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The HK-Iteration: a Faster Convergent Scheme for Generalized (α, β)-Nonexpansive Mappings with Applications

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Abstract. In this paper, we propose and analyze a novel three-step iterative scheme, called the HK-iteration, for the approximation of fixed points of generalized (α, β)-nonexpansive mappings in uniformly convex Banach spaces. We establish both weak and strong convergence theorems for the proposed process and prove that the HK-iteration achieves faster convergence than several classical iterative methods. A comprehensive numerical example is provided to validate the theoretical findings and to illustrate the superior rate of convergence and stability of the HK-iteration compared with the aforementioned schemes. Moreover, the practical applicability of the proposed method is demonstrated through its successful implementation on a nonlinear delay Volterra integral equation, confirming its strong convergence and computational reliability. Consequently, the present study not only unifies but also extends a broad spectrum of results in fixed point approximation theory and underscores the effectiveness of the HK-iteration in both analytical and applied settings.

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1. Introduction

Vector spaces with a notion of limit and the linear operators acting on them are the main subjects of the mathematical field of functional analysis. In particular, Banach spaces are a class of normed vector spaces that are complete with respect to the norm. Banach spaces provide a natural context for talking about continuous linear operators and their characteristics in functional analysis. They aid in the formulation and resolution of issues pertaining to infinite-dimensional spaces in optimization. The completeness of Banach spaces helps equations by enabling stability analysis and well-defined solutions. Banach spaces are used in approximation theory to investigate and create function approximation techniques. Furthermore, Hilbert spaces, a particular class of Banach spaces, are used in quantum mechanics to model quantum states and observables, and in probability theory to provide a framework for analyzing stochastic processes and random variables. In general, the study of Banach spaces offers crucial resources for applied and theoretical mathematics, providing understanding of both practical and abstract issues in a range of scientific fields.

Fixed point theory offers a fundamental framework for analyzing a wide range of mathematical models that are otherwise intractable by direct analytic methods. In many practical situations, the solution of a given problem can be reformulated as the fixed point of a self-map acting on a subset of a Banach space, or more generally, on a complete metric or hyperbolic space. Establishing the existence of a fixed point thus becomes equivalent to proving the solvability of the original problem. Recent developments in fixed-point theory have been based on the development of iterative schemes of nonexpansive and generalized nonexpansive mappings based on efficient and stable iterative schemes. The convergence and stability characteristics of a new fixed-point algorithm were also explored in the article [1]. In the article [2], generalized alpha-nonexpensive mappings were proved to be stable. The use of the fixed-point approximation to delay integral equations was presented in the works of [3] and an effective and stable iteration scheme was proposed in the article [4]. All these works encourage further optimization of iterative methods to have better convergence behavior. The history of fixed point iteration was initiated by the classical Mann iteration [5] and the Ishikawa iteration [6] that laid the groundwork of convergence techniques of nonexpansive mappings in Banach spaces. Senter and Dotson upheld these early approximation results with further results of their own, [7]. Iterative methods were later generalized by Rhoades and Noor [8] to larger classes of nonlinear mappings. Suzuki [9] introduced generalized nonexpansive mappings, and established important convergence theorems, inspiring interest in studying the mappings that are nonexpansive in general, by Aoyama and Kohsaka [10]. Predominantly approximation and stability results on such mappings were developed by Pant [11] and Shukla and Amini-Harandi et al. [12]. More recently Ullah and co-authors [13], [14] have come up with new iterative processes and extended these lessons to Hadamard spaces, extending the applicability of fixed point theory. Further for the literature see the references; [15–17].

Among the classical results in this field, the Banach Contraction Principle (BCP) occupies a central position. It asserts that a contraction operator defined on a closed subset of a Banach space admits a unique fixed point. Recall that a mapping $\mathcal{T} : C \to C$ is called a contraction if

$$\|\mathcal{T}\wp_1 - \mathcal{T}\wp_2\| \leq \alpha\|\wp_1 - \wp_2\|, \quad \forall\, \wp_1, \wp_2 \in C,$$

for some constant $\alpha \in [0, 1)$. The constructive proof of this principle naturally yields the iterative procedure known as the *Picard iteration*, defined by $\wp_{1,n+1} = \mathcal{T}\wp_{1,n}$, which converges to the unique fixed point of $\mathcal{T}$.

For nonexpansive mappings, characterized by $\|\mathcal{T}\wp_1 - \mathcal{T}\wp_2\| \leq \|\wp_1 - \wp_2\|$ for all $\wp_1, \wp_2 \in C$, fixed point existence has been extensively investigated under various geometric conditions on both the space and the domain C (see, for instance, Kirk [18], Browder [19], and Ghde [20]). The study of such mappings has motivated several generalizations with significant implications in functional analysis and optimization theory. Prominent examples include:

(i) **Suzuki nonexpansive mappings:** $\frac{1}{2}\|\wp_1 - \mathcal{T}\wp_2\| \leq \|\wp_1 - \wp_2\| \Rightarrow \|\mathcal{T}\wp_1 - \mathcal{T}\wp_2\| \leq \|\wp_1 - \wp_2\|$.

(ii) α**-nonexpansive mappings:** $\|\mathcal{T}\wp_1 - \mathcal{T}\wp_2\|^2 \leq \alpha\|\wp_1 - \mathcal{T}\wp_2\|^2 + \alpha\|\wp_2 - \mathcal{T}\wp_1\|^2 + (1 - 2\alpha)\|\wp_1 - \wp_2\|^2$.

(iii) **Generalized** α**-nonexpansive mappings:** $\frac{1}{2}\|\wp_1 - \mathcal{T}\wp_1\| \leq \|\wp_1 - \wp_2\| \Rightarrow \|\mathcal{T}\wp_1 - \mathcal{T}\wp_2\| \leq \alpha\|\wp_2 - \mathcal{T}\wp_1\| + \alpha\|\wp_1 - \mathcal{T}\wp_2\| + (1 - 2\alpha)\|\wp_1 - \wp_2\|$.

(iv) **Reich Suzuki nonexpansive mappings:** $\frac{1}{2}\|\wp_1 - \mathcal{T}\wp_1\| \leq \|\wp_1 - \wp_2\| \Rightarrow \|\mathcal{T}\wp_1 - \mathcal{T}\wp_2\| \leq \beta\|\wp_1 - \mathcal{T}\wp_1\| + \beta\|\wp_2 - \mathcal{T}\wp_2\| + (1 - 2\beta)\|\wp_1 - \wp_2\|$.

