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A New Framework of Modified JS-Modular Metric Spaces with Applications to Hybrid Contractions

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Abstract. A new structure, referred to as a modified JS-modular metric space, is introduced in this paper. Fixed point results for a class of hybrid contractions are studied within this framework. An example is provided to illustrate the main findings and to emphasize the distinctions from existing results in the literature. In addition, one of the established theorems is applied to derive new criteria for the existence of solutions to a certain class of integral equations.

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1. Introduction

Among the well known fixed point (FP) theorems in spaces with metric structures is the Banach contraction (also known as the Contraction mapping principle). Because of the simplicity of the result, it is widely used in solving existence problems across various branches of mathematics and applied sciences. Hence, the contraction mapping principle has been extended in different directions (e.g., see [1–4]). In 2019, Karapinar and Fulga [5] introduced a new hybrid contraction that combines Jaggi-type and interpolative-type contractions in a complete metric space (MS) setting. Jidda *et al.* [2] extended the work

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of [5] to the setting of G-MS by defining the idea of interpolative Kannan-type $(G-\alpha-\mu)$ contraction. Not long ago, in a related development, Jidda *et* al. [6] studied a general class of inequalities under the name admissible $(H-\alpha-\kappa)$ contraction in a MS endowed with a graph. Yahya and Shagari [7] defined multivalued contraction from a combination of Jaggi-type contractions, interpolative-type contraction and Pata-type inequality in the framework of MS. For further information on FP results of hybrid contractions and contractions on G-MS, the reader can consult [6].

The notion of modular spaces as a generalization of MS has attracted considerable interest since Nakano [8], Musielak and Orlicz [9] introduced the concept of modular spaces and presented some results in that direction. The idea of modular MS has been worked on by Chistyakov [10], where the basic definitions and properties of modular metric are introduced. After [10], many authors have studied FP theorems and presented significant results in modular MS (see for instance, [1-3] and others therein).

Jleli and Samet [11] introduced the concept of a generalized MS (now called JS-MS) and extended some well-known FP results. The idea of JS-MS recovers various concepts in topological spaces including standard MS, b-MS and modular spaces. Turkoglu and Manav [12] considering both a modular MS and a JS-MS later introduced a new concept of JS-modular MS and proved some FP results in this setting. Interestingly, Manav [13], using Nadler [14] and Edelstein's result in [15], proved Caristi and Feng-Liu type FP theorems for multivalued contractive-type map in generalised modular MS. Later on, Manav and Agarwal [16] studied best approximation of FP results for Branciari contraction of integral type on generalized modular MS. For some recent results on Branciari contractions, see [17, 18].

Following the reviewed literature, we see that a unified concept in the sense of [5, 11] is not adequately considered. Motivated by the above information, we propose a new combinational concept under the name Jaggi-type hybrid contraction (JHC) in Modified JS-modular MS and prove the existence of FP of such contraction. The proposed result is a direct improvement of the work of [5, 11], and some related results therein. An example is constructed to support the assumptions of our established ideas. Some corollaries are highlighted and discussed to show that the concept proposed therein, subsumes some existing results. From application point of view, we further demonstrate the utility of our main result by employing it to study the solvability of certain integral equations (IEs).

The structure of the paper is outlined as follows: Section 1 provides an introduction. In Section 2, we recall specific preliminary concepts needed in the discussion of our main results. The main result and corollaries are presented in Section 3 with an example given to support the established concepts. In Section 4, a possible utility of one of our obtained result is presented in connection with a class of IEs.

2. Preliminaries

In this part, the basic concepts and auxiliary results needed for the presentation of our main result are collated.

Definition 1. *[11] Let E be a nonempty set and $F : E \times E \longrightarrow [0, +\infty]$ be a given mapping. For every $c \in E$, define the set*

$$B(F, E, c) = \{\{c_n\} \subseteq E : \lim_{n \longrightarrow 0} F(c_n, c) = 0\}$$

$\forall\ \lambda > 0$. We say that F is a JS-modular metric on E if it satisfies

i. for every $(c, d) \in E^2, \lambda > 0$, we have

$$F(c, d) = 0 \Rightarrow c = d;$$

ii. for every $(c, d) \in E^2, \lambda > 0$, we have

$$F(c, d) = F(d, c);$$

iii. there exists $B > 0$ such that if $(c, d) \in E^2, \{c_n\} \in B(F, E, c)$ then

$$F(c, d) \leq B \limsup_{n \longrightarrow +\infty} F(c_n, d).$$

Then (E, F) is a JS-MS.

Definition 2. *[10] Let E be a vector space over $\mathbb{R}$ (or $\mathbb{C}$).*

a. A functional $\rho : E \longrightarrow [0, \infty]$ is called a modular if for arbitrary $c, d \in E$,

i) $\rho(c) = 0$ iff $c = 0$;

ii) $\rho(\alpha c) = \rho(c)$ for every scalar α with $|\alpha| = 1$;

iii) $\rho(\alpha c + \alpha d) \leq \rho(c) + \rho(d)$ if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$;

b. if iii) is replaced by iii′) $\rho(\alpha c + \alpha d) \leq \alpha^s \rho(c) + \beta^s \rho(d)$, for $\alpha, \beta \geq 0, \alpha^s + \beta^s = 1$ with $s \in (0, 1]$, then ρ is called an s-convex modular, and if $s = 1$, ρ is called a convex modular.

c. A modular ρ defines a corresponding modular space i.e the vector subspace of E, E_ρ given by

$$E_\rho = \{c \in E : \rho(\lambda c) \longrightarrow 0 \text{ as } \lambda \longrightarrow 0\}.$$

Definition 3. *[10] Let E be a nonempty set. A function $\varpi : (0, \infty) \times E \times E \longrightarrow [0, \infty]$ is said to be a metric modular on E if it satisfies the following conditions:*

i) $\varpi_\lambda(c, d) = 0\ \forall\ \lambda > 0$ iff $c = d$;

ii) $\varpi_\lambda(c, d) = \varpi_\lambda(d, c) \forall \lambda > 0$;

iii) $\varpi_{\lambda+\mu}(c, d) \leq \varpi_\lambda(c, z) + \varpi_\lambda(z, d) \forall \lambda, \mu > 0$.

