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A Simple Bound on the ℓ-Rank of Finite Gyrogroups

Teerapong Suksumran

Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand

Abstract. In this article, we modify the definitions of left and right generating sets for a gyrogroup, a group-like structure whose binary operation is generally non-associative. We then establish an upper bound for the minimal number of left generators of a finite gyrogroup, called the ℓ-rank of G, in terms of its order and the orders of its elements. This leads to the Maximum-Element-Order Problem for gyrogroups: given a finite gyrogroup G, determine the maximum order of elements in G.

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Key Words and Phrases: Generating set, gyrogroup, maximum order, rank, gyro-invariant set.

1. Introduction

The minimal number of generators of a group Γ represents the smallest number of elements needed to generate Γ, denoted by $d(\Gamma)$. This algebraic invariant reflects the structural complexity of a group and plays a significant role in both computational and combinatorial group theory (see, for instance, [1–5]). In particular, evaluating $d(\Gamma)$ is essential for analyzing the efficiency of group presentations and the generative behavior of finitely generated groups. It turns out that this notion can be naturally extended to the framework of gyrogroups, which are group-like structures whose binary operations are generally non-associative [6, 7]. However, studying generating sets for gyrogroups is quite subtle because of the non-associativity of their operations (see, for instance, [8]). In this article, we attempt to reduce the complexity by proving that any left generating set for a gyrogroup can be enlarged to one that is invariant under gyroautomorphisms. This result leads to a simplified description of the subgyrogroup generated by a set. Consequently, several standard results from group theory can be directly extended to the setting of gyrogroups, as we will see in the sequel. For a comprehensive introduction to gyrogroup theory, we refer the reader to [7].

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Email address: `teerapong.suksumran@cmu.ac.th` (T. Suksumran)

2. Preliminaries

The basic theory of gyrogroups can be found in, for instance, [6, 7]. Let G be a non-empty set endowed with a binary operation $\oplus$. The system $(G, \oplus)$ is called a *gyrogroup* if (i) there exists a (unique) element e, called the *identity*, such that $e \oplus a = a$ for all $a \in G$; (ii) for each $a \in G$, there exists a (unique) element $\ominus a$ in G, called the *inverse*, such that $\ominus a \oplus a = e$; (iii) for each pair of elements a and b, there exists a (unique) automorphism $\mathrm{gyr}[a, b]$ of $(G, \oplus)$, called the *gyroautomorphism generated by a and b*, such that $a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b](c)$ for all $c \in G$; and (iv) for all $a, b \in G$, $\mathrm{gyr}[a \oplus b, b] = \mathrm{gyr}[a, b]$. In general, the operation of a gyrogroup is non-associative, and furthermore every group can be viewed as a gyrogroup whose gyroautomorphisms are trivial. Property (iii) is referred to as the *left gyroassociative law*, which plays the same role as the associative law in groups. Property (iv) is referred to as the *left loop property*, which plays a fundamental role in studying gyrogroup structures. A gyrogroup is called *degenerate* if all of its gyroautomorphisms are trivial; in this case, it forms a group under the same operation.

A *gyrogroup homomorphism* is defined to be a function that preserves the gyrogroup operations. Let G be a gyrogroup. A subset H of G is called a *subgyrogroup* if H forms a gyrogroup under the operation inherited from G. If H is a subgyrogroup of G, then the *left coset* $a \oplus H$ is defined as $a \oplus H = \{a \oplus h \colon h \in H\}$. The *index* of H in G is defined as the number of distinct left cosets of H in G. A subset A of G is called a *generating set* for G if the smallest subgyrogroup of G containing A is equal to G, that is, if $G = \bigcap\{H : A \subseteq H \text{ and } H \text{ is a subgyrogroup of } G\}$. The subgyrogroup on the right-hand side is denoted by $\langle A \rangle$. The symmetric group on G is denoted by $\mathrm{Sym}(G)$. For each $a \in G$, the *left gyrotranslation* by a, written L_a, is defined by $L_a(x) = a \oplus x$ for all $x \in G$. Using a left cancellation law in a gyrogroup, one can prove that L_a belongs to $\mathrm{Sym}(G)$ for all $a \in G$. For each non-empty subset A of G, define $\ominus A = \{\ominus a \colon a \in A\}$. Also, define $\ominus\emptyset = \emptyset$. For convenience, define $\oplus A = A \cup \ominus A$. Note that $\mathrm{gyr}[a, b](A) \subseteq A$ implies $\mathrm{gyr}[a, b](\oplus A) \subseteq \oplus A$. However, the converse is not, in general, true.