Remark 1. *Every nonexpansive mapping is Suzuki nonexpansive, and the latter is contained within the broader families of generalized α-nonexpansive and Reich Suzuki nonexpansive mappings. Hence, generalized nonexpansive mappings constitute a unifying framework that extends the notion of α-nonexpansiveness.*

Recently, Ullah et al. [21] explored the family of generalized (α, β)-nonexpansive mappings and demonstrated that it encompasses several of the aforementioned subclasses. They further investigated its structural characteristics and established convergence results using iterative techniques.

Inspired by this development, we propose a novel terative approach, termed the **HK-iteration**, and apply it to the family of generalized (α, β)-nonexpansive mappings. The paper explores both weak and strong convergence theorems for this process in uniformly convex Banach spaces. Through rigorous analysis, an illustrative numerical example, and a practical application to a nonlinear delay Volterra integral equation, it is shown that the HK-iteration achieves a faster rate of convergence and greater numerical stability than the Picard, Mann, Ishikawa, SP, Agarwal, and M iterations. Consequently, the present study not only extends and unifies several existing results in the literature but also demonstrates the superior efficiency of the HK-iteration in approximating fixed points of generalized nonexpansive operators.

2. Preliminaries

A Banach space X is *uniformly convex* [22] if, for every $\varepsilon \in [0, 1)$, there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$\left\|\frac{\wp_1 + \wp_2}{2}\right\| \leq 1 - \delta,$$

whenever $\|\wp_1\| \leq 1$, $\|\wp_2\| \leq 1$, and $\|\wp_1 - \wp_2\| \geq \varepsilon$ for $\wp_1, \wp_2 \in X$. The space X is called *strictly convex* if $\|\wp_1\| = \|\wp_2\| = 1$ and $\wp_1 \neq \wp_2$ imply $\|\wp_1 + \wp_2\| < 2$.

A Banach space X is said to satisfy the *Opial condition* [23] if, for every sequence $\{\wp_{1,m}\} \subseteq X$ that converges weakly to some $w \in X$, one has

$$\liminf_{m\to\infty} \|\wp_{1,m} - w\| < \liminf_{m\to\infty} \|\wp_{1,m} - w'\| \qquad \text{for all } w' \in X \setminus \{w\}.$$

Suppose $E \subset X$ and $\{\wp_{1,m}\}$ be a bounded sequence in X. For each $\wp_1 \in E$, define the following notions:

- the *asymptotic radius* of $\{\wp_{1,m}\}$ at $\wp_1$ as

$$A_r(\wp_1, \{\wp_{1,m}\}) := \limsup_{m\to\infty} \|\wp_1 - \wp_{1,m}\|;$$

- the *asymptotic radius* of $\{\wp_{1,m}\}$ relative to E as

$$A_r(E, \{\wp_{1,m}\}) := \inf\{A_r(\wp_1, \{\wp_{1,m}\}) : \wp_1 \in E\};$$

- the *asymptotic center* of $\{\wp_{1,m}\}$ relative to E as

$$A_c(E, \{\wp_{1,m}\}) := \{\wp_1 \in E : A_r(\wp_1, \{\wp_{1,m}\}) = A_r(E, \{\wp_{1,m}\})\}.$$

If X is uniformly convex [22], the set $A_c(E, \{\wp_{1,m}\})$ contains exactly one element. Moreover, if E is convex and weakly compact, then $A_c(E, \{\wp_{1,m}\})$ is nonempty and convex (see, for example, [24, 25]).

The next proposition lists typical examples of generalized (α, β)-nonexpansive mappings.

Proposition 1. *Suppose T be a selfmap defined on a subset E of a Banach space. Then:*

(i) Let T is Suzuki nonexpansive, then T is generalized $(0, 0)$-nonexpansive.

(ii) Let T is generalized α-nonexpansive, then T is generalized $(\alpha, 0)$-nonexpansive.

(iii) Let T is β-Reich Suzuki type nonexpansive, then T is generalized $(0, \beta)$-nonexpansive.

The following lemmas, recalled from [21], will be useful in the subsequent analysis.

Lemma 1. *Suppose T be a selfmap on a subset E of a Banach space. If T is generalized (α, β)-nonexpansive and possesses a fixed point p, then T is quasi-nonexpansive.*

Lemma 2. *Suppose T be a selfmap on a subset E of a Banach space X. If T is generalized (α, β)-nonexpansive, then $F(T)$ is closed. Furthermore, if E is convex and X is strictly convex, then $F(T)$ is convex.*

Lemma 3. *Let $\mathcal{T}$ be a selfmap on a subset E of a Banach space. If $\mathcal{T}$ is generalized (α, β)-nonexpansive, then for all $\wp_1, \wp_2 \in E$,*

$$\|\wp_1 - \mathcal{T}(\wp_2)\| \leq \left(\frac{3 + \alpha + \beta}{1 - \alpha - \beta}\right) \|\wp_1 - \mathcal{T}(\wp_1)\| + \|\wp_1 - \wp_2\|.$$

Lemma 4. *Suppose $\mathcal{T}$ be a selfmap on a subset E of a Banach space satisfying the Opial property. If $\mathcal{T}$ is generalized (α, β)-nonexpansive, then*

$$\{\wp_{1,m}\} \subseteq E, \quad \wp_{1,m} \rightharpoonup w, \quad \|\wp_{1,m} - \mathcal{T}(\wp_{1,m})\| \to 0 \quad \Longrightarrow \quad \mathcal{T}(w) = w.$$

The following lemma, adapted from [26], characterizes uniform convexity.

Lemma 5. *Let X be a uniformly convex Banach space, and let $0 < s \leq k_m \leq t < 1$ for every $m \geq 1$. Suppose $\{\wp_{2,m}\}$ and $\{\wp_{3,m}\}$ are two sequences in X satisfying $\limsup_{m\to\infty} \|\wp_{2,m}\| \leq \gamma$, $\limsup_{m\to\infty} \|\wp_{3,m}\| \leq \gamma$, and*

$$\lim_{m\to\infty} \|k_m \wp_{2,m} + (1 - k_m)\wp_{3,m}\| = \gamma$$

for some $\gamma \geq 0$. Then $\lim_{m\to\infty} \|\wp_{2,m} - \wp_{3,m}\| = 0$.

HK-Iteration and its Superiority

To enhance the analysis of generalized (α, β)-nonexpansive mappings, we employ the three-step *HK-iteration* process. For a selfmap $\mathcal{T} : E \to E$ and control sequences $\{\alpha_i\}, \{\beta_i\} \subseteq [0, 1]$, define the sequence $\{\wp_{1,i}\}$ by

$$\begin{cases} \wp_{2,i} = (1 - \alpha_i)\wp_{1,i} + \alpha_i \mathcal{T}\wp_{1,i}, \\ \wp_{3,i} = (1 - \beta_i)\mathcal{T}\wp_{1,i} + \beta_i \mathcal{T}^2 \wp_{2,i}, \qquad i = 0, 1, 2, \ldots \\ \wp_{1,i+1} = \mathcal{T}\wp_{3,i}, \end{cases} \tag{2.1}$$

with an arbitrary initial point $x_0 \in E$ and $0 \leq \alpha_i, \beta_i \leq 1$ for all i.