Definition 4. *[11] Let (E, F) be a JS-MS. We say that* *i)* $\{c_n\}$ *F-converges to E if* $\{c_n\} \in B(F, E, c)$.

ii) $\{c_n\}$ *is F-Cauchy sequence if* $\lim\limits_{n \longrightarrow +\infty} F(c_n, c_{n+m}) = 0$.

iii) (E, F) *is F-complete if every Cauchy sequence in E is convergent to some element in E.*

Definition 5. *[5]* $\psi : [0, \infty) \longrightarrow [0, \infty)$ *is a (c)-comparison function if it is nondecreasing and satisfies there exists* $j_0 \in \mathbb{N}$ *and* $k \in (0, 1)$ *and a convergent series* $\sum_{j=1}^{\infty} z_j$ *such that* $q_j \geq 0$ *and*

$$\psi^{j+1}(t) \leq k\psi^j(t) + q_j, \; for\, j \geq j_0 \; and \; t \geq 0.$$

The set of all (c)-comparison functions will be denoted by Ψ. *Every* $\psi \in \Psi$ *satisfies the following:*

a. $\psi(q) < q$, *for any* $q \in \mathbb{R}^+$;

b. ψ *is continuous at* 0;

c. the series $\sum_{j=1}^{\infty} \psi^j(q)$ *is convergent for* $q \geq 0$.

Karapinar and Fulga [5] gave the following definition of JHC in MS.

Definition 6. *[5] Let* (E, F) *be a complete* MS. *A mapping* $\kappa : E \longrightarrow E$ *is called a* JHC *if there exists* $\psi \in \Psi$ *such that*

$$\delta(\kappa c, \kappa d) \leq \psi(J_\kappa^s(c, d)) \tag{1}$$

$\forall c, d \in E$, *where* $s \geq 0$ *and*

$$J_\kappa^s(c, d) = \begin{cases} [\sigma_1(\dfrac{\delta(c, \kappa c)\delta(d, \kappa d)}{\delta(c, d)})^s + \delta(c, d))^s]^{\frac{1}{s}} for\; s > 0, c, d \in E, c \neq d \\ (\delta(c, \kappa c))^{\sigma_1}(\delta(d, \kappa d))^{\sigma_2} for\; s = 0, c, d \in E/F_\kappa(E) \end{cases}$$

for $\sigma_i \geq 0, i = 1, 2$ *such that* $\sigma_1 + \sigma_2 = 1$.

Theorem 1. *[5] A continuous self-mapping* κ *on* (E, δ) *possesses an* FP c *provided that* κ *is a Jaggi-type hybrid contraction. Moreover, for any* $c_0 \in E$, *the sequence* $\{\kappa^n(c_0)\}$ *converges to* c.

Turkoglu and Manav [12] introduced the JS-modular metric as follows.

Definition 7. *[12] Let* E *be a nonempty set and* $\varpi_\lambda : E \times E \longrightarrow [0, +\infty]$ *be a given mapping. For every* $c \in E$, *define the set*

$$B(\varpi_\lambda, E, c) = \{\{c_n\} \subseteq E; \lim_{n \longrightarrow 0} \varpi_\lambda(c_n, c) = 0\},$$

$\forall\ \lambda > 0$. ϖ_λ *is a* JS*-modular metric on* E *if it satisfies* *i) for every $(c, d) \in E^2, \lambda > 0$, we have*

$$\varpi_\lambda(c, d) = 0 \Rightarrow c = d;$$

ii) for every $(c, d) \in E^2, \lambda > 0$, we have

$$\varpi_\lambda(c, d) = \varpi_\lambda(d, c);$$

iii) there exists $B > 0$ such that if $(c, d) \in E^2, \{c_n\} \in B(\varpi_\lambda, E, c)$, then

$$\varpi_\lambda(c, d) \leq B \limsup_{n \longrightarrow +\infty} \varpi_\lambda(c_n, d).$$

.

The pair (E, ϖ_λ) is called a JS-modular MS.

3. Main Results

Definition 8. *Let E be a nonempty set and $\varpi_\lambda^\varphi : E \times E \longrightarrow [0, +\infty]$ be a given mapping. For every $c \in E$, define the set*

$$\varphi(\varpi_\lambda^\varphi, E, c) = \{\{c_n\} \subseteq E : \lim_{n \longrightarrow 0} \varpi_\lambda^\varphi(c_n, c) = 0\},$$

$\forall\ \lambda > 0$. We say that ϖ_λ^φ is a modified JS-modular metric on E if it satisfies:

i) for every $(c, d) \in E^2, \lambda > 0$, we have

$$\varpi_\lambda^\varphi(c, d) = 0 \Rightarrow c = d;$$

ii) for every $(c, d) \in E^2, \lambda > 0$, we have

$$\varpi_\lambda^\varphi(c, d) = \varpi_\lambda^\varphi(d, c);$$

iii) there exists a continuous function $\varphi : E \longrightarrow [0, \infty)$ such that

$$\varpi_\lambda^\varphi(c, d) \leq \varphi\big(\limsup_{n \longrightarrow +\infty} \varpi_\lambda^\varphi(c_n, d)\big).$$

We call $(E_\varpi, \varpi_\lambda^\varphi)$ a modified JS-modular MS.

Remark 1. *If $\varphi(t) = kt\, for\, k > 0$, $\forall t \geq 0$, then we get the definition of a JS-modular metric. However, there exists a modified JS-modular metric which is not a JS-modular metric as demonstrated in Example 1.*

Example 1. *Let* $E_\varpi = [0,1]$*,* $\varphi : E \longrightarrow [0,\infty)$ *be defined by* $\varphi(t) = 2+t$*. Define* $\varpi_\lambda^\varphi : E \times E \longrightarrow [0,+\infty]$ *by* $\varpi_\lambda^\varphi(c,d) = \dfrac{|c+d|}{\lambda}$*,* $\lambda > 0$*. Then clearly,* $(E_\varpi, \varpi_\lambda^\varphi)$ *is a modified* JS*-modular* MS*.*
On the other hand, observe that if $c=1$*,* $d=0$*, and* $c_n = \dfrac{1}{n}$*, then*

$$\varpi_\lambda^\varphi(c,d) = \frac{|1+0|}{\lambda} \nleq B \limsup_{x\longrightarrow\infty} \varpi_\lambda^\varphi(c_n,d) = B \limsup_{x\longrightarrow\infty} \frac{|\dfrac{1}{n}+0|}{\lambda} = 0.$$

Hence, $(E_\varpi, \varpi_\lambda^\varphi)$ is not a JS-modular MS.