3. Main results

In Section 3.1 of [8], the authors characterize the subgyrogroup generated by a subset of a gyrogroup in a rather complicated manner. In this section, we reduce the complexity by proving that any generating set for a gyrogroup can be enlarged to one that is invariant under gyroautomorphisms. This result leads to a simplified description of the subgyrogroup generated by a set. Consequently, several standard results from group theory can be naturally extended to the setting of gyrogroups. We then turn our attention to algebraic properties of left and right generating sets in gyrogroups. This leads to the Maximum-Element-Order Problem for finite gyrogroups.

Let G be an arbitrary gyrogroup, and suppose that $a_i \in G$ for all $i \in \mathbb{N}$. We inductively define the following notations: $S^\ell_1(a_1) = a_1$, $S^r_1(a_1) = a_1$, $S^\ell_n(a_1, a_2, \dots, a_n) = a_1 \oplus S^\ell_{n-1}(a_2, a_3, \dots, a_n)$, and $S^r_n(a_1, a_2, \dots, a_n) = S^r_{n-1}(a_1, a_2, \dots, a_{n-1}) \oplus a_n$. Then S^ℓ_n and S^r_n are functions from G^n to G for all $n \in \mathbb{N}$. In the case when G is degenerate,

$S_n^\ell = S_n^r$ for all $n \in \mathbb{N}$. Since S_n^ℓ and S_n^r are defined entirely in terms of the gyrogroup operation, we obtain the following proposition, which shows that S_n^ℓ and S_n^r are respected by gyrogroup homomorphisms.

Proposition 1. *If φ is a homomorphism from a gyrogroup G to a gyrogroup H, then*

$$\varphi(S_n^\ell(a_1, a_2, \ldots, a_n)) = S_n^\ell(\varphi(a_1), \varphi(a_2), \ldots, \varphi(a_n))$$

and

$$\varphi(S_n^r(a_1, a_2, \ldots, a_n)) = S_n^r(\varphi(a_1), \varphi(a_2), \ldots, \varphi(a_n))$$

for all $n \in \mathbb{N}$ and for all $a_1, a_2, \ldots, a_n \in G$.

Proof. The proof proceeds by induction on n, based on the fact that S_n^ℓ and S_n^r are defined solely in terms of gyrogroup operations. □

The following two propositions deal with algebraic combinations of $S_m^\ell, S_m^r, S_n^\ell$, and S_n^r for various values of m and n. The proof of the next proposition appropriately employs the left and right gyroassociative laws.

Proposition 2. *Let G be a gyrogroup with $\emptyset \neq A \subseteq G$. Suppose that* $\mathrm{gyr}[a, b](\oplus A) \subseteq \oplus A$ *for all $a, b \in G$.*

1. *For all $m, n \in \mathbb{N}$, if $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n$ are elements in $\oplus A$, then there are elements $k_1, k_2, \ldots, k_{m+n}$ in $\oplus A$ such that*

$$S_m^\ell(a_1, a_2, \ldots, a_m) \oplus S_n^\ell(b_1, b_2, \ldots, b_n) = S_{m+n}^\ell(k_1, k_2, \ldots, k_{m+n}).$$