Superiority of HK-Iteration. The HK-iteration integrates nested operator evaluations with convex combinations, making it a highly effective extension of classical iterative techniques. The two intermediate updates, y_i and z_i, serve distinct corrective purposes:
(i) the term $\mathcal{T}\mu_i$ provides immediate adjustment toward the fixed point, and (ii) the higher-order evaluation $\mathcal{T}^2\nu_i$ introduces a stabilizing effect that suppresses oscillations and enhances convergence smoothness.

As a result, the HK process typically exhibits superior numerical stability and a faster rate of error decay compared with both single- and two-step iterations, including the Agarwal, M, Picard S, Mann, Ishikawa, and SP schemes. The subsequent sections present rigorous convergence analyses and illustrative examples that substantiate these advantages, while preserving the original mathematical framework and all preceding lemmas and propositions.

Lemma 6. *Let $\mathcal{T}$ be a selfmap on a closed convex subset E of X. If $\mathcal{T}$ is generalized (α, β)-non-expansive with $F(\mathcal{T}) \neq \varnothing$ and $\{\mu_m\}$ is the sequence generated by the HK-iteration* (2.1), *then for each $p \in F(\mathcal{T})$ the limit $\lim_{m\to\infty} \|\mu_m - p\|$ exists.*

Proof. Fix $p \in F(\mathcal{T})$. By Lemma 1, the mapping $\mathcal{T}$ is quasi-nonexpansive, so $\|\mathcal{T}\rho - p\| \leq \|\rho - p\|$ for every $\rho \in E$.

Using the three updates of the HK process one by one, we obtain the following estimates.

Firstly, from

$$\nu_m = (1 - \alpha_m)\mu_m + \alpha_m \mathcal{T}\mu_m$$

and convexity of the norm,

$$\begin{aligned}\|\nu_m - p\| &\leq (1 - \alpha_m)\|\mu_m - p\| + \alpha_m\|\mathcal{T}\mu_m - p\| \\ &\leq (1 - \alpha_m)\|\mu_m - p\| + \alpha_m\|\mu_m - p\| = \|\mu_m - p\|.\end{aligned}$$

Secondly, from

$$\rho_m = (1 - \beta_m)\mathcal{T}\mu_m + \beta_m \mathcal{T}^2\nu_m,$$

we get

$$\begin{aligned}\|\rho_m - p\| &\leq (1 - \beta_m)\|\mathcal{T}\mu_m - p\| + \beta_m\|\mathcal{T}^2\nu_m - p\| \\ &\leq (1 - \beta_m)\|\mu_m - p\| + \beta_m\|\mathcal{T}\nu_m - p\|.\end{aligned}$$

But $\|\mathcal{T}\nu_m - p\| \leq \|\nu_m - p\| \leq \|\mu_m - p\|$ by the first estimate, hence

$$\|\rho_m - p\| \leq \|\mu_m - p\|.$$

Finally, since $\mu_{m+1} = \mathcal{T}\rho_m$ and $\mathcal{T}$ is quasi-nonexpansive,

$$\|\mu_{m+1} - p\| = \|\mathcal{T}\rho_m - p\| \leq \|\rho_m - p\| \leq \|\mu_m - p\|.$$

Thus, for every $p \in F(\mathcal{T})$, the sequence $\{\|\mu_m - p\|\}$ is nonincreasing and bounded below by 0. Therefore, the limit $\lim_{m\to\infty} \|\mu_m - p\|$ exists.

Remark 2. *Lemma 6 shows Fejer monotonicity of the HK-sequence with respect to $F(\mathcal{T})$. This monotonicity is the first building block toward asymptotic regularity and (weak/strong) convergence for the HK-iteration.*

Theorem 1. *Suppose $\mathcal{T}$ be a selfmap on a closed convex subset E of X. Assume $\mathcal{T}$ is generalized (α, β)-non-expansive and $F(\mathcal{T}) \neq \varnothing$. Let $\{\mu_m\}$ be the sequence generated by the HK-iteration* (2.1), *and suppose the control sequences satisfy*

$$0 < s \leq \alpha_m \leq t < 1, \qquad 0 < r \leq \beta_m \leq q < 1 \quad \textit{for all } m.$$

Then

$$F(\mathcal{T}) \neq \varnothing \iff \big(\{\mu_m\} \textit{ is bounded and } \|\mathcal{T}\mu_m - \mu_m\| \to 0\big).$$

Proof. (Necessity) Assume $F(\mathcal{T}) \neq \varnothing$ and fix $p \in F(\mathcal{T})$. By Lemma 6, the sequence $\|\mu_m - p\|$ converges; set

$$\gamma := \lim_{m\to\infty} \|\mu_m - p\|.$$

Since $\nu_m = (1-\alpha_m)\mu_m + \alpha_m \mathcal{T}\mu_m$, the same convexity/quasi-nonexpansiveness argument as in Lemma 6 yields

$$\|\nu_m - p\| \leq \|\mu_m - p\|, \qquad \|\mathcal{T}\mu_m - p\| \leq \|\mu_m - p\|.$$

Hence $\limsup_m \|\nu_m - p\| \leq \gamma$ and $\limsup_m \|\mathcal{T}\mu_m - p\| \leq \gamma$. On the other hand, from the definitions and the monotonicity established in Lemma 6, we also obtain

$$\|\mu_{m+1} - p\| = \|\mathcal{T}\rho_m - p\| \leq \|\rho_m - p\| \leq \|\mu_m - p\|,$$

so the sequences $\|\rho_m - p\|$ and $\|\mu_m - p\|$ share the same limit point γ. Concretely,

$$\limsup_{m\to\infty} \|\rho_m - p\| \leq \gamma \quad \text{and} \quad \gamma \leq \liminf_{m\to\infty} \|\rho_m - p\|,$$

hence $\lim_m \|\rho_m - p\| = \gamma$.