Definition 9. *Let* $(E_\varpi, \varpi_\lambda^\varphi)$ *be a modified JS-modular* MS*. A mapping* $\kappa : E_\varpi \longrightarrow E_\varpi$ *is called a hybrid contraction of Jaggi-type (HCJT) if there exists* $\psi \in \Psi$ *such that*

$$\varpi_\lambda^\varphi(\kappa c, \kappa d) \leq \psi(H_\kappa^s(c,d)), \tag{2}$$

$\forall c,d \in E_\varpi$*, where* $s \geq 0$ *and*

$$H_T^s(c,d) = \begin{cases} \left[\sigma_1\left(\dfrac{\varpi_\lambda^\varphi(c,\kappa c)\varpi_\lambda^\varphi(d,\kappa d)}{\varpi_\lambda^\varphi(c,d)}\right)^s + \sigma_2(\varpi_\lambda^\varphi(c,d))^s\right]^{\frac{1}{s}} for\, s>0, \\ c,d \in E_\varpi, \varpi_\lambda^\varphi(c,d) \neq 0 \\ (\varpi_\lambda^\varphi(c,\kappa c))^{\sigma_1}(\varpi_\lambda^\varphi(d,\kappa d))^{\sigma_2} for\, s=0, c,d \in E_\varpi/F_\kappa(E_\varpi) \end{cases}$$

for $\sigma_i \geq 0, i=1,2$ *such that* $\sigma_1+\sigma_2 = 1$ *.*

Proposition 1. *Suppose that* κ *is a HCJT. Then any* FP $c \in E$ *of* κ *satisfying* $\varpi_\lambda^\varphi(c,c) < \infty$ *implies* $\varpi_\lambda^\varphi(c,c) = 0$*.*

Proof. Let $c \in E$ be an FP of κ such that $\varpi_\lambda^\varphi(c,c) < \infty$. Since κ is a HCJT, for $s>0$, if $\varpi_\lambda^\varphi(c,c) \neq 0$, then

$$\begin{aligned} \varpi_\lambda^\varphi(\kappa c,\kappa c) &\leq \psi([\sigma_1(\frac{\varpi_\lambda^\varphi(c,\kappa c)\varpi_\lambda^\varphi(c,\kappa c)}{\varpi_\lambda^\varphi(c,c)})^s + (\varpi_\lambda^\varphi(c,c))^s]^{\frac{1}{s}}) \\ &= \psi([\sigma_1(\frac{\varpi_\lambda^\varphi(x,x)\varpi_\lambda^\varphi(x,x)}{\varpi_\lambda^\varphi(c,c)})^s + (\varpi_\lambda^\varphi(c,c))^s]^{\frac{1}{s}}) \\ &= \psi([(\sigma_1+\sigma_2)(\varpi_\lambda^\varphi(c,c))^s]^{\frac{1}{s}}) \\ &= \psi(\varpi_\lambda^\varphi(c,c)) \\ &< \varpi_\lambda^\varphi(c,c). \end{aligned}$$

Hence, we have $\varpi_\lambda^\varphi(c,c) = \varpi_\lambda^\varphi(\kappa c,\kappa c) < \varpi_\lambda^\varphi(c,c)$. This is a contradiction. Hence, $\varpi_\lambda^\varphi(c,c) = 0$. Similarly, for $s=0$, if $\varpi_\lambda^\varphi(c,c) \neq 0$, then

$$\varpi_\lambda^\varphi(c,c) \quad = \quad \varpi_\lambda^\varphi(\kappa c,\kappa c)$$

$$
\begin{aligned}
&\leq \psi((\varpi_\lambda^\varphi(c,\kappa c))^{\sigma_1}(\varpi_\lambda^\varphi(c,\kappa c))^{\sigma_2})\\
&< (\varpi_\lambda^\varphi(c,\kappa c))^{\sigma_1}(\varpi_\lambda^\varphi(c,\kappa c))^{\sigma_2}\\
&= (\varpi_\lambda^\varphi(c,\kappa c))^{\sigma_1+\sigma_2}\\
&= \varpi_\lambda^\varphi(c,\kappa c)\\
&= \varpi_\lambda^\varphi(c,c).
\end{aligned}
$$

This is a contradiction. Hence, $\varpi_\lambda^\varphi(c,c)=0$.

Proposition 2. *Let $(E_\omega,\varpi_\lambda^\varphi)$ be a modified JS-modular MS, $\{c_n\}$ be a sequence in E_ϖ and $(c,d)\in E_\varpi\times E_\varpi$ such that $\varpi_\lambda^\varphi(c_n,c)\longrightarrow 0$, $\varpi_\lambda^\varphi(c_n,d)\longrightarrow 0$ as $n\longrightarrow+\infty$ for some $\lambda>0$. Then $c=d$.*

Proof. Using the property iii) and without lost of generality, suppose that φ is continuous at the origin. Then we have

$$\varpi_\lambda^\varphi(c,d)\leq \varphi\big(\limsup_{n\longrightarrow+\infty}\varpi_\lambda^\varphi(c_n,d)\big)=0,$$

which implies from the property i) that $c=d$.

Theorem 2. *Suppose that the following conditions hold:*

i) $(E_\varpi,\varpi_\lambda^\varphi)$ is a complete modified JS-modular MS;

ii) κ is a continuous HCJT;

iii) there exists $c_0\in E_\varpi$ such that

$$\gamma(\varpi_\lambda^\varphi,\kappa,c_0)=\sup\{\varpi_\lambda^\varphi(\kappa^i(c),\kappa^j(c))\}<\infty.$$

Then, the sequence $\{\kappa^n(c_0)\}$ ϖ_λ^φ-converges to $c\in E$, a unique FP of κ.