2. *For each $n \in \mathbb{N}$, if $a_1, a_2, \ldots, a_n$ are elements in $\oplus A$, then there are elements k_1, k_2, $\ldots, k_n$ in $\oplus A$ such that $\ominus S_n^\ell(a_1, a_2, \ldots, a_n) = S_n^\ell(k_1, k_2, \ldots, k_n)$.*

Proof. By induction on m, one can prove that if $a_1, a_2, \ldots, a_m, b_1 \in \oplus A$, then there are elements $c_1, c_2, \ldots, c_{m+1}$ in $\oplus A$ such that

$$S_m^\ell(a_1, a_2, \ldots, a_m) \oplus S_1^\ell(b_1) = S_{m+1}^\ell(c_1, c_2, \ldots, c_{m+1}),$$

where the right gyroassociative law (see Theorem 2.35 of [7]) is appropriately applied. Next, we proceed by induction on n. Fix $m \in \mathbb{N}$. Let $a_1, \ldots, a_m, b_1, \ldots, b_{n+1} \in \oplus A$. Then

$$S_m^\ell(a_1, \ldots, a_m) \oplus S_{n+1}^\ell(b_1, \ldots, b_{n+1}) = S_m^\ell(a_1, \ldots, a_m) \oplus (b_1 \oplus S_n^\ell(b_2, \ldots, b_{n+1})).$$

Set $u = S_m^\ell(a_1, a_2, \ldots, a_m)$ and $v = S_n^\ell(b_2, \ldots, b_{n+1})$. Then, by Proposition 1 and the left gyroassociative law,

$$\begin{aligned} u \oplus (b_1 \oplus v) &= (u \oplus b_1) \oplus \mathrm{gyr}[u, b_1](v) \\ &= (u \oplus b_1) \oplus S_n^\ell(\mathrm{gyr}[u, b_1](b_2), \ldots, \mathrm{gyr}[u, b_1](b_{n+1})) \\ &= S_{m+1}^\ell(c_1, \ldots, c_{m+1}) \oplus S_n^\ell(\mathrm{gyr}[u, b_1](b_2), \ldots, \mathrm{gyr}[u, b_1](b_{n+1})) \end{aligned}$$

for some $c_1, \ldots, c_{m+1} \in \oplus A$. Set $x = S^\ell_m(c_2, \ldots, c_{m+1})$, and set

$$y = S^\ell_n(\mathrm{gyr}[u, b_1](b_2), \ldots, \mathrm{gyr}[u, b_1](b_{n+1})).$$

Then

$$\begin{aligned} S^\ell_{m+1}(c_1, \ldots, c_{m+1}) \oplus S^\ell_n(\mathrm{gyr}[u, b_1](b_2), \ldots, \mathrm{gyr}[u, b_1](b_{n+1})) &= (c_1 \oplus x) \oplus y \\ &= c_1 \oplus (x \oplus \mathrm{gyr}[x, c_1](y)). \end{aligned}$$

Note that $\mathrm{gyr}[x, c_1](y) = S^\ell_n(\mathrm{gyr}[x, c_1](\mathrm{gyr}[u, b_1](b_2)), \ldots, \mathrm{gyr}[x, c_1](\mathrm{gyr}[u, b_1](b_{n+1})))$. By inductive hypothesis, $x \oplus \mathrm{gyr}[x, c_1](y) = S^\ell_{m+n}(d_1, \ldots, d_{m+n})$ for some $d_1, \ldots, d_{m+n}$ in $\oplus A$. It follows that

$$c_1 \oplus (x \oplus \mathrm{gyr}[x, c_1](y)) = c_1 \oplus S^\ell_{m+n}(d_1, \ldots, d_{m+n}) = S^\ell_{m+n+1}(c_1, d_1, \ldots, d_{m+n}),$$

which completes the induction. Part 2 can be proved by induction on n, utilizing part 1 and the identity $\ominus(a \oplus b) = \mathrm{gyr}[a, b](\ominus b \ominus a)$ (see Theorem 2.11 of [7]). □