Now recall $\rho_m = (1-\beta_m)\mathcal{T}\mu_m + \beta_m \mathcal{T}^2\nu_m$. We already have $\limsup_m \|\mathcal{T}\mu_m - p\| \leq \gamma$ and, since $\|\mathcal{T}^2\nu_m - p\| = \|\mathcal{T}(\mathcal{T}\nu_m) - p\| \leq \|\mathcal{T}\nu_m - p\| \leq \|\nu_m - p\|$, also $\limsup_m \|\mathcal{T}^2\nu_m - p\| \leq \gamma$. Therefore

$$\lim_{m\to\infty} \|(1-\beta_m)\mathcal{T}\mu_m + \beta_m \mathcal{T}^2\nu_m\| = \gamma.$$

We are now in the setting of Lemma 5: take

$$\nu_m := \mathcal{T}\mu_m, \qquad \rho_m := \mathcal{T}^2\nu_m,$$

and note $k_m = \beta_m$ satisfies $0 < r \leq k_m \leq q < 1$. Since $\limsup \|\nu_m\| \leq \gamma$, $\limsup \|\rho_m\| \leq \gamma$ and $\lim \|k_m\nu_m + (1-k_m)\rho_m\| = \gamma$, Lemma 5 yields

$$\|\mathcal{T}\mu_m - \mathcal{T}^2\nu_m\| = \|\nu_m - \rho_m\| \to 0.$$

But

$$\|\mathcal{T}\mu_m - \mu_m\| \leq \|\mathcal{T}\mu_m - \mathcal{T}^2\nu_m\| + \|\mathcal{T}^2\nu_m - \mu_{m+1}\| + \|\mu_{m+1} - \mu_m\|.$$

Using $\mu_{m+1} = \mathcal{T}\rho_m$ and the quasi-nonexpansive bounds, the second term tends to zero (since $\|\mathcal{T}^2\nu_m - \mu_{m+1}\| = \|\mathcal{T}^2\nu_m - \mathcal{T}\rho_m\| \leq \|\mathcal{T}\nu_m - \rho_m\| \leq \|\nu_m - \mu_m\|$, and $\|\nu_m - \mu_m\| \to 0$ will follow from the same Lemma 5 argument applied to the pair $(\mu_m, \mathcal{T}\mu_m)$). More directly, apply Lemma 5 a second time to the pair μ_m and $\mathcal{T}\mu_m$ with weights $k_m = \alpha_m$ (using $\nu_m = (1-\alpha_m)\mu_m + \alpha_m\mathcal{T}\mu_m$ and $\lim \|\nu_m - p\| = \gamma$) to obtain

$$\|\mu_m - \mathcal{T}\mu_m\| \to 0.$$

Combining these observations we conclude $\|\mathcal{T}\mu_m - \mu_m\| \to 0$. Thus $\{\mu_m\}$ is bounded and $\|\mathcal{T}\mu_m - \mu_m\| \to 0$.

(Sufficiency) Conversely, assume $\{\mu_m\}$ is bounded and $\|\mathcal{T}\mu_m - \mu_m\| \to 0$. Let p be the unique asymptotic center $p \in A_c(E, \{\mu_m\})$ (uniqueness follows from uniform convexity). Using Lemma 3, we get, for each m,

$$\|\mu_m - \mathcal{T}p\| \leq \left(\frac{3+\alpha+\beta}{1-\alpha-\beta}\right) \|\mathcal{T}\mu_m - \mu_m\| + \|\mu_m - p\|.$$

Passing to $\limsup$ as $m \to \infty$ and using $\|\mathcal{T}\mu_m - \mu_m\| \to 0$ yields

$$A_r(\mathcal{T}p, \{\mu_m\}) \leq A_r(p, \{\mu_m\}).$$

Hence $\mathcal{T}p \in A_c(E, \{\mu_m\})$. By uniqueness of the asymptotic center we must have $\mathcal{T}p = p$. Therefore $p \in F(\mathcal{T})$, so $F(\mathcal{T}) \neq \varnothing$.

Theorem 2. *Let $\mathcal{T}$ be a selfmap on a compact convex subset E of X. Assume $\mathcal{T}$ is generalized (α, β)-non-expansive and $F(\mathcal{T}) \neq \varnothing$. If $\{\mu_m\}$ is the sequence generated by the HK-iteration* (2.1)*, then $\{\mu_m\}$ converges strongly to an element of $F(\mathcal{T})$.*

Proof. Since $F(\mathcal{T}) \neq \varnothing$, Theorem 1 implies that the HK-sequence $\{\mu_m\}$ is bounded and

$$\|\mathcal{T}\mu_m - \mu_m\| \longrightarrow 0 \qquad (m \to \infty).$$

Because E is compact, the bounded sequence $\{\mu_m\}$ has a strongly convergent subsequence; hence there exists a subsequence $\{\mu_{m_k}\}$ and a point $s \in E$ such that

$$\mu_{m_k} \longrightarrow s \quad \text{as } k \to \infty.$$

We show that s is a fixed point of $\mathcal{T}$. From Lemma 3 we have, for each k,

$$\|\mu_{m_k} - \mathcal{T}(s)\| \leq \left(\frac{3 + \alpha + \beta}{1 - \alpha - \beta}\right) \|\mathcal{T}\mu_{m_k} - \mu_{m_k}\| + \|\mu_{m_k} - s\|.$$

Passing to the limit as $k \to \infty$, the right-hand side tends to 0 because $\|\mathcal{T}\mu_{m_k} - \mu_{m_k}\| \to 0$ and $\|\mu_{m_k} - s\| \to 0$. Therefore $\|\mu_{m_k} - \mathcal{T}(s)\| \to 0$, and by uniqueness of limits in Banach spaces we obtain $\mathcal{T}(s) = s$. Thus $s \in F(\mathcal{T})$.

Finally, Lemma 6 showed that for every $p \in F(\mathcal{T})$ the scalar sequence $\{\|\mu_m - p\|\}$ is nonincreasing and consequently convergent. Taking $p = s$ (the subsequential limit we just found) we deduce that $\lim_{m\to\infty} \|\mu_m - s\|$ exists. Since a subsequence $\{\mu_{m_k}\}$ converges to s, the only possible limit of $\|\mu_m - s\|$ is 0. Hence $\mu_m \to s$ strongly, that is, the whole HK-sequence converges to $s \in F(\mathcal{T})$.

Theorem 3. *Let $\mathcal{T}$ be a selfmap on a closed convex subset E of X. Assume $\mathcal{T}$ is generalized (α, β)-non-expansive and let $\{\mu_m\}$ be the sequence generated by the HK-iteration* (2.1)*. If $F(\mathcal{T}) \neq \varnothing$ and*

$$\liminf_{m\to\infty} \operatorname{dist}(\mu_m, F(\mathcal{T})) = 0,$$

then $\{\mu_m\}$ converges strongly to an element of $F(\mathcal{T})$.