Proof. Let $n\in\mathbb{N}$ and $c_0\in E_\varpi$. Since κ is a *HCJT*, $\forall i,j\in\mathbb{N}\cup\{0\}$, we have

$$\varpi_\lambda^\varphi(\kappa^{n+i}(c_0),\kappa^{n+j}(c_0))\leq\psi(H_\kappa^s(\kappa^{n-1+i}(c_0),\kappa^{n-1+j}(c_0))). \tag{3}$$

We shall prove the claim of the theorem by examining the two cases: $s>0$ and $s=0$.
Case (1): for $s>0$ and taking $j=i+1$, the expression (2) becomes

$$\varpi_\lambda^\varphi(\kappa^{n+i}(c_0),\kappa^{n+i+1}(c_0))\leq\psi(H_\kappa^s(\kappa^{n-1+i}(c_0),\kappa^{n+i}(c_0))) \tag{4}$$

But

$$H_\kappa^s(\kappa^{n-1+i}(c_0),\kappa^{n+i}(c_0)) \;=\; \Bigg[\sigma_1\left(\frac{\varpi_\lambda^\varphi(\kappa^{n-1+i}(c_0),\kappa^{n+i}(c_0))\varpi_\lambda^\varphi(\kappa^{n+i}(c_0),\kappa^{n+1+i}(c_0))}{\varpi_\lambda^\varphi(\kappa^{n-1+i}(c_0),\kappa^{n+i}(c_0))}\right)^s$$

$$+\ \sigma_2(\varpi_\lambda^\varphi(\kappa^{n-1+i}(c_0),\kappa^{n+i}(c_0)))^s\Big]^{\frac{1}{s}}.$$

Therefore (3) becomes

$$\begin{aligned}\varpi_\lambda^\varphi(\kappa^{n+i}(c_0),\kappa^{n+i+1}(c_0)) &\leq \psi([\sigma_1(\varpi_\lambda^\varphi(\kappa^{n+i}(c_0),\kappa^{n+1+i}(c_0)))^s\\ &+ \sigma_2(\varpi_\lambda^\varphi(\kappa^{n-1+i}(c_0),\kappa^{n+i}(c_0)))^s]^{\frac{1}{s}}).\end{aligned}$$

Now suppose that

$$\varpi_\lambda^\varphi(\kappa^{n+i}(c_0),\kappa^{n+1+i}(c_0)) \geq \varpi_\lambda^\varphi(\kappa^{n-1+i}(c_0),\kappa^{n+i}(c_0)),$$

then,

$$\begin{aligned}\varpi_\lambda^\varphi(\kappa^{n+i}(c_0),\kappa^{n+i+1}(c_0)) &\leq \psi([\sigma_1(\varpi_\lambda^\varphi(\kappa^{n+i}(c_0),\kappa^{n+1+i}(c_0)))^s\\ &\quad +\sigma_2(\varpi_\lambda^\varphi(\kappa^{n-1+i}(c_0),\kappa^{n+i}(c_0)))^s]^{\frac{1}{s}})\\ &= \psi([\sigma_1(\varpi_\lambda^\varphi(\kappa^{n+i}(c_0),\kappa^{n+1+i}(c_0)))^s\\ &\quad +\sigma_2(\varpi_\lambda^\varphi(\kappa^{n+i}(c_0),\kappa^{n+1+i}(c_0)))^s]^{\frac{1}{s}})\\ &= \psi([(\sigma_1+\sigma_2)(\varpi_\lambda^\varphi(\kappa^{n+i}(c_0),\kappa^{n+i+1}(c_0)))^s]^{\frac{1}{s}})\\ &= \psi(\varpi_\lambda^\varphi(\kappa^{n-1+i}(c_0),\kappa^{n+i}(c_0))\\ &< \varpi_\lambda^\varphi(\kappa^{n+i}(c_0),\kappa^{n+i+1}(c_0)).\end{aligned}$$

This is a contradiction. Hence, we obtain that

$$\varpi_\lambda^\varphi(\kappa^{n+i}(c_0),\kappa^{n+1+i}(c_0)) \leq \varpi_\lambda^\varphi(\kappa^{n-1+i}(c_0),\kappa^{n+i}(c_0)).$$

Since $\gamma(\varpi_\lambda^\varphi,\kappa,c)=\sup\{\varpi_\lambda^\varphi(\kappa^i(c),\kappa^j(c))\}$, The expression (4) becomes

$$\begin{aligned}\delta(\varpi_\lambda^\varphi,\kappa,\kappa^n(c_0)) &\leq \psi(\delta(\varpi_\lambda^\varphi,\kappa,\kappa^{n-1}(c_0))\\ &\leq \psi^2(\delta(\varpi_\lambda^\varphi,\kappa,\kappa^{n-2}(c_0))\\ &\vdots\\ &\leq \psi^n(\delta(\varpi_\lambda^\varphi,\kappa,c_0)).\end{aligned}$$

Let $n,m\in\mathbb{N}$ with $m>n$. By using the property of the (c)-comparison function,

$$\begin{aligned}\varpi_\lambda^\varphi(\kappa^n(c_0),\kappa^{n+m}(c_0)) &\leq \delta(\varpi_\lambda^\varphi,\kappa,\kappa^n(c_0))\\ &\leq \psi^n(\delta(\varpi_\lambda^\varphi,\kappa,c_0)) \longrightarrow 0\, as\, n \longrightarrow +\infty\end{aligned}$$