Proposition 3. *Let G be a gyrogroup with $g \in G$, and suppose that $\emptyset \neq A \subseteq G$. If $\mathrm{gyr}[a, b](\oplus A) \subseteq \oplus A$ for all $a, b \in G$, then for all $a_1, a_2, \ldots, a_n \in \oplus A$, there are elements $b_1, b_2, \ldots, b_n$ in $\oplus A$ such that*

$$S^\ell_{n+1}(g, a_1, a_2, \ldots, a_n) = S^r_{n+1}(g, b_1, b_2, \ldots, b_n).$$

Proof. We proceed by induction on n. Let $a_1, a_2, \ldots, a_n, a_{n+1} \in \oplus A$ with $n \geq 2$. Then, using the inductive hypothesis three times, we obtain that

$$\begin{aligned} S^\ell_{n+2}(g, a_1, a_2, \ldots, a_n, a_{n+1}) &= g \oplus S^\ell_{n+1}(a_1, a_2, \ldots, a_n, a_{n+1}) \\ &= g \oplus S^r_{n+1}(a_1, a'_2, \ldots, a'_n, a'_{n+1}) \\ &= g \oplus (S^r_n(a_1, a'_2, \ldots, a'_n) \oplus a'_{n+1}) \\ &= g \oplus (S^\ell_n(a_1, a''_2, \ldots, a''_n) \oplus a'_{n+1}) \\ &= (g \oplus S^\ell_n(a_1, a''_2, \ldots, a''_n)) \oplus \mathrm{gyr}[g, S^\ell_n(a_1, a''_2, \ldots, a''_n)](a'_{n+1}) \\ &= S^\ell_{n+1}(g, a_1, a''_2, \ldots, a''_n) \oplus \mathrm{gyr}[g, S^\ell_n(a_1, a''_2, \ldots, a''_n)](a'_{n+1}) \\ &= S^r_{n+1}(g, b_1, b_2, \ldots, b_n) \oplus b_{n+1} \\ &= S^r_{n+2}(g, b_1, b_2, \ldots, b_n, b_{n+1}) \end{aligned}$$

for some $a'_i, a''_i, b_i \in \oplus A$, where the seventh equality is obtained by the invariance of $\oplus A$ under gyroautomorphisms. This completes the induction. □

As a consequence of Proposition 2, we obtain a characterization of the smallest subgyrogroup of a gyrogroup G containing a subset A of G, denoted by $\langle A \rangle$, in the case when $\oplus A$ is invariant under all the gyroautomorphisms of G. Furthermore, Proposition 3 shows that S^ℓ_n can be approximately transformed into S^r_n, and vice versa, using elements from the same set $\oplus A$.

Proposition 4. *Let G be a gyrogroup. Suppose that $\emptyset \neq A \subseteq G$. If* $\mathrm{gyr}[a,b](\oplus A) \subseteq \oplus A$ *for all $a, b \in G$, then $\langle A \rangle = \{S_n^\ell(a_1, a_2, \ldots, a_n) \colon n \in \mathbb{N}, a_1, a_2, \ldots, a_n \in \oplus A\}$.*

Proof. Set $H = \{S_n^\ell(a_1, a_2, \ldots, a_n) \colon n \in \mathbb{N}, a_1, a_2, \ldots, a_n \in \oplus A\}$. Clearly, $\oplus A \subseteq H$. Thus, $H \neq \emptyset$. Let $x, y \in H$. By Proposition 2, $\ominus x \in H$ and $x \oplus y \in H$. By the subgyrogroup criterion (see Proposition 21 in [6]), $H \leq G$. By minimality of $\langle A \rangle$, $\langle A \rangle \subseteq H$. Let $h \in H$. Then $h = S_n^\ell(a_1, a_2, \ldots, a_n)$ for some $a_1, a_2, \ldots, a_n \in \oplus A$, and so $h \in \langle A \rangle$ by the closure property. This shows that $H \subseteq \langle A \rangle$, and so equality holds. □

Proposition 4 motivates the following definition, which is a modified version of the previous definition of a left generating set for a gyrogroup found in the literature (see, for instance, page 6 of [8]).