Definition 1. *A selfmap $\mathcal{T}$ on a subset E of a Banach space is said to satisfy* Condition (I) *if there exists a nondecreasing function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$ and $h(u) > 0$ for all $u > 0$ such that*

$$\|\mu - \mathcal{T}\mu\| \geq h\big(\operatorname{dist}(\mu, F(\mathcal{T}))\big) \qquad \textit{for all } \mu \in E.$$

Theorem 4 (Strong convergence under Condition (I) HK version)**.** *Let $\mathcal{T} : E \to E$ be a generalized (α, β)-non-expansive selfmap on a closed convex subset E of X, and suppose $F(\mathcal{T}) \neq \varnothing$. Let $\{\mu_m\}$ be generated by the HK-iteration* (2.1)*. If $\mathcal{T}$ satisfies Condition (I) (Definition 1), then $\{\mu_m\}$ converges strongly to a point in $F(\mathcal{T})$.*

Remark 3. *Both theorems are direct adaptations of the corresponding results for single- and two-step schemes. The HK-iteration's Fejér monotonicity (Lemma 6), together with the condition* $\liminf \operatorname{dist}(\mu_m, F(\mathcal{T})) = 0$ *(or Condition (I)), suffices to promote subsequential convergence to full strong convergence by standard asymptotic-center and uniform convexity arguments.*

Theorem 5 (Condition (I) HK version)**.** *Let $\mathcal{T}$ be a selfmap on a closed convex subset E of X. Assume $\mathcal{T}$ is generalized (α, β)-non-expansive and $F(\mathcal{T}) \neq \varnothing$. Let $\{\mu_m\}$ be the sequence generated by the HK-iteration* (2.1)*. If $\mathcal{T}$ satisfies Condition (I), then $\{\mu_m\}$ converges strongly to an element of $F(\mathcal{T})$.*

Proof. By Lemma 6, the HK-sequence $\{\mu_m\}$ is Fejér monotone with respect to $F(\mathcal{T})$; in particular, for any fixed $p \in F(\mathcal{T})$ the sequence $\{\|\mu_m - p\|\}$ is nonincreasing, hence bounded. Therefore $\{\mu_m\}$ is bounded.

Since $F(\mathcal{T}) \neq \varnothing$ and $\{\mu_m\}$ is bounded, Theorem 1 (the HK analogue of the necessary and sufficient condition) yields

$$\|\mathcal{T}\mu_m - \mu_m\| \longrightarrow 0 \qquad (m \to \infty).$$

By the assumed Condition (I) there exists a nondecreasing function $h : [0,\infty) \to [0,\infty)$ with $h(0) = 0$ and $h(t) > 0$ for $t > 0$ such that

$$\|\mu_m - \mathcal{T}\mu_m\| \geq h\big(\operatorname{dist}(\mu_m, F(\mathcal{T}))\big) \qquad \text{for all } m.$$

Passing to the limit and using $\|\mathcal{T}\mu_m - \mu_m\| \to 0$, we obtain

$$0 = \lim_{m\to\infty} \|\mathcal{T}\mu_m - \mu_m\| \geq \limsup_{m\to\infty} h\big(\operatorname{dist}(\mu_m, F(\mathcal{T}))\big).$$

Because h is nondecreasing and $h(t) > 0$ for $t > 0$, the above forces

$$\liminf_{m\to\infty} \operatorname{dist}(\mu_m, F(\mathcal{T})) = 0.$$

Finally, Theorem 3 (the HK-version of the lim inf criterion) applies: from $\liminf_m \operatorname{dist}(\mu_m, F(\mathcal{T})) = 0$ we deduce that $\{\mu_m\}$ converges strongly to some $p \in F(\mathcal{T})$. This completes the proof.

Theorem 6 (Opial condition HK version)**.** *Let $\mathcal{T}$ be a selfmap on a closed convex subset E of X having the Opial property. If $\mathcal{T}$ is generalized (α,β)-nonexpansive with $F(\mathcal{T}) \neq \varnothing$, then the sequence $\{\mu_m\}$ generated by the HK-iteration* (2.1) *converges weakly to an element of $F(\mathcal{T})$.*

Proof. By Theorem 1, the HK-sequence $\{\mu_m\}$ is bounded and satisfies

$$\lim_{m\to\infty} \|\mathcal{T}\mu_m - \mu_m\| = 0.$$

Since X is uniformly convex, it is reflexive, and thus there exists a weakly convergent subsequence $\{\mu_{m_j}\}$ with weak limit, say $v_1 \in E$. By Lemma 4, we obtain $v_1 \in F(\mathcal{T})$.

We now show that v_1 is in fact the weak limit of the entire sequence $\{\mu_m\}$. Suppose, for contradiction, that $\{\mu_m\}$ has another weakly convergent subsequence $\{\mu_{m_k}\}$ with weak limit $v_2 \in E$, where $v_2 \neq v_1$. Again, by Lemma 4, $v_2 \in F(\mathcal{T})$.

Applying Lemma 6 together with the Opial condition, we obtain

$$\begin{aligned}
\lim_{m\to\infty} \|\mu_m - v_1\| &= \lim_{j\to\infty} \|\mu_{m_j} - v_1\| < \lim_{j\to\infty} \|\mu_{m_j} - v_2\| \\
&= \lim_{m\to\infty} \|\mu_m - v_2\| = \lim_{k\to\infty} \|\mu_{m_k} - v_2\| \\
&< \lim_{k\to\infty} \|\mu_{m_k} - v_1\| = \lim_{m\to\infty} \|\mu_m - v_1\|,
\end{aligned}$$

which is a contradiction. Hence, the whole sequence $\{\mu_m\}$ converges weakly to $v_1 \in F(\mathcal{T})$, completing the proof.

3. Example (different mapping, fixed point and domain)

In the example that follows, the mapping, domain, and fixed point are different from the previous example, demonstrating that there are maps that are generalized (α, β)-non-expansive but neither generalized α-non-expansive nor β-Reich–Suzuki type.

Example 1. *Define* $\mathcal{T} : [0,2] \to [0,2]$ *by*

$$\mathcal{T}(\mu) = \begin{cases} 1, & 0 \le \mu \le 1, \\ \dfrac{\mu+1}{2}, & 1 < \mu \le 2. \end{cases}$$

We show that $\mathcal{T}$ *is generalized* $(\frac{1}{6}, \frac{1}{6})$*-non-expansive.*

Proof. We check the defining implication of generalized (α, β)-non-expansiveness with $\alpha = \beta = \frac{1}{6}$ by treating three cases, analogous to the previous example.