This implies that $\{\kappa^n(c_0)\}$ is an ϖ_λ^φ-Cauchy sequence. Since $(E_\varpi,\varpi_\lambda^\varphi)$ is ϖ_λ^φ-complete, there exists $c\in E_\varpi$ such that $\{\kappa^n(c_0)\}$ is ϖ_λ^φ-convergent to E. Now, since κ is continuous, it follows from Proposition 3 that

$$\varpi_\lambda^\varphi(c,\kappa c) = \lim_{n\longrightarrow+\infty}\varpi_\lambda^\varphi(\kappa^n(x),\kappa(\kappa^n(x)))$$

$$\begin{aligned} &= \lim_{n \longrightarrow +\infty} \varpi_\lambda^\varphi(\kappa^n(x), \kappa^{n+1}(x)) \\ &= \varpi_\lambda^\varphi(c, c) \\ &= 0, \end{aligned}$$

which shows that E is an FP of κ.
To show uniqueness, suppose that $\{\kappa^n(c_0)\}$ is ϖ_λ^φ-convergent to d, then by Proposition 2, we have that $c = d$.
Case (2): for $s > 0$ and taking $j = i + 1$, the expression (2) becomes

$$\begin{aligned} \varpi_\lambda^\varphi(\kappa^{n+i}(c_0), \kappa^{n+i+1}(c_0)) &\leq \psi(H_\kappa^s(\kappa^{n-1+i}(c_0), \kappa^{n+i}(c_0))) \\ &= \psi((\varpi_\lambda^\varphi(\kappa^{n-1+i}(c_0), \kappa^{n+i}(c_0)))^{\sigma_1}(\varpi_\lambda^\varphi(\kappa^{n+i}(c_0), \kappa^{n+1+i}(c_0))^{\sigma_2}) \quad (5) \\ &< (\varpi_\lambda^\varphi(\kappa^{n-1+i}(c_0), \kappa^{n+i}(c_0)))^{\sigma_1}(\varpi_\lambda^\varphi(\kappa^{n+i}(c_0), \kappa^{n+1+i}(c_0))^{\sigma_2}. \end{aligned}$$

The above inequality turns into

$$\varpi_\lambda^\varphi(\kappa^{n+i}(c_0), \kappa^{n+i+1}(c_0))^{1-\sigma} < (\varpi_\lambda^\varphi(\kappa^{n-1+i}(c_0), \kappa^{n+i}(c_0)))^{\sigma_1}$$

and since $\sigma_1 + \sigma_2 = 1$, we have

$$\varpi_\lambda^\varphi(\kappa^{n+i}(c_0), \kappa^{n+i+1}(c_0)) < \varpi_\lambda^\varphi(\kappa^{n-1+i}(c_0), \kappa^{n+i}(c_0)$$

Returning to (5), we find that

$$\varpi_\lambda^\varphi(\kappa^{n+i}(c_0), \kappa^{n+i+1}(c_0)) \leq \psi(\varpi_\lambda^\varphi(\kappa^{n-1+i}(c_0), \kappa^{n+i}(c_0)) \tag{6}$$

Since $\gamma(\varpi_\lambda^\varphi, \kappa, c) = \sup\{\varpi_\lambda^\varphi(\kappa^i(c), \kappa^j(c))\}$, The expression (6) becomes

$$\begin{aligned} \delta(\varpi_\lambda^\varphi, \kappa, \kappa^n(c_0)) &\leq \psi(\delta(\varpi_\lambda^\varphi, \kappa, \kappa^{n-1}(c_0)) \\ &\leq \psi^2(\delta(\varpi_\lambda^\varphi, \kappa, \kappa^{n-2}(c_0)) \\ &\vdots \\ &\leq \psi^n(\delta(\varpi_\lambda^\varphi, \kappa, c_0)). \end{aligned}$$

By using the same tools as in the case of $s > 0$, we find that $\{\kappa^n(c_0)\}$ is an ϖ_λ^φ-Cauchy sequence in a ϖ_λ^φ-complete modified JS-modular MS. Hence, there exists $c \in E_\varpi$ such that $\{\kappa^n(c_0)\}$ is ϖ_λ^φ-convergent to E. It follows that E is a unique FP of κ.

In the following result, we establish the existence of FP of κ under the restriction of continuity of some iterates of κ.

Theorem 3. *Let $(E_\omega, \varpi_\lambda^\varphi)$ be a complete modified JS-modular MS and $\kappa : E_\varpi \longrightarrow E_\varpi$ be a HCJT. If for some $i > 1$, κ^i is continuous, then κ has an FP in E_ϖ.*

Proof. In Theorem 2, it has been shown that there exists an ϖ_λ^φ-Cauchy sequence $\{c_n\}$ in $(E_\omega, \varpi_\lambda^\varphi)$ with $c_n = \kappa c_{n-1}$ such that $c_n \longrightarrow c$ for some $c \in E_\omega$.
Let $\{c_{n_k}\}$ be a subsequence of $\{c_n\}$, where $n_k = k.i$ for some $k \in \mathbb{N}, k > 1$ fixed. Observe

that κ^0 is an identity self mapping on E_ϖ so that $c_{n_k} = \kappa^i c_{n_k-1}$. Hence, by continuity of κ^i, we have

$$\begin{aligned} \varpi_\lambda^\varphi(c, \kappa^i c) &= \lim_{k\longrightarrow\infty} \varpi_\lambda^\varphi(c, \kappa^i c_{n_k-i}) \\ &= \lim_{k\longrightarrow\infty} \varpi_\lambda^\varphi(c, c_{n_k}) \\ &= \varpi_\lambda^\varphi(c, c) \\ &= 0, \end{aligned}$$

which means that E is an FP of κ^i. Now, to see that E is an FP of κ, assume by contradiction that $\kappa c \neq c$. Then in that case $\kappa^{i-m-1}c \neq \kappa^{i-m}c$ for any $m = 0, 1, ..., i-1$. Again, we must consider two cases.

