$

where

$$\rho = \frac{1}{4}, \ \xi = \frac{1}{3}, \ T = \frac{\pi}{4}, \ y_0 = \frac{1}{7}, \ z_0 = \frac{1}{9}, \ a = 0, \ b = 1.$$

$$\begin{aligned} \nu\left((t, y(t), \int_0^t \phi_1(t, s, y(s))ds, \int_0^T \phi_2(t, s, y(s))ds\right) &= \frac{1}{2}\left(\frac{1}{4}\int_0^t \frac{7}{(2+t^2)} y(s)d(s) \right. \\ &\quad \left. + \frac{1}{5}\int_0^{\frac{\pi}{4}} \cos(t) y(s) d(s)\right) \end{aligned}$$

$$\phi_1(t, s, y(s)) = \frac{1}{4}\left(\frac{7}{(2+t^2)} y(s)\right), \quad \phi_2(t, s, y(s)) = \frac{1}{5}\cos(t) y(s).$$

Now,

$$| \phi_1(t, s, y(s))ds - \phi_1(t, s, y_1(s))ds | \leq \frac{7}{12}(\|y - y_1\|),$$

$$| \phi_2(t, s, y(s))ds - \phi_2(t, s, y_1(s))ds | \leq \frac{1}{5}(\|y - y_1\|),$$

$$\left| \nu(t, y(t), I_1 y(t), I_2 y(t)) - \nu(t, y_1(t), I_1 y_1(t), I_2 y_1(t)) \right| \leq 0.19165\|y - y_1\|.$$

Therefore assumption $(\mathscr{A}_1)$, $(\mathscr{A}_2)$, and $(\mathscr{A}_3)$ are satisfied. Thus, according to Theorem (3.2), There exists at least one solution of $Eq.(24)$.
Also, $\Omega = \frac{\mu(b^\rho - a^\rho)^\xi}{\rho^\xi \Gamma(\xi+1)} \approx 0.62350 < 1$. Hence, by Theorem 4.1, the given problem (24) has a unique solution. Furthermore, the hypotheses of Theorem 5.1 are satisfied, and consequently, $Eq.(24)$ possesses HU stability.

7. Conclusion

To summarize, we attained the conditions for showing the existence and uniqueness of the solution for the considered fractional integro-differential equations involving the generalized Caputo operator. Utilizing the fixed point theorem, we examined the existence of solutions, whereas the application of the Banach contraction mapping principle technique obtains the uniqueness of the solution. Moreover, we discussed the stability of this fractional differential equation in the Ulam–Hyers sense. Later, we provided an application to verify the accuracy of the obtained findings. The obtained results are new and generalize many existing results in the literature. This field is active in research, and hence we recommend continuing in this line of study to more qualitative analysis of such systems and using generalized fractional derivatives.

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