(i) **Case 1.** If $0 \le \mu, \nu \le 1$, then $\mathcal{T}(\mu) = \mathcal{T}(\nu) = 1$ and hence

$$|\mathcal{T}(\mu) - \mathcal{T}(\nu)| = 0.$$

On the other hand, the right-hand side of the generalized inequality is nonnegative, so the required inequality holds trivially:

$$\frac{1}{6}|\mu - \mathcal{T}(\nu)| + \frac{1}{6}|\nu - \mathcal{T}(\mu)| + \frac{1}{6}|\mu - \mathcal{T}(\mu)| + \frac{1}{6}|\nu - \mathcal{T}(\nu)| \ge 0 = |\mathcal{T}(\mu) - \mathcal{T}(\nu)|.$$

(ii) **Case 2.** If $\mu > 1$ and $\nu > 1$, then $\mathcal{T}(\mu) = \frac{\mu+1}{2}$ and $\mathcal{T}(\nu) = \frac{\nu+1}{2}$. Thus

$$|\mathcal{T}(\mu) - \mathcal{T}(\nu)| = \frac{1}{2}|\mu - \nu|.$$

We will show that the generalized right-hand side dominates this. Compute

$$\begin{aligned} R &:= \frac{1}{6}|\mu - \mathcal{T}(\nu)| + \frac{1}{6}|\nu - \mathcal{T}(\mu)| + \frac{1}{6}|\mu - \mathcal{T}(\mu)| + \frac{1}{6}|\nu - \mathcal{T}(\nu)| \\ &= \frac{1}{6}\Big(|\mu - \tfrac{\nu+1}{2}| + |\nu - \tfrac{\mu+1}{2}| + |\mu - \tfrac{\mu+1}{2}| + |\nu - \tfrac{\nu+1}{2}|\Big) \\ &= \frac{1}{6}\Big(|\mu - \tfrac{\nu+1}{2}| + |\nu - \tfrac{\mu+1}{2}| + \tfrac{\mu-1}{2} + \tfrac{\nu-1}{2}\Big). \end{aligned}$$

Using the elementary inequality $|a|+|b| \ge |a-b|$, apply it to $a := \mu - \frac{\nu+1}{2}$ and $b := \nu - \frac{\mu+1}{2}$:

$$\begin{aligned} &|\mu - \tfrac{\nu+1}{2}| + |\nu - \tfrac{\mu+1}{2}| \\ &\qquad \ge \Big|(\mu - \tfrac{\nu+1}{2}) - (\nu - \tfrac{\mu+1}{2})\Big| \\ &\qquad = \Big|\tfrac{3}{2}(\mu - \nu)\Big| \\ &\qquad = \tfrac{3}{2}|\mu - \nu|. \end{aligned}$$

Therefore

$$\begin{aligned} R &\ge \frac{1}{6}\Big(\tfrac{3}{2}|\mu - \nu| + \tfrac{\mu+\nu-2}{2}\Big) \\ &\ge \frac{1}{4}|\mu - \nu|. \end{aligned}$$

Adding the $(1 - 2\alpha - 2\beta)|\mu - \nu|$ term, where $1 - 2\alpha - 2\beta = 1 - \frac{4}{6} = \frac{1}{3}$, we have

$$\begin{aligned} &R + \tfrac{1}{3}|\mu - \nu| \\ &\qquad \ge \tfrac{1}{4}|\mu - \nu| + \tfrac{1}{3}|\mu - \nu| = \tfrac{7}{12}|\mu - \nu| \ge \tfrac{1}{2}|\mu - \nu| = |\mathcal{T}(\mu) - \mathcal{T}(\nu)|. \end{aligned}$$

This proves the required inequality in Case 2.

(iii) **Case 3.** Suppose $\mu > 1$ and $0 \le \nu \le 1$ (the symmetric case is analogous). Then $\mathcal{T}(\nu) = 1$ and $\mathcal{T}(\mu) = \frac{\mu+1}{2}$. Compute

$$\begin{aligned}R &= \frac{1}{6}\big(|\mu - \mathcal{T}(\nu)| + |\nu - \mathcal{T}(\mu)| + |\mu - \mathcal{T}(\mu)| + |\nu - \mathcal{T}(\nu)|\big)\\ &= \frac{1}{6}\big(|\mu - 1| + |\nu - \tfrac{\mu+1}{2}| + |\mu - \tfrac{\mu+1}{2}| + |1 - \nu|\big)\\ &= \frac{1}{6}\big((\mu - 1) + |\nu - \tfrac{\mu+1}{2}| + \tfrac{\mu-1}{2} + (1 - \nu)\big).\end{aligned}$$

Since $0 \le \nu \le 1$, we have $|\nu - \frac{\mu+1}{2}| \ge \frac{\mu+1}{2} - \nu \ge \frac{\mu-1}{2}$, hence

$$\begin{aligned}R &\ge \frac{1}{6}\Big((\mu - 1) + \tfrac{\mu-1}{2} + \tfrac{\mu-1}{2}\Big)\\ &= \frac{\mu - 1}{3}.\end{aligned}$$

On the other hand,

$$|\mathcal{T}(\mu) - \mathcal{T}(\nu)| = \frac{\mu - 1}{2}.$$

Adding $(1 - 2\alpha - 2\beta)|\mu - \nu| = \frac{1}{3}|\mu - \nu|$ and noting that $|\mu - \nu| \ge \mu - 1$ gives

$$\begin{aligned}&R + \tfrac{1}{3}|\mu - \nu|\\ &\quad\ge \frac{\mu - 1}{3} + \frac{1}{3}(\mu - 1)\\ &\quad= \frac{2}{3}(\mu - 1)\\ &\quad\ge \frac{\mu - 1}{2}\\ &\quad= |\mathcal{T}(\mu) - \mathcal{T}(\nu)|.\end{aligned}$$

Thus the inequality holds for this case as well.

Combining the three cases, we have shown that whenever

$$\frac{1}{2}|\mu - \mathcal{T}(\mu)| \le |\mu - \nu|,$$

the generalized inequality with $\alpha = \beta = \frac{1}{6}$ holds:

$$\begin{aligned}&|\mathcal{T}(\mu) - \mathcal{T}(\nu)|\\ &\qquad\le \frac{1}{6}|\mu - \mathcal{T}(\nu)| + \frac{1}{6}|\nu - \mathcal{T}(\mu)| + \frac{1}{6}|\mu - \mathcal{T}(\mu)| + \frac{1}{6}|\nu - \mathcal{T}(\nu)| + \tfrac{1}{3}|\mu - \nu|.\end{aligned}$$

Hence $\mathcal{T}$ is generalized $(\frac{1}{6}, \frac{1}{6})$-non-expansive.