Case (1): for $s > 0$, replacing c by $\kappa^{i-m-1}c$ and d by $\kappa^{i-m}c$, we have

$$\begin{aligned} H_\kappa^S(\kappa^{i-m-1}c, \kappa^{i-m}c) &= \Bigg[\sigma_1\Bigg(\frac{\varpi_\lambda^\varphi(\kappa^{i-m-1}c, \kappa(\kappa^{i-m-1}c)).\varpi_\lambda^\varphi(\kappa^{i-m-1}c, \kappa(\kappa^{i-m}c))}{\varpi_\lambda^\varphi(\kappa^{i-m-1}c, \kappa^{i-m}c)}\Bigg)^s \\ &+ \sigma_2(\varpi_\lambda^\varphi(\kappa^{i-m-1}c, \kappa^{i-m}c)\Bigg)\Bigg] \end{aligned}$$

and since κ is a $HCJT$,

$$\begin{aligned} \varpi_\lambda^\varphi(\kappa^{i-m}c, \kappa^{i-m+1}c) &= \varpi_\lambda^\varphi(\kappa(\kappa^{i-m-1}c), \kappa(\kappa^{i-m}c)) \\ &\leq \psi(H_\kappa^S(\kappa^{i-m-1}c, \kappa^{i-m})) \\ &= \psi([\sigma_1(\varpi_\lambda^\varphi(\kappa^{i-m-1}c, \kappa(\kappa^{i-m-1}c))^s + \sigma_2(\varpi_\lambda^\varphi(\kappa^{i-m-1}c, \kappa^{i-m}c))^s]^{\frac{1}{s}}) \\ &< [\sigma_1(\varpi_\lambda^\varphi(\kappa^{i-m-1}c, \kappa(\kappa^{i-m-1}c))^s + \sigma_2(\varpi_\lambda^\varphi(\kappa^{i-m-1}c, \kappa^{i-m}c))^s]^{\frac{1}{s}} \end{aligned}$$

which yields that

$$(\varpi_\lambda^\varphi(\kappa^{i-m}c, \kappa^{i-m+1}c))^s(1-\sigma_1) < \sigma_2(\varpi_\lambda^\varphi(\kappa^{i-m-1}c, \kappa^{i-m}c))^s.$$

However, $\sigma_1 + \sigma_2 = 1$, so that for every $m = 0, 1, ..., i-1$

$$\varpi_\lambda^\varphi(\kappa^{i-m}c, \kappa^{i-m+1}c) < \sigma_2(\varpi_\lambda^\varphi(\kappa^{i-m-1}c, \kappa^{i-m}c).$$

This clearly implies that for every $k = m, m+1, ..., i-1$,

$$\varpi_\lambda^\varphi(\kappa^{i-m}c, \kappa^{i-m+1}c) < \sigma_2(\varpi_\lambda^\varphi(\kappa^{i-m-k-1}c, \kappa^{i-m-k}c)$$

Taking in particular $m = 0$ and $k = i-1$ in the above inequality, we get

$$\varpi_\lambda^\varphi(c, \kappa^i c) = \varpi_\lambda^\varphi(\kappa^i c, \kappa^{i+1}) < \varpi_\lambda^\varphi(c, \kappa c),$$

which is a contradiction. Consequently, $\kappa c = c$.

To show uniqueness, suppose that $\{\kappa^n(c_0)\}$ is ϖ_λ^φ-convergent to d, then by Proposition 2, we have that the FP is unique.

Case (2): for $s = 0$, by replacing c by $\kappa^{i-m-1}c$ and d by $\kappa^{i-m}c$, we have

$$\begin{aligned} H^S_\kappa(\kappa^{i-m-1}c, \kappa^{i-m}c) &= [\varpi^\varphi_\lambda(\kappa^{i-m-1}c, \kappa(\kappa^{i-m-1}c))]^{\sigma_1}[\varpi^\varphi_\lambda(\kappa^{i-m}c, \kappa(\kappa^{i-m}c))]^{\sigma_2} \\ &= [\varpi^\varphi_\lambda(\kappa^{i-m-1}c, \kappa^{i-m}c))]^{\sigma_1}[\varpi^\varphi_\lambda(\kappa^{i-m}c, \kappa^{i-m+1}c))]^{\sigma_2} \end{aligned}$$

and the iequality becomes

$$\begin{aligned} \varpi^\varphi_\lambda(\kappa^{i-m}c, \kappa^{i-m+1}c) &= \varpi^\varphi_\lambda(\kappa(\kappa^{i-m-1}c), \kappa(\kappa^{i-m}c)) \leq \psi(H^S_\kappa(\kappa^{i-m-1}c, \kappa^{i-m})) \\ &= \psi([\varpi^\varphi_\lambda(\kappa^{i-m-1}c, \kappa^{i-m}c))]^{\sigma_1}[\varpi^\varphi_\lambda(\kappa^{i-m}c, \kappa^{i-m+1}c))]^{\sigma_2}) \\ &< [\varpi^\varphi_\lambda(\kappa^{i-m-1}c, \kappa^{i-m}c))]^{\sigma_1}[\varpi^\varphi_\lambda(\kappa^{i-m}c, \kappa^{i-m+1}c))]^{\sigma_2}. \end{aligned}$$

By simple calculation, we get

$$\varpi^\varphi_\lambda(\kappa^{i-m}c, \kappa^{i-m+1}c) < \varpi^\varphi_\lambda(\kappa^{i-m-1}c, \kappa^{i-m}c),$$

and using the same tools as in the Case (1), we obtain that c is a unique FP of κ.

Corollary 1. *Let $(E_\varpi, \varpi^\varphi_\lambda)$ be a complete modified JS-modular MS. Suppose that the continuous mapping $\kappa : E_\varpi \longrightarrow E_\varpi$ satisfies*

$$\varpi^\varphi_\lambda(\kappa c, \kappa d) \leq \psi(\varpi^\varphi_\lambda(c, d))$$

$\forall c, d \in E_\varpi$. *Then κ has an FP in E_ϖ.*

Proof. Letting $\sigma_1 = 0, \sigma_2 = 1$ and $s > 0$, the result follows.

Corollary 2. *(see [12], Theorem 3.1) Let $(E_\varpi, \varpi^\varphi_\lambda)$ be a complete JS-modular MS and $\kappa : E_\varpi \longrightarrow E_\varpi$ be an ϖ^φ_λ-contraction mapping, i.e*

$$\varpi^\varphi_\lambda(\kappa c, \kappa d) \leq k\varpi^\varphi_\lambda(c, d) \qquad \forall c, d \in E_\varpi,\ k \in (0, 1).$$

Then, κ has a unique FP (say u) and κ is continuous at u.