Definition 1. *Let G be a gyrogroup. A non-empty subset A of G is a left generating set for G if for each element $g \in G$, there are elements $a_1, a_2, \ldots, a_n$ in $\oplus A$ such that $g = S_n^\ell(a_1, a_2, \ldots, a_n)$.*

A *right generating set* for G is defined in a similar fashion by replacing the condition that $g = S_n^\ell(a_1, a_2, \ldots, a_n)$ with the condition that $g = S_n^r(a_1, a_2, \ldots, a_n)$. We also define that the empty set is a left generating set and a right generating for the trivial gyrogroup. An application of Proposition 3 yields the following corollary immediately.

Corollary 1. *Let G be a gyrogroup with $\emptyset \neq A \subseteq G$. If* $\mathrm{gyr}[a,b](\oplus A) \subseteq \oplus A$ *for all $a, b \in G$, then A is a left generating set for G if and only if A is a right generating set for G.*

Using the closure property, one can show that every left or right generating set for a gyrogroup is also a generating set for the gyrogroup. Furthermore, by Proposition 4, if A is a generating set for a gyrogroup G such that $\oplus A$ is invariant under all the gyroautomorphisms of G, then A is a left generating set for G. The existence of a left generating set in a gyrogroup plays a crucial role in understanding its structure, and thus we introduce the following definition, analogous to the group case.

Definition 2. *The ℓ-rank of a gyrogroup G, denoted by $d_\ell(G)$, is defined as the smallest cardinality of a left generating set for G.*

Let G be a gyrogroup. The *r-rank* of G, denoted by $d_r(G)$, is defined in the same way. If $d(G)$ denotes the smallest cardinality of a generating set for G, then it is easy to see that $d(G) \leq d_\ell(G)$ and $d(G) \leq d_r(G)$. By definition, if A is a left (respectively, right) generating set for G and if $A \subseteq B \subseteq G$, then B is a left (respectively, right) generating set for G. Furthermore, generating sets, left generating sets, and right generating sets are invariant under gyrogroup homomorphisms. The condition of invariance under gyroautomorphisms is fundamental to the structure of a gyrogroup, giving rise to the concept of a gyro-invariant subset: a subset S of G is *gyro-invariant* if $\mathrm{gyr}[a,b](S) = S$ for all $a, b \in G$. This definition generalizes the notion of a strong subgyrogroup found in the topology literature (see, for instance, Definition 3.9 in [9]). The next theorem provides a method to modify a left generating set for a gyrogroup into a gyro-invariant left generating set.

Theorem 1. *Let G be a gyrogroup with a left generating set S. Then there exists a proper gyro-invariant left generating set for G containing $S \setminus \{e\}$.*

Proof. Note that $S\setminus\{e\}$ is a left generating set for G. Let $\mathrm{GYR}(G)$ be the subgroup of $\mathrm{Sym}(G)$ generated by all the gyroautomorphisms of G. Set $T = \bigcup_{\gamma\in \mathrm{GYR}(G)} \gamma(S\setminus\{e\})$. Then T is a left generating set for G. Note that $e \notin T$ since $\gamma(e) = e$ for all $\gamma \in \mathrm{GYR}(G)$. By construction, $\mathrm{gyr}[a, b](T) \subseteq T$ for all $a, b \in G$. By Proposition 6 of [10], $\mathrm{gyr}[a, b](T) = T$ for all $a, b \in G$. □