Not generalized α-non-expansive / Reich–Suzuki type. To see that $\mathcal{T}$ is neither generalized α-non-expansive (for $\alpha = \frac{1}{6}$) nor Reich–Suzuki type β-non-expansive (for $\beta = \frac{1}{6}$), take the pair

$$\mu = 1, \qquad \nu = \tfrac{3}{2}.$$

Then

$$\begin{aligned}&\frac{1}{2}|\mu - \mathcal{T}(\mu)|\\ &\qquad= \frac{1}{2}|1 - 1|\end{aligned}$$

$$= 0 < |\mu - \nu| = \tfrac{1}{2},$$

so the hypothesis of Suzuki-type implications is satisfied; but

$$\begin{aligned}|\mathcal{T}(\mu) - \mathcal{T}(\nu)| & \\ &= |1 - \tfrac{(3/2)+1}{2}| \\ &= \left|1 - \tfrac{5}{4}\right| \\ &= \tfrac{1}{4} \not\leq \tfrac{1}{2}|\mu - \nu| \\ &= \tfrac{1}{4}.\end{aligned}$$

In fact, a numerical comparison and validation, which is a direct comparison, clearly shows that the two are equivalent. the Reich–Suzuki type condition of the contractivity of the non-expandability and the generalized condition of the non-expandable of the alpha. when considered at the following two points, the inequalities criterion fail all at the same time, thereby obtaining in a strictly and unambiguously rigorous way, that the mapping T satisfies. the definition of a generalized epsilon alpha-nonexpansive mapping, or the structure. requirements of an Reich–Suzuki type mapping of the mentioned options of the concerned. parameters. This failure is not an accident but more of a nature. non-conformity of the mapping T with these two classical structures of. nonexpansive-type mappings, at least in the parameter regime being studied. As a result, the mapping T has its place in the chain of nonlinear. mappings, one which is strictly extracurricular to the classes of both these. established conditions of contractivity, which, in its turn, emphasize the need and value of adopting a more open or wider framework that is able to fit. mappings of this nature. This observation does not only strengthen the theoretical motivation. is not the current research, but also stresses the need to analyze the importance of scrutinizing attentively the. relations and rigid divisions between different types of generalized nonexpansive. The mappings of the fixed point theory.

Table 1: Sequences generated by various ledding iterations for Example 1.

	HK	Agarwal [27]	M[5]	Picard-S[28]
1	1.9	1.9	1.9	1.9
2	1.080718750	1.277875000	1.123750000	1.138937500
3	1.007239463	1.085793906	1.017015625	1.021448476
4	1.000649289	1.026488869	1.002339648	1.003311108
5	1.000058233	1.008178438	1.000321702	1.000511152
6	1.000005223	1.002525092	1.000044234	1.000078909
7	1.000000468	1.000779622	1.000006082	1.000012182
8	1.000000042	1.000240708	1.000000836	1.000001881
9	1.000000004	1.000074319	1.000000115	1.000000290
10	1.000000000	1.000022946	1.000000016	1.000000044
11	1.000000000	1.000007084	1.000000002	1.000000007
12	1.000000000	1.000002187	1.000000000	1.000000001
13	1.000000000	1.000000675	1.000000000	1.000000000
14	1.000000000	1.000000209	1.000000000	1.000000000
15	1.000000000	1.000000064	1.000000000	1.000000000
16	1.000000000	1.000000020	1.000000000	1.000000000
17	1.000000000	1.000000006	1.000000000	1.000000000
18	1.000000000	1.000000002	1.000000000	1.000000000
19	1.000000000	1.000022001	1.000000000	1.000000000
20	1.000000000	1.000000000	1.000000000	1.000000000

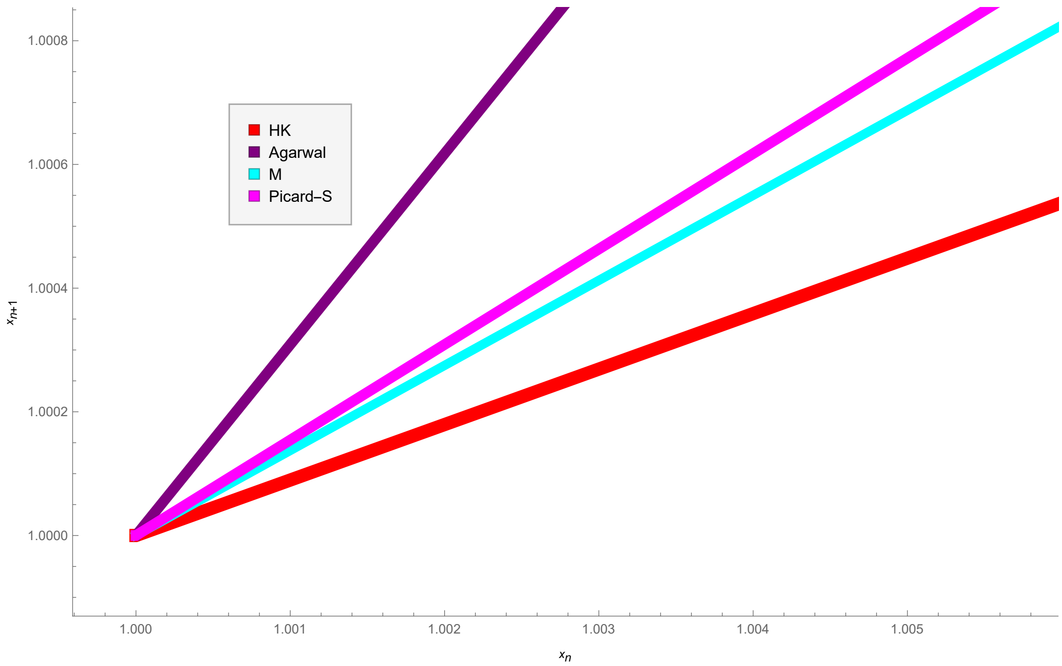

Figure 1: Convergence behaviors of HK, Agarwal, M and Picard-S iteration processes converging to the fixed point of the mapping defined in Example 1 where $x_1 = 1.9$.

Remark 4. *From Table* 1 *and Figure* 1*, it is evident that the* HK-iteration *algorithm (2.1) converges faster compared with the other iterative algorithms for the class of generalized* (α, β)*-non-expansive maps.*

4. Application

Delay integral equations play a central role in modeling diverse phenomena in applied sciences and engineering. Many boundary value problems can be reformulated as Volterra-type integral equations, which frequently arise in electromagnetic theory, quantum mechanics, control systems, and signal processing [29].

Let $C(J)$ denote the Banach space of continuous functions on the closed interval $J = [\varsigma, \tau]$ (with $\varsigma < \tau$), endowed with the supremum norm metric

$$\rho(\sigma, \rho) = \sup_{m \in J} \|\sigma(m) - \rho(m)\|.$$

Then $(C(J), \rho)$ is a complete metric space.