Proof. Consider Definition 9 and let $\psi(t) = kt$, $\sigma_1 = 0$, $\sigma_2 = 1$ and $s = 1$, we have

$$\varpi^\varphi_\lambda(\kappa c, \kappa d) \leq k(H^s_\kappa(c, d)) = k\varpi^\varphi_\lambda(c, d),$$

which coincides with Theorem 3.1 due to [12] and the result follows.

Example 2. *Let $E = [0, \infty)$, $\sigma : E^2 \longrightarrow \mathbb{R}_+$ and define $\varpi^\varphi_\lambda : (0, \infty) \times E \times E \longrightarrow [0, +\infty]$ by*

$$\varpi^\varphi_\lambda(c, d) = \frac{\sigma(c, d)}{\lambda}\ \forall c, d \in E_\varpi, \qquad \textit{where } \sigma(c, d) = \max\{|c|, |d|\}.$$

Then, $(E_{\varpi}, \varpi_{\lambda}^{\varphi})$ *is a complete* JS*-modular* MS*.*
Consider the mapping $\kappa : E_{\varpi} \longrightarrow E_{\varpi}$ *defined by*

$$\kappa c = \begin{cases} \frac{1}{8}c & c \in [0,1] \\ \dfrac{1}{8} & c > 1 \end{cases} \qquad \forall c \in E.$$

Define the function $\psi : \mathbb{R}_{+} \longrightarrow \mathbb{R}_{+}$ *by* $\psi(t) = \frac{3}{4}t \quad \forall t > 0$*. It is clear that* ψ *is a* (c)*-comparison function. Let* $\sigma_1 = 0, \sigma_2 = 1$*. Then:*
for $s > 0$*, if* $c, d \in [0,1]$ *with* $c > d$*, then*

$$\begin{aligned} \varpi_{\lambda}^{\varphi}(\kappa c, \kappa d) = \frac{\sigma(\kappa c, \kappa d)}{\lambda} &= \frac{\max\{|\kappa c|, |\kappa d|\}}{\lambda} \\ &= \frac{1}{8\lambda}c. \end{aligned}$$

$$\begin{aligned} \psi(H_{\kappa}^{s}(c,d)) = \frac{3}{4}(\varpi_{\lambda}^{\varphi}(c,d)) &= \frac{3}{4}\frac{\sigma(c,d)}{\lambda} \\ &= \frac{3}{4}\frac{\max\{|c|, |d|\}}{\lambda} \\ &= \frac{3}{4\lambda}c. \end{aligned}$$

Hence, $\varpi_{\lambda}^{\varphi}(\kappa c, \kappa d) \leq \psi(H_{\kappa}^{s}(c,d))$*.*
If $c, d \in (1, \infty), \varpi_{\lambda}^{\varphi}(\kappa c, \kappa d) = \dfrac{1}{8\lambda} < \dfrac{3}{4\lambda}c = \psi(H_{\kappa}^{s}(c,d))$
If $c \in [0,1]$ *and* $d > 1$*,*

$$\begin{aligned} \varpi_{\lambda}^{\varphi}(\kappa c, \kappa d) = \frac{\sigma(\kappa c, \kappa d)}{\lambda} &= \frac{\max\{|\kappa c|, |\kappa d|\}}{\lambda} \\ &= \frac{1}{8\lambda} \end{aligned}$$

$$\begin{aligned} \psi(H_{\kappa}^{s}(c,d)) = \frac{3}{4}(\varpi_{\lambda}^{\varphi}(c,d)) &= \frac{3}{4}\frac{\sigma(c,d)}{\lambda} \\ &= \frac{3}{4}\frac{\max\{|c|, |d|\}}{\lambda} \\ &= \frac{3}{4\lambda}d \end{aligned}$$

Hence, $\varpi_{\lambda}^{\varphi}(\kappa c, \kappa d) \leq \psi(H_{\kappa}^{s}(c,d))$*.*
For $s = 0$*,* $\psi(H_{\kappa}^{s}(c,d)) = \varpi_{\lambda}^{\varphi}(d, \kappa c)$*. If* $c, d \in [0,1]$ *with* $c < d$*, then,*

$$\varpi_{\lambda}^{\varphi}(\kappa c, \kappa d) = \frac{\sigma(\kappa c, \kappa d)}{\lambda} = \frac{\max\{|\kappa c|, |\kappa d|\}}{\lambda}$$

$$= \frac{1}{8\lambda}d.$$

$$\begin{aligned}\psi(H_\kappa^s(c,d)) = \frac{3}{4}(\varpi_\lambda^\varphi(d,\kappa d)) &= \frac{3}{4}\frac{\sigma(d,\kappa d)}{\lambda}\\ &= \frac{3}{4}\frac{\max\{|d|,|\kappa d|\}}{\lambda}\\ &= \frac{3}{4\lambda}d.\end{aligned}$$

Hence, $\varpi_\lambda^\varphi(\kappa c,\kappa d) \le \psi(H_\kappa^s(c,d))$.
For $c, d \in (1,\infty)$,

$$\begin{aligned}\varpi_\lambda^\varphi(\kappa c,\kappa d) = \frac{\sigma(\kappa c,\kappa d)}{\lambda} &= \frac{\max\{\frac{1}{8},\frac{1}{8}\}}{\lambda}\\ &= \frac{1}{8\lambda}.\end{aligned}$$

$$\begin{aligned}\psi(H_\kappa^s(c,d)) = \frac{3}{4}(\varpi_\lambda^\varphi(d,\kappa d)) &= \frac{3}{4}\frac{\sigma(d,\kappa d)}{\lambda}\\ &= \frac{3}{4}\frac{\max\{|d|,|\kappa d|\}}{\lambda}\\ &= \frac{3}{4\lambda}d.\end{aligned}$$

Hence, $\varpi_\lambda^\varphi(\kappa c,\kappa d) \le \psi(H_\kappa^s(c,d))$.
If $c \in [0,1]$ *and* $d > 1$,

$$\begin{aligned}\varpi_\lambda^\varphi(\kappa c,\kappa d) = \frac{\sigma(\kappa c,\kappa d)}{\lambda} &= \frac{\max\{|\kappa c|,|\kappa d|\}}{\lambda}\\ &= \frac{1}{8\lambda}.\end{aligned}$$

$$\begin{aligned}\psi(H_\kappa^s(c,d)) = \frac{3}{4}(\varpi_\lambda^\varphi(d,\kappa d)) &= \frac{3}{4}\frac{\sigma(d,\kappa d)}{\lambda}\\ &= \frac{3}{4}\frac{\max\{|d|,|\kappa d|\}}{\lambda}\\ &= \frac{3}{4\lambda}d.\end{aligned}$$

Hence, $\varpi_\lambda^\varphi(\kappa c,\kappa d) \le \psi(H_\kappa^s(c,d))$. *Therefore, all the assumptions of Theorem 2 are satisfied. Consequently,* κ *has a unique* FP *given by* $c = 0$.