Lemma 1. *Let G be a gyrogroup. For all $a, g \in G$,*

$$S^{\ell}_m(a, a, \dots, a) \oplus g = S^{\ell}_{m+1}(a, a, \dots, a, g).$$

Proof. We proceed by induction on m. Let $a, g \in G$. Note that $S^{\ell}_1(a) \oplus g = a \oplus g = S^{\ell}_2(a, g)$. Assume that $S^{\ell}_m(a, a, \dots, a) \oplus g = S^{\ell}_{m+1}(a, a, \dots, a, g)$. By Lemma 2 of [6], $\mathrm{gyr}[S^{\ell}_m(a, a, \dots, a), a]$ is the identity map on G. Hence,

$$\begin{aligned}
S^{\ell}_{m+1}(a, a, \dots, a) \oplus g &= (a \oplus S^{\ell}_m(a, a, \dots, a)) \oplus g \\
&= a \oplus (S^{\ell}_m(a, a, \dots, a) \oplus \mathrm{gyr}[S^{\ell}_m(a, a, \dots, a), a](g)) \\
&= a \oplus (S^{\ell}_m(a, a, \dots, a) \oplus g) \\
&= a \oplus S^{\ell}_{m+1}(a, a, \dots, a, g) \\
&= S^{\ell}_{m+2}(a, a, \dots, a, g),
\end{aligned}$$

which completes the induction. □

The following proposition provides a simple upper bound for the minimal number of left generators of a finite gyrogroup in terms of its order and the orders of its elements.

Proposition 5. *If G is a finite gyrogroup, then*

$$d_{\ell}(G) \le \min\left\{\frac{|G|}{|a|} : a \in G\right\}. \tag{1}$$

Proof. As an application of Theorem 4.1 of [11], $|a|$ divides $|G|$ for all $a \in G$. Let b be an element in G such that $\frac{|G|}{|b|} = \min\left\{\frac{|G|}{|a|} : a \in G\right\}$. By Theorem 4.1 of [11], $\langle b\rangle = \{mb : m \in \mathbb{Z}\}$ partitions G into disjoint right cosets, namely $\langle b\rangle, \langle b\rangle \oplus g_2, \dots, \langle b\rangle \oplus g_n$, where $n = \frac{|G|}{|b|}$. We claim that $\{b, g_2, \dots, g_n\}$ is a left generating set for G. Let $g \in G$. Since $G = \langle b\rangle \cup \bigcup_{i=2}^{n}(\langle b\rangle \oplus g_i)$, we obtain that $g \in \langle b\rangle$ or $g \in \langle b\rangle \oplus g_j$. In the former case, $g = S^{\ell}_m(b, b, \dots, b)$ for some $m \in \mathbb{N}$. In the latter case, by Lemma 1, $g = (mb) \oplus g_j = S^{\ell}_m(b, b, \dots, b) \oplus g_j = S^{\ell}_{m+1}(b, b, \dots, b, g_j)$, which proves the claim. Hence, by minimality of $d_{\ell}(G)$, we obtain that $d_{\ell}(G) \le n$. □

In light of Theorem 1 and Proposition 5, we obtain that every finite gyrogroup has a (proper) gyro-invariant left generating set whose size is bounded above by a quantity determined by an intrinsic property of the corresponding gyrogroup.

Theorem 2. *Let G be an arbitrary finite gyrogroup. Then there is a gyro-invariant left generating set T for G such that*

$$d_\ell(G) \leq |T| \leq |\mathrm{GYR}(G)| \min\left\{\frac{|G|}{|a|} : a \in G\right\}. \tag{2}$$

Proof. According to Proposition 5, we can pick a left generating set A for G such that $|A| \leq \min\left\{\frac{|G|}{|a|} : a \in G\right\}$. Furthermore, we assume that $e \notin A$. As in Theorem 1, define $T = \bigcup_{\gamma \in \mathrm{GYR}(G)} \gamma(A)$. Then T is a gyro-invariant left generating set for G, and so $d_\ell(G) \leq |T|$ by minimality of $d_\ell(G)$. A direct computation shows that