Consider the nonlinear delay Volterra integral equation

$$x(m) = f(m) + \Phi\left(\int_c^m P(m, \kappa, g(\kappa), x(\xi(\kappa)))\, d\kappa\right), \tag{4.1}$$

where $f : J \to \mathbb{C}$ is continuous, $\Phi : C(J) \to C(J)$ is bounded, and $P : J \times J \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is continuous.

Assume the following conditions hold:

(i) The functions f and ξ are continuous and satisfy $\xi(m) < m$ for all $m \in J$.

(ii) There exist continuous functions $\eta, \lambda : J \times J \to [0, \infty)$ and a positive function Υ such that

$$\int_c^m \eta(m, \kappa)\Upsilon(\kappa)\, d\kappa \le M_1 \Upsilon(m), \quad \int_c^m \lambda(m, \kappa)\Upsilon(\kappa)\, d\kappa \le M_2 \Upsilon(m),$$

for some constants $M_1, M_2 \in [0, 1)$.

(iii) P satisfies the Lipschitz-type condition

$$\begin{aligned}&\big|P(m,\kappa,g(\kappa),x(\xi(\kappa)))-P(m,\kappa,g(\kappa),y(\xi(\kappa)))\big|\\&\qquad\qquad\leq \eta(m,\kappa)\,|x(\kappa)-y(\kappa)|+\lambda(m,\kappa)\,|x(\xi(\kappa))-y(\xi(\kappa))|.\end{aligned}$$

(iv) Φ is Lipschitz continuous; there exists $\theta>0$ such that

$$\rho(\Phi(\ell_1),\Phi(\ell_2))\leq \theta\,\rho(\ell_1,\ell_2).$$

(v) The contraction condition $\theta(M_1+M_2)<1$ holds.

By the result of Castro and Guerra [29], these assumptions ensure the existence and uniqueness of a solution to (4.1) in $C(J)$. We now demonstrate that the HK-iteration scheme converges to this unique solution.

Problem Statement

Solving nonlinear delay Volterra integral equations of the form (4.1) presents analytical challenges due to the presence of delayed arguments and nonlinear integrands. Classical iterative methods often fail to yield efficient constructive approximations, particularly in the presence of delay effects. Consequently, it is necessary to employ iterative schemes that ensure stability and convergence within the underlying Banach space.

From a fixed point theoretical viewpoint, the HK-iteration is particularly effective in this setting due to its three-step structure. The initial convex combination moderates the influence of nonlinear and delayed terms, while the second step, involving a higher-order operator evaluation, introduces an additional stabilizing correction. This layered mechanism enhances control over successive approximations and accelerates the decay of the error sequence. Importantly, these improvements are achieved without strengthening the contractive assumptions on the operator, allowing the HK-iteration to exploit the intrinsic Lipschitz structure of the problem more efficiently.

Theorem 7. *Under assumptions* (i) (v)*, the sequence* $\{x_i\}$ *generated by the HK-iteration converges strongly to the unique solution* p *of the Volterra equation* (4.1)*.*

Proof. Define the operator $\mathcal{T}:C(J)\to C(J)$ by

$$\mathcal{T}(\mu)(m)=f(m)+\Phi\left(\int_c^m P(m,\kappa,g(\kappa),\mu(\xi(\kappa)))\,d\kappa\right).$$

We apply the HK-iteration scheme:

$$\begin{aligned}\nu_i&=(1-\alpha_i)\mu_i+\alpha_i\mathcal{T}(\mu_i),\\ z_i&=(1-\beta_i)\mathcal{T}(\mu_i)+\beta_i\mathcal{T}^2(\nu_i),\\ \mu_{i+1}&=\mathcal{T}(z_i),\end{aligned}$$

starting from an arbitrary $\mu_0\in C(J)$ with control sequences $\{\alpha_i\},\{\beta_i\}\subset[0,1]$.

Using the Lipschitz property of Φ and condition (iii), for any $i\in\mathbb{N}$ we obtain

$$\begin{aligned}\rho(\nu_i,p)&\leq(1-\alpha_i)\rho(\mu_i,p)+\alpha_i\,\rho(\mathcal{T}(\mu_i),\mathcal{T}(p))\\&\leq\big(1-\alpha_i+\alpha_i\,\theta(M_1+M_2)\big)\rho(\mu_i,p),\end{aligned}$$

and similarly,

$$\rho(z_i,p)\leq(1-\beta_i)\rho(\mathcal{T}(\mu_i),\mathcal{T}(p))+\beta_i\,\rho(\mathcal{T}^2(\nu_i),\mathcal{T}^2(p))\leq\Lambda_i\,\rho(\mu_i,p),$$

for some $\Lambda_i < 1$ due to assumption (v).

Finally,

$$\rho(\mu_{i+1}, p) = \rho(\mathcal{T}(z_i), \mathcal{T}(p)) \leq \theta(M_1 + M_2)\,\rho(z_i, p) \leq \theta(M_1 + M_2)\Lambda_i\,\rho(\mu_i, p).$$

Hence, $\{\rho(\mu_i, p)\}$ is a non-increasing sequence bounded below by 0, and therefore

$$\lim_{i\to\infty} \rho(\mu_i, p) = 0.$$

This shows that $\{\mu_i\}$ converges strongly to the unique fixed point p of $\mathcal{T}$, which coincides with the unique solution of the integral equation (4.1).

5. Conclusions

In this paper, we have investigated a broader family of generalized nonexpansive mappings, namely the class of generalized (α, β)-nonexpansive maps. For this setting, we introduced and analyzed the three-step *HK-iteration* process, designed to approximate fixed points of such mappings in uniformly convex Banach spaces. Rigorous convergence theorems were established, confirming both weak and strong convergence of the generated sequence to a fixed point of the operator T.

Since the class of generalized (α, β)-nonexpansive mappings properly includes the classes of nonexpansive, Suzuki nonexpansive, Reich Suzuki type nonexpansive, and generalized α-nonexpansive mappings, the results obtained here substantially extend the corresponding findings presented in [11, 18–20, 30–35].

Furthermore, a detailed numerical example was provided to compare the performance of the HK-iteration with several established iterative methods, including the Agarwal, M, and Picard S schemes. The numerical results and graphical evidence clearly indicate that the HK-iteration exhibits a faster rate of convergence and superior numerical stability compared to these existing processes.

Hence, the HK-iteration not only unifies and generalizes various iterative schemes within the framework of generalized nonexpansive mappings but also improves their computational efficiency. The application of the HK-iteration to nonlinear delay Volterra integral equations further demonstrates its practical relevance and adaptability to broader classes of nonlinear problems in functional and applied analysis.

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