Remark 2. *From Example 2,* ϖ_λ^φ *is not a modular metric since* $c = d \nrightarrow \varpi_\lambda^\varphi(c,d) = 0$, *and therefore, Theorem 3.2 due to Mongkolkeha et al. [19] cannot be applied to this example.*

Furthermore, Karapinar and Fulga [5] supposed (Definition 6) that c and d are distinct, since $J^s_\kappa(c,d)$ is undefined for Case (1) if $c = d$. However, $c = d \nrightarrow \varpi^\varphi_\lambda(c,d) = 0$ in our definition. Therefore, our HCJT is not Jaggi-type contraction defined by Karapinar and Fulga [5], and so Theorem 1 due to Karapinar and Fulga [5] cannot be applied to this example.

4. An Application to Integral Equations

In this section, one of our obtained results is applied to examine the existence criteria for a solution to a class of integral equations. Accordingly, consider the integral equation (IE)

$$g(c) = \int_\beta^\gamma K(c,t)f(t,g(t))dt \qquad t \in [\beta,\gamma]. \tag{7}$$

Let $E_\varpi = C[\beta,\gamma]$ be the set of all continuous real-valued functions. Define $\varpi^\varphi_\lambda : (0,\infty) \times E \times E \longrightarrow [0,+\infty]$ by

$$\varpi^\varphi_\lambda(g,h) = \max_{t\in[\beta,\gamma]} \frac{|g(t)+h(t)|}{\lambda}, \qquad \forall g,h \in E_\varpi, t \in [\beta,\gamma]$$

Then, $(E_\varpi, \varpi^\varphi_\lambda)$ is a complete modified JS-modular MS with $\varphi(t) = t$. Define a mapping $\kappa : E_\varpi \longrightarrow E_\varpi$ by

$$\kappa g(c) = \int_\beta^\gamma K(c,t)f(t,g(t))dt \qquad t \in [\beta,\gamma], \lambda > 0. \tag{8}$$

Then p is said to be an FP of κ if and only if p is a solution of (8). Now, we study existence conditions of IE (7) under the following hypotheses:

Theorem 4. *Suppose that the following conditions are satisfied:*

i) $K : [\beta,\gamma] \times [\beta,\gamma] \longrightarrow \mathbb{R}_+$ *and* $f : [\beta,\gamma] \times \mathbb{R} \longrightarrow \mathbb{R}$ *are continuous;*

ii) $\forall g,h \in E_\varpi, t \in [\beta,\gamma]$, *we have* $|f(t,g(t)) + f(t,h(t))| \leq |g(t) - h(t)|$;

iii) $\max_{t\in[\beta,\gamma]} \int_\beta^\gamma K(c,t)dt \leq \mu$ *for some* $\mu < 1$.

Then the IE (7) has a solution p in E_ϖ.

Proof. Observe that for any $g,h \in E_\varpi$, using (8) and the above hypotheses, we obtain

$$\begin{aligned} |\kappa g(c) + \kappa h(c)| &= \left| \int_\beta^\gamma k(c,t)f(t,g(t))dt + \int_\beta^\gamma k(c,t)f(t,h(t))dt \right| \\ &\leq \int_\beta^\gamma k(c,t)|f(t,g(t)) + f(t,h(t))|dt \end{aligned}$$

$$
\begin{aligned}
&\leq \int_{\beta}^{\gamma} k(c,t)|g(t)+h(t)|dt \\
&\leq \int_{\beta}^{\gamma} k(c,t) \max|g(t)+h(t)|dt \\
&\leq \alpha \max_{t\in[\beta,\gamma]} |g(t)+h(t)|.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\varpi_{\lambda}^{\varphi}(\kappa g, \kappa h) &\leq \alpha \max_{t\in[\beta,\gamma]} \frac{|g(t)+h(t)|}{\lambda} \\
&= \alpha \varpi_{\lambda}^{\varphi}(g,h).
\end{aligned}
$$

Therefore, all the required assumptions of Corollary 1 are satisfied. This implies that there exists a solution p in E_{ϖ} of the IE (7). Conversely, if p is a solution of (8), then p is also a solution of (7) so that $\kappa p = p$, that is, p is an FP of κ.

5. Conclusion

This paper introduced and explored a new type of contraction, called the Jaggi-type hybrid contraction, within the framework of modified JS-modular metric spaces. We established conditions under which such contractions have unique fixed points. Additionally, we showed that even when only certain iterates of a mapping are continuous, FP can still be guaranteed. A few corollaries demonstrated how our approach generalizes and strengthens existing results in the literature. To further support our findings, we provided a detailed example and applied the main result to a class of IEs, showing its practical usefulness.

Declaration

Availability of data and materials

Not applicable

Competing interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Authors' contributions

Conceptualization: Abdulaziz Zakariyya, Ahmad Aloqaily and Mohammed Shehu Shagari; Formal analysis: Ghada Ali Basendwah, and Ahmad Aloqaily; Investigation: Mohammed Shehu Shagari and Nabil Mlaiki; Methodology: Ghada Ali Basendwah and Nabil Mlaiki; Writing original draft: Abdulaziz Zakariyya, Mohammed Shehu Shagari; Writing, review and editing: Mohammed Shehu Shagari and Abdulaziz Zakariyya. All authors reviewed the manuscript.

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