$$|T| \leq \sum_{\gamma \in \mathrm{GYR}(G)} |\gamma(A)| \leq |\mathrm{GYR}(G)||A| \leq |\mathrm{GYR}(G)| \min\left\{\frac{|G|}{|a|} : a \in G\right\},$$

as required. □

Example 1. *In the gyrogroup G_8 (see page 404 of [6]), it can be verified that $A = \{1, 4\}$ is a left generating set for G_8. Furthermore, since G_8 is not cyclic, it follows that $d_\ell(G_8) = 2$. Note that $\min\left\{\frac{|G_8|}{|a|} : a \in G_8\right\} = 2$ since G_8 has an element of order 4 but has no element of order 8. As in Example 2 of [12], $|\mathrm{GYR}(G_8)| = 2$. Hence, $|\mathrm{GYR}(G_8)| \min\left\{\frac{|G_8|}{|a|} : a \in G_8\right\} = 4$. It can be verified that $T = \{1, 4, 6\}$ is a gyro-invariant left generating set for G_8. This shows that*

$$d_\ell(G_8) < |T| < |\mathrm{GYR}(G_8)| \min\left\{\frac{|G_8|}{|a|} : a \in G_8\right\}.$$

This example also indicates that the factor $|\mathrm{GYR}(G)|$ in (2) cannot be replaced with $d(\mathrm{GYR}(G))$ because $d(\mathrm{GYR}(G_8)) = 1$ and G_8 has no gyro-invariant left generating set of order 2.

Proposition 5 and Theorem 2 motivate the following problem for finite gyrogroups:

Problem 1 (Maximum-Element-Order Problem)**.** *Given a finite gyrogroup G, determine the maximum order of its elements, that is, determine*

$$\mathrm{meo}(G) = \max\{|a| : a \in G\}. \tag{3}$$

The Maximum-Element-Order Problem seeks to compute $\mathrm{meo}(G)$ for a given finite gyrogroup G, or to characterize the elements achieving this maximal order. In the case of finite groups, this problem remains an active area of research—particularly in the context of finite simple groups, permutation groups, and matrix groups—due to its applications in cryptography, combinatorial designs, and the analysis of algorithms that rely on the structure and properties of finite groups. To see a nice application of Theorem 2, recall that the left-gyrotranslation group of a gyrogroup G is the subgroup of $\mathrm{Sym}(G)$ generated by the left gyrotranslations of G, denoted by $\mathrm{LTrs}(G)$. If G is finite, then the order of $\mathrm{LTrs}(G)$ is equal to $|G||\mathrm{GYR}(G)|$ (see Corollary 1 in [12]). Hence, Theorem 2 states that there exists a gyro-invariant left generating set T for a finite gyrogroup G such that

$$d_\ell(G) \leq |T| \leq \frac{|\mathrm{LTrs}(G)|}{\mathrm{meo}(G)}. \tag{4}$$

In fact, if G is a finite gyrogroup, equation (4) relates the ℓ-rank of G, the maximum order of its elements, and the size of the left-gyrotranslation group of G.

We conclude this section by highlighting a potential connection between the algebraic results on gyrogroups established in this article and permutation groups. As shown earlier, the left-gyrotranslation group $\mathrm{LTrs}(G)$ of a finite gyrogroup G is a subgroup of the symmetric group $\mathrm{Sym}(G)$ [12]. Consequently, elements of a gyrogroup naturally induce permutations of the underlying set, and structural properties of G, such as the ℓ-rank $d_\ell(G)$ and the maximum element order $\mathrm{meo}(G)$, can be interpreted in terms of the cycle structure and generating behavior of the associated permutation group.

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Conflict of interest

The author declares no conflict of interest.

Data availability

Data sharing is not applicable to this article, as no datasets were generated or analyzed during the current study.

Use of Artificial Intelligence (AI) tools declaration

The author declares that Artificial Intelligence (AI) tools were used in the preparation of this manuscript only for language editing and grammar correction. The author takes full responsibility for the content, interpretation, and conclusions of this work